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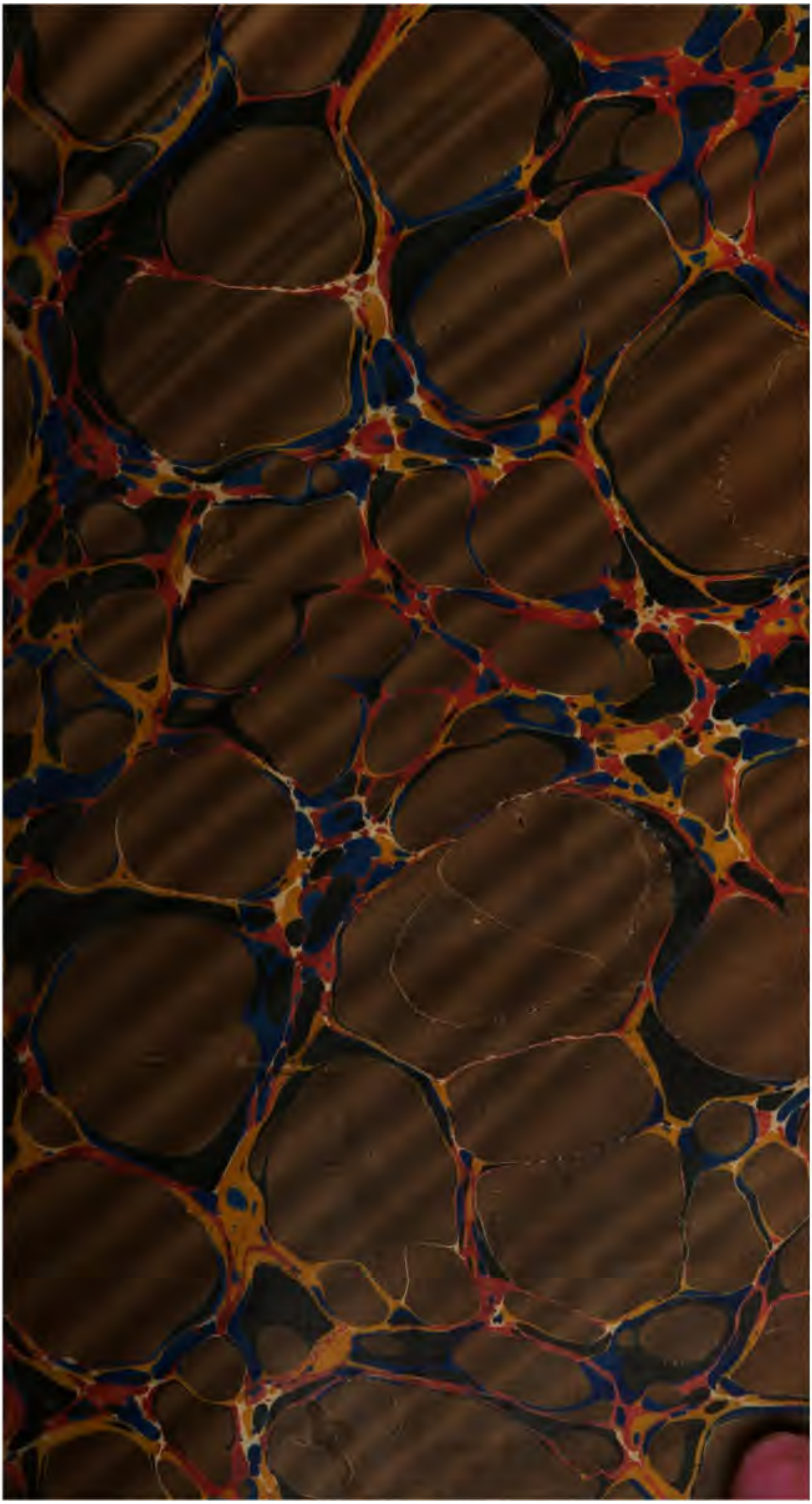

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NEW SERIES

OF THE

MATHEMATICAL

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BY THOMAS LEYBOURN,

Of the Royal Military College.

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DIRECTIONS TO THE BINDER,

I. The Binder is desired to observe, that the Vol. consists of different sets of Signatures. The first set A, B, C, D, contains the questions, and is to be placed at the beginning of the Vol. to serve as a Table of Contents to the First Part.

II. The First, Second, and Third Parts of the Vol. must follow in their order. The two sets of questions of Vol. VI. (viz. the Signatures A, B) which are stitched up with the numbers forming the Fifth Vol. must be preserved apart till the Sixth Vol. is completed.

III. The Cambridge Problems to be placed after the Third Part.

ERRATA IN THE FIFTH VOLUME.

First Part. Page 6, Formula (14) and (16) *for* tan $\frac{a}{2}$, *read* sin $\frac{a}{2}$.
Formula (15) ………… *for* tan R, *read* tan r.
Formula (20) ………… *for* $\frac{A}{4}$, *read* $\frac{A}{2}$, and
for $\frac{1}{4}$A, *read* $\frac{1}{2}$A.

In the Second Part. Page 41 *for* Article VI, *read* Article VI (*).

THE
MATHEMATICAL REPOSITORY.

MATHEMATICAL QUESTIONS.

To be answered in Number XVIII.

I. QUESTION 411, *by* THEODOSIUS.

Having given the three sides of a spherical triangle, it is required to find, in terms of these sides, the radii of the inscribed and circumscribing circles and the distance of their centres.

II. QUESTION 412, *by* GEOMETRICUS.

In a given polygon to inscribe another of the same number of sides, having its angles respectively equal to given angles, and either its perimeter or area a given magnitude.

III. QUESTION 413 *by* ANALYTICUS.

Two vessels capable of containing a and b gallons are filled with mixtures of wine and water, in given proportions for each vessel; two equal measures, c gallons each, are filled at the same time, namely one from each vessel, and then emptied into the other. Supposing this operation to be repeated n times, what would then be the proportion of wine and water in each vessel?

IV. QUESTION 414, *by* GALLICUS.

Find the position of a plane so that if a given triangle be projected orthographically upon it, the projection may be a triangle similar to a given triangle.

V. QUESTION 415, *by* MECHANICUS.

The chord of an arc described by the lower extremity of a simple pendulum being given in length, it is required to determine the length of the pendulum, when its vibrations through this arc are performed in the least time.

VI. QUESTION 416, *by* MECHANICUS.

If the point of suspension of a simple pendulum, instead of being fixed, be constrained to move along a straight horizontal line with a given uniform velocity, it is required to assign the nature of the curve described by its lower extremity, and determine the other circumstances of its motion.

VII. QUESTION 417, *by* ASTRONOMICUS.

Knowing the declination of a star, and the time of its rising or setting for any determinate place on the globe, it is required to determine the correction which must be made to give the time of rising or setting for any other place near to the former.

VIII. QUESTION 418, *by* EDINBURGENSIS.

An obelisk of a given height stands on an inclined plane given by position: What is the *locus* of that point on the plane at which a statue of a given height on the top of the obelisk subtends a given angle?

IX. QUESTION 419, *by* PALABA.

The distance of the centre of gravity of the surface of a certain solid from the vertex is equal to half the abscissa: determine the nature of the curve by the revolution of which round its axis the surface was generated.

X. QUESTION 420, *by* PROTEUS.

It is required to inscribe a triangle in a given circle, such, that the ratio of two of its sides shall be given, and the rectangle of the segments of the other side, made by a perpendicular, from the opposite angle shall be the greatest possible.

XI. QUESTION 421, *by* SCANDINIANUS.

Suppose two straight lines and a point without them to be given by position; two points, given by position, may be found, such, that if any two parallels be drawn through them, to meet the lines given by position, one of the diagonals of the trapezium thus formed shall pass through the other given point.

XII. QUESTION 422, *by* LAPUTIENSIS.

It is required to draw two right lines, from the focus of a given conic section, to terminate in the curve, so that these lines shall have a given ratio to each other, and shall also include a given angle.

XIII. QUESTION 423, *by* C. B.

In the circle of the nth order whose equation is $x^{2n} + y^{2n} = a^{2n}$, any point C being taken, and the abscissa and ordinate OF and CF being drawn, if the triangle OCF is placed in the situation OEB, then the point B shall be situated in the curve, and the sum of the two arcs DB and DC shall always be constant.

XIV. QUESTION 424, *by Mr.* CUNLIFFE, R. M. C.

Given the segments of the base, made by a perpendicular from the vertical angle, and the rectangle of the other two sides, to construct the plane triangle by a method purely geometrical.

XV QUESTION 425, *by Mr.* CUNLIFFE, R. M. C.

If circles are described upon the sides of a plane triangle as diameters; and perpendiculars drawn from the angles to the sides; the segments of these perpendiculars, intercepted by each side, and the circumference of its respective circle, will be equal to each other: required the demonstration?

XVI. QUESTION 426, *by* LAPUTIENSIS.

Suppose two given weights, appended at given points, to a flexible line of a given length, considered without weight and fastened at the ends to two given points in the same horizontal line, and imagine a third given weight, appended to the line, between the other two, but at liberty to slide freely thereon by means of a ring; it is required to determine the position of the ring, and the points in the line, to which the other two weights are appended, when at rest, in the above mentioned circumstances.

XVII. QUESTION 427, *by* C. B.

Prove that $\frac{1}{2} = \frac{1}{\cos\theta} - \frac{1}{\cos 2\theta} + \frac{1}{\cos 3\theta} - \&c.$

XVIII. QUESTION 428, *by* C. B.

Prove that the sum of the series

$$1\frac{1}{.3\ldots\ldots 2n+1} - \frac{1}{3.5\ldots\ldots 2n+3} + \&c. \text{ is equal to}$$

$$\frac{2^{-1}}{1.2\ldots\ldots n}\left\{\frac{\pi}{2} - \frac{1}{1} - \frac{1}{1.3} - \frac{1.2}{1.3.5} - \&c. \text{ to } n \text{ terms}\right\}.$$

XIX. QUESTION 429, *by* C. B.

Prove that the series

$$\frac{1}{1.2.3\ldots n} - \frac{1}{2.3.4\ldots\ldots n+1} + \frac{1}{3.4\ldots\ldots n+2} - \&c. \text{ is equal to}$$

$$\frac{1.}{1.2\ldots(n-1)}\left\{2^{n-1}\log.2 - \frac{2^{n-2}}{1} - \frac{2^{n-3}}{2} - \&c - \frac{2^0}{n-1}\right\};$$

XX. PRIZE QUESTION 430, *by* MECHANICUS.

If two bodies, connected by an inflexible and inextensible line, be placed on an horizontal plane, and one of them be constrained to move in a straight line given by position, with a uniform velocity; it is required to determine the circumstances of the motion of the other body, and the nature of the curve it describes.

* * *

Solutions to these questions must come to hand (post paid) by the first day of December, 1819.

THE

MATHEMATICAL REPOSITORY.

MATHEMATICAL QUESTIONS.

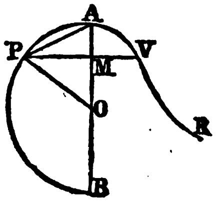
To be answered in Number XIX.

I. QUESTION 431, *by* PALABA.

A and B travelled on the same road and at the same rate from H to L. At the 50th mile stone from L, A overtook a drove of geese, which were proceeding at the rate of three miles in two hours; and two hours afterwards met a stage waggon, which was moving at the rate of nine miles in four hours. B overtook the same drove of geese at the 45th mile stone and met the same stage waggon exactly forty minutes before he came to the 31st mile stone. Where was B when A reached L?

II. QUESTION 432, *by* PALABA.

The curve AVR and semicircle APB have the same abscissa, the ordinate MV = tan $\frac{1}{2}$ arc AP: prove that the area AMV = twice the segment AP.



III. QUESTION 433, *by* PALABA.

A parabola revolves round its axis, which is vertical, in a given time, and the angular motion will just prevent a body at any point of the curve from descending: required the parameter of the parabola?

IV. QUESTION 434, *by* PALABA.

To divide a given arc (A), less then a quadrant, into two such parts (p) and (q) that $\tan^m p \times \tan^n q$ may = max.

V. QUESTION 435, *by* G. V.

A normal at any point of an equilateral hyperbola is equal to the distance of that point from the centre. Required the demonstration.

VI. QUESTION 436, *by* G. V.

C is the centre of an ellipse and F either focus, PH a tangent at P: draw the diameter PCp and pF meeting PH in H. Prove that pH = the transverse diameter.

VII. QUESTION 437, *by* G. V.

P is any point in the diameter of a circle and PB a perpendicular to the diameter: draw any chord PA, and a tangent AB meeting PB in B, and draw BD and CE (C the centre) perpendicular to PA; then PE = DA. Required a demonstration.

VIII. QUESTION 438, *by* PROTEUS.

Within a given triangle suppose another triangle to be inscribed, by joining the middle points of its sides; and again, within this triangle, suppose another triangle to be inscribed, by joining the middle points of its sides, and so on ad infinitum: What will be the limit of the aggregate, of the sum of the squares of the sides of all the triangles so formed?

IX. QUESTION 439, *by Mr.* CUNLIFFE.

Let any right line be drawn through the focus of a given conic section, terminating in the curve; then a fourth proportional to the whole line and the two segments thereof, made by the focus, will always be of the same constant length. Required a demonstration.

X. QUESTION 440, *by* ———.

A triangle being given, it is required to describe three circles, so that each circle shall touch the other two and a side of the triangle at the point of bisection.

XI. QUESTION 441, *by* ———.

Required the curve that has at each point the radius of curvature a fourth proportional to the abscissa, the ordinate, and a given straight line.

XII. QUESTION 442, *by* Palaba.

To determine the nature of the curve such that the perpendicular from a given point upon the tangent shall be a mean proportional between a given line and the segment of the axis intercepted between the tangent and this same given point.

XIII. QUESTION 443, *by* Palaba.

To determine the equation to the curve whose tangent is a mean proportional between the segment of the axis intercepted between it and a given point, and that same segment augmented by a given line.

XIV. QUESTION 444, *by* Palaba.

A body, urged by a force perpendicular to the horizon, describes the quadrant of a circle. Required the law of force which will make it recede uniformly from the horizontal radius, and the time elapsed and the velocity acquired at any point of the descent.

XV. QUESTION 445, *by* ———.

The characteristic property of the circle is, that all the chords which pass through a certain determinate point in its plane are equal: but there exists an indefinite number of curves which possess this property. It is required to find the most general equation to curves of this nature.

XVI. QUESTION 446, *by Mr.* Cunliffe.

A given rod or beam, has one end suspended by a cord of a given length, fixed at a given point above an inclined plane of a given inclination; it is required to determine the position of the beam, weight sustained by the chord, and pressure against the inclined plane, when the beam is in equilibrio.

XVII. QUESTION 447, *by* G. V.

Find the rectification and quadrature of the magnetic curve, which is such, that if lines be drawn from the poles to any point in the curve and ϕ and ψ be the angles they make with the line joining them, $\cos\phi + \cos\psi = c$, a constant quantity.

XVIII. QUESTION 448, *by* Eratosthenes.

Required a number consisting of six digits $abcdef$ such that, its multiples by 1, 2, 3, 4, 5, 6 may contain amongst them the following arrangements:

$$\begin{array}{cccccc} a & b & c & d & e & f \\ b & c & d & e & f & a \\ c & d & e & f & a & b \\ d & e & f & a & b & c \\ e & f & a & b & c & d \\ f & a & b & c & d & e. \end{array}$$

XIX. QUESTION 449, *by* Eratosthenes.

It is well-known that the formula $x^2 + x + 41$ contains a great number of primes: prove that when x is of either of the two following forms it cannot be a prime, $x = 53a + 3$ and $x = 97a - 8$.

XX. PRIZE QUESTION 450, *by* Mechanicus.

If the beam in question 401 instead of resting upon the curve, be at liberty to slide freely over the prop and descend by the force of gravity, it is required to determine the motion of its centre of gravity.

Solutions to these questions must come to hand (post paid) by the first day of June, 1820.

THE

MATHEMATICAL REPOSITORY.

MATHEMATICAL QUESTIONS.

To be answered in Number XX.

I. QUESTION 451, *by Mr.* John Baines, *Dewsbury.*

A gentleman has a triangular field, a mile round, and a perpendicular from one angle cuts the opposite side in extreme and mean proportion. Required the sides and area, when the latter is a maximum.

II. QUESTION 452, *by Mr.* Nostt.

If (R) and (r) be the radii of the circles inscribed and circumscribed about plane triangle. Prove $Rr = \frac{abc}{2(a+b+c)}$; a, b, c being the sides.

III. QUESTION 453, *by Mr.* Nostt.

Find the sum of the recurring decimal $\cdot qppp$, &c. in infinitum. (q) and (p) containing (m) and (n) digits respectively.

IV. QUESTION 454, *by* A. D.

AB is a common tangent to two circles AH, BK, bisected in E. EF is perpendicular to the line of the centres; then if any point G is taken in EF, the tangents GH, GK, are equal. Required the investigation?

V. QUESTION 455, *by* A. D.

AB is a chord in a given circle, bisected in C. DE, FG are any two chords, FE, DG are joined. Shew that CH = CK, *analytically.*

VI. QUESTION 456, *by the Rev. Mr.* W. Callow.

Solve the following equations:

1. Given to find x and z the two equations

$$z - ax = x^4 + 3b, \text{ and } \frac{az}{x} = \left(\frac{b}{x^2}\right)^2.$$

2. Given to find x and z the two equations

$$z + ab \cdot \left(\frac{b}{x}\right)^3 = 5x - 5b, \text{ and } abz = \overline{1+a} \cdot x^2.$$

3. Given to find x, y and z, the three equations.

$$x^{12} + ax^9y^3 + 3bx^7y^5 = \frac{b^3}{a}y^{12}$$

$$x^4 + xz = 5cx^2 - 5c^2$$

and

$$y^2z = d.$$

VII. QUESTION 457, *by Mr.* John Butterworth.

Let ACB be a given semicircle, CD any ordinate, and let AC be joined. It is required to determine the quadrature of the curve which is the *locus* of the centre of the circle inscribed in the triangle ACD.

VIII. QUESTION 458, *by the Rev. Mr.* L Evans.

It is recorded that Menechmus was one of those ancient mathematicians who solved the delian problem by two methods; one of which was, by the construction of two parabolas, having one common summit with their axes perpendicular to each other, and for their respective parameters the sides of the given cube and double that side; then the two ordinates drawn to the point of intersection of the two curves are two mean proportionals between those parameters, the first of which, Menechmus discovered, would be the side of the cube sought. Required a demonstration? See Bossut's General History of mathematics, whence the above originated.

IX. QUESTION 459, *by Mr.* John Baines.

Admit the matter of a planet, whose axis is to its equatorial diameter as 399 to 400, were in a state of fluidity at the commencement of rotation. Required the time of one revolution on its axis.

X. QUESTION 460, *by Mr.* Cunliffe, *R. M. C.*

Having a given, even, square integer, it is required to determine a certain number of integers, such, that if they are severally augmented or diminished by the given square, the several results shall all be rational squares.

XI. QUESTION 461, *by Mr.* NOSYT.

Suppose T and t to be parts of the tangent at the vertex A of a rectangular hyperbola (whose semi-axis = 1) cut off by lines CP, Cp, drawn from the centre to the curve: shew that if sector CAP = (n) times sector CAp,

$$\frac{1-T}{1+T} = \left\{\frac{1-t}{1+t}\right\}^n.$$

XII. QUESTION 462, *by Mr.* THOMAS WHITE, *Dumfries,*

Required the sum S of the infinite series

$$1 + \frac{x}{a} + \frac{x^2}{b} + \frac{x^3}{c} + \frac{x^4}{d} + \text{\&c.}$$

converging ever so slowly, by a direct method purely algebraical.

XIII. QUESTION 463, *by Mr* ADLINGFLEET.

If D and D′ be the lengths of a degree of a meridian at the equator and at the latitude λ respectively; (a) and (b) the equatorial and polar diameters:

Then $$\frac{a}{b} = \frac{\text{sine } \lambda}{\sqrt{\left\{\left(\frac{D}{D'}\right)^{\frac{2}{3}} - \cos^2 \lambda\right\}}}.$$

XIV. QUESTION 464, *by* A. D.

If any number of plane areas P, Q, R, &c. are projected on the same plane, the sum of the projections, when a maximum, is equal to the square root of the following expression:

$$P^2 + Q^2 + R^2 + \ldots + 2PQ \cos \widehat{PQ} + 2PR \cos \widehat{PR} + \ldots + 2QR \cos \widehat{QR} + \ldots\ldots$$

Required proof.

XV. QUESTION 465, *by Mr.* CUNLIFFE.

Let two given weights P and Q, be fastened to the given points C and D in the string ACD; and suppose one end of the string to be fastened to a given point A, and another part of the string to pass over the pulley B, fixed at a given point B, in the horizontal line AB; it is required to find the nature or equation of the curve which is the locus of D, while the part of the string DB, is slowly drawn over the pulley, the weights P and Q all the time hanging freely in the vertical plane ACDB.

XVI. QUESTION 466, *by Mr.* CUNLIFFE,

To find the nature of such a curve, that a tangent being drawn to any point thereof, and a perpendicular to the tangent, from a given point in the axis, the part of the tangent intercepted between the point of contact and the perpendicular, shall always be of the same given length.

XVII. QUESTION 467, *by Mr* CUNLIFFE.

Prove that the series

$$\frac{1}{(1+x)^2} \times \left\{1 + \frac{2x}{(1+x)^2} + \frac{4x^2}{(1+x)^4} + \frac{8x^3}{(1+x)^6} + \text{\&c.}\right\}$$

continued in infinitum is equal to the series

$$\frac{1}{(1-x)^2} \times \left\{1 - \frac{2x}{(1-x)^2} + \frac{4x^2}{(1-x)^4} - \frac{8x^3}{(1-x)^6} - \text{\&c.}\right\}$$

continued in infinitum; and find the sum of n leading terms of each series.

XVIII. QUESTION 468, *by Mr.* NOSYT.

Required the ratio of the time of describing the parabolic arc AP to the time of falling down PS, in terms of SP and SA. A being the vertex.

XIX. QUESTION 469, *by Mr.* NOSYT.

Sum (n) terms of the series

$$\cos \theta + 2^2 . \cos 2\theta + 3^2 . \cos 3\theta + \text{\&c.}$$

XX. PRIZE QUESTION 470, *by* C. B.

Two persons, A and B, play at "*heads and tails*". A has the privilege of regulating the bets at each throw which he does thus: To the constant sum u he adds the n^{th} part of all his winnings by all the previous throws, or, if he have lost, he subtracts the n^{th} part of his loss from the same sum u. After $p + q$ throws, out of which p have been won and q lost, what is the amount of his profit?

Solutions to these questions must come to hand (post paid) by the first day of December, 1821.

THE
MATHEMATICAL REPOSITORY.

MATHEMATICAL QUESTIONS.

To be answered in Number XXI.

I. QUESTION 471, *by* G. V.

Draw a normal to a given parabola from a given point.

II. QUESTION 472, *by* PHANTOM.

In a plane triangle let r denote the radius of the inscribed circle, R, R′ and R″ the radii of the three circles touching the sides of the triangle externally; then will $\frac{1}{r} = \frac{1}{R} + \frac{1}{R'} + \frac{1}{R''}$; required a demonstration.

III. QUESTION 473, *by* G. V.

Let the sides of a triangle be a, b, c and its angles A, B, C.

Prove that $\frac{(a+b+c)^2}{4 . \text{area}} = \frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C - 1}$.

IV. QUESTION 474, *by* X.

Determine x, y, and z from the three equations

$$x^2 + xy + y^2 = a$$
$$x^2 + xz + z^2 = b$$
$$y^2 + yz + z^2 = c$$

a, b, c being given numbers.

V. QUESTION 475, *by* G. V.

Shew how a straight line may be drawn to touch a given circle from a given point either in the circumference or without the circle by means of a ruler only, without employing compasses.

VI. QUESTION 476, *by* A. B.

If from the extremities of the base of a spherical triangle perpendiculars be drawn to the arc which bisects the vertical angle: Prove that the rectangle contained by the sines of the perpendiculars is equal to the rectangle of the sines of the segments of the base made by the contact of the inscribed circle.

VII. QUESTION 477, *by* A—B.

To find an expression for the area of a spherical quadrilateral inscribed in a small circle of the sphere in terms of the four sides of the quadrilateral.

VIII. QUESTION 478, *by* G V.

If on A, one end of the diameter of a circle an arc, DHE be described meeting the circle in D and E, and the chord DE be drawn meeting the diameter in F. Prove that the space between the arcs DBE and DHE is equal to AC . DF + CF . arc BD.

IX. QUESTION 479, *by* G. V.

Let AB, BC be two straight lines given in position, and A, B given points in these lines: if AV, BV be inflected to a point V, so that the tangents of the angles VAC, VBC may have a given ratio; what is the locus of the point V?

X. QUESTION 480, *by* IGNOTUS.

Investigate the nature of the curve in which a line drawn from the foot of the ordinate perpendicular on the tangent is an invariable magnitude.

XI. QUESTION 481, *by* IGNOTUS.

Find the area and length of a curve whose equation is $(x^2 - bx + y^2)^2 = a^2(x^2 + y^2)$.

XII. QUESTION 482, *by* PHANTOM.

Find the sum of n terms of the series

$$\sin^3 x + \sin^3 2x + \sin^3 3x + \sin^3 4x + \&c.$$

where x denotes a circular arc radius 1.

XIII. QUESTION 483, *by Mr* T. S. DAVIES, *Bath.*

Let any three circles (A) (B) (C) be given, and let them be cut by any other circle, the three chords which join the extremities of the segments cut off will form a triangle whose angles are on three given lines: and if any two circles cut these, the two chords which are in (A) intersecting in a, those in (B) intersecting in b, and those in (C) intersecting in c, the points a, b, c are in a straight line.

XIV. QUESTION 484, *by* G. V.

Find the nature of a curve such, that the curvilineal spaces contained between any three ordinates to the axis and the axis, shall be to one another as the arcs of the curve between these ordinates.

XV. QUESTION 485, *by* A. B.

Supposing the form of the earth to be perfectly spherical, what curve must a person describe on its surface so, that at every point of his journey, he may be at the same distance from London that he is from the equator.

XVI. QUESTION 486, *by* A. B.

If from a point perpendiculars be drawn to four straight lines given by position, it is required to describe the locus of that point when the rectangle of two of the perpendiculars has to the sum of the squares of the other two a given ratio.

XVII. QUESTION 487, *by Mr.* T. GALLOWAY, A. M. *R. M. College*

Let A and B be two points without a circle of which O is the centre, and let BO × OC = square of radius and AO × OD = square of radius; and let BD and AC meet in E. Then, if any two chords FG and HK be drawn through E, making respectively equal angles with OA and OB, the four points A, F, C, G, as also the four points B, H, D, K, are situated in the circumference of a circle, and the diameter of the circle AFCG is to the diameter of the circle BHDK in a constant ratio. Required the demonstration?

XVIII. QUESTION 488, *by Mr.* GALLOWAY.

Let AB, AC be two given straight lines, and $a : b$ a given ratio; two other straight lines AD, AE may be found, such, that if from any point whatever P in the plane of the given straight lines, there be drawn PB, PC, PD, PE respectively perpendicular to AB, AC, AD, AE, then $PD^2 + \frac{b}{a} PE^2$ is to PA × PB in a constant ratio.

XIX. QUESTION 489, *by* PERIGRINATOR.

Let there be a system of four points, accessible (and either visible or invisible from one another). Let there be a system of four other points, inaccessible, but each of them visible from all the four points of the first system. The angles are observed under which the distances of all the four points of the second system are seen from each point of the first system: required the positions of these eight points in respect to one another.

XX. PRIZE QUESTION 490, *by* ———.

Let a portion of the surface of an oblate spheroid be projected on the surface of a given sphere in such a manner that its differential elements and their projections may be similar: It is required 1st. to assign a relation between any point on the spheroid and its corresponding point on the sphere in terms of their respective latitudes and longitudes; and 2nd. to determine at a given latitude the ratio of the corresponding linear elements of the two surfaces, supposing that at another given latitude it approaches the nearest possible to a ratio of equality.

Solutions to these questions must come to hand (post paid) by the first day of September, 1828.

THE
MATHEMATICAL REPOSITORY.

VOL. V. PART I.

MATHEMATICAL QUESTIONS.

ARTICLE I.

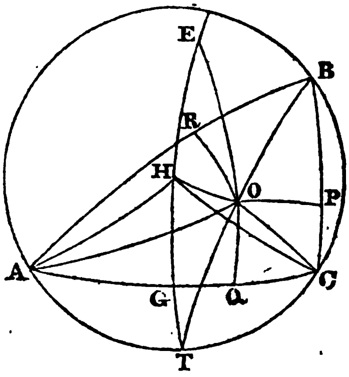
Solutions to Questions proposed in Number XVI.

I. QUESTION 411, *by* THEODOSIUS.

Having given the three sides of a spherical triangle, it is required to find, in terms of these sides, the radii of the inscribed and circumscribing circles, and the distance of their centres.

SOLUTION, *by Mr.* LOWRY, *R. M. College.*

Let ABC be a spherical triangle, O the centre of the inscribed, and H the centre of the circumscribing circle.



Draw the arcs OP, OQ, OR, and HG perpendicular to the sides of the triangle, and join AH, HC; CO, BO and AO by arcs of great circles.

Let a, b and c denote the sides opposite to the angles A, B, and C, and put $s = \frac{a+b+c}{2}$; then $\text{BP} = \text{BR} = \frac{a+c-b}{2} = s-b$, $\text{CQ} = \text{CP} = \frac{a+b-c}{2} = s-c$, and $\text{AQ} = \text{AR} = \frac{b+c-a}{2} = s-a$.

By the rule in spherical trigonometry for finding an angle from the three sides, we have

$$\tan \frac{\mathrm{B}}{2} \text{ or } \tan \mathrm{OBP} = \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s \sin(s-b)}},$$

and in the right-angled triangle OPB,

$$\begin{aligned}\tan \mathrm{OP} &= \tan \mathrm{OBP} \sin \mathrm{BP} \\ &= \tan \mathrm{OBP} \sin (s-b);\end{aligned}$$

therefore, by putting r for the radius OP,

$$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}. \qquad (\alpha)$$

Again, to find the radius of the circumscribing circle. It is evident that this circle also circumscribes the rectilineal triangle formed by the chords a', b' and c' of the arcs a, b and c, and that the angle a' B c' made by the chords a', c' is equal to half the spherical angle AHC, for it is equal to half the plane angle at the centre of the circle described about a' b' c', and this is equal to the spherical angle AHC, because H is the pole of the small circle which passes through A, B and C.

$$\text{Now} \quad \begin{aligned} a' &= 2 \sin \tfrac{1}{2}a \\ b' &= 2 \sin \tfrac{1}{2}b \\ c' &= 2 \sin \tfrac{1}{2}c, \end{aligned}$$

and, by plane trigonometry,

sin a' B c' (angle made by the chords a' c',)

$$\text{or } \sin \mathrm{GHC} = \frac{\sqrt{\left\{4a'^2c'^2 - (a'^2 + c'^2 - b'^2)^2\right\}}}{2a'c'}$$

$$= \frac{\sqrt{\left\{16 \sin^2 \frac{a}{2} \sin^2 \frac{c}{2} - 4\left(\sin^2 \frac{a}{2} + \sin^2 \frac{c}{2} - \sin^2 \frac{b}{2}\right)\right\}}}{4 \sin \frac{a}{2} \sin \frac{c}{2}}.$$

By substituting $1 - \cos a$, $1 - \cos b$, and $1 - \cos c$ for their equals, $2\left(\sin \frac{a}{2}\right)^2$, $2\left(\sin \frac{b}{2}\right)^2$ and $2\left(\sin \frac{c}{2}\right)^2$, and putting

$$f = 2\sqrt{\left\{\sin s \sin (s-a) \sin (s-b) \sin (s-c)\right\}},$$

the numerator in the above expression becomes

$$\sqrt{\left\{f^2 + 2(1-\cos a)(1-\cos b)(1-\cos c)\right\}} \text{ or}$$

$$\sqrt{\left\{f^2 + 16\left(\sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2}\right)^2\right\}};$$

therefore $$\sin \mathrm{GCH} = \frac{\sqrt{\left\{f^2 + 16\left(\sin\frac{a}{2}\ \sin\frac{b}{2}\ \sin\frac{c}{2}\right)^2\right\}}}{4\sin\frac{a}{2}\ \sin\frac{c}{2}};$$

But $$\sin \mathrm{HC} = \frac{\sin \mathrm{CG}}{\sin \mathrm{GHC}} = \frac{\sin\frac{b}{2}}{\sin \mathrm{GHC}};$$

therefore, putting R for the radius of the circumscribing circle, we have,

$$\sin \mathrm{R} = \frac{4\sin\frac{a}{2}\ \sin\frac{b}{2}\ \sin\frac{c}{2}}{\sqrt{\left\{f^2 + 16\left(\sin\frac{a}{2}\ \sin\frac{b}{2}\ \sin\frac{c}{2}\right)^2\right\}}}; \qquad (\beta)$$

whence we derive

$$\cos \mathrm{R} = \frac{f}{\sqrt{\left\{f^2 + 16\left(\sin\frac{a}{2}\ \sin\frac{b}{2}\ \sin\frac{c}{2}\right)\right\}}} \qquad (\gamma)$$

and

$$\tan \mathrm{R} = \frac{4\sin\frac{a}{2}\ \sin\frac{b}{2}\ \sin\frac{c}{2}}{f}. \qquad (\delta)$$

These expressions are the same as those found by Lagrange, in a memoir on spherical triangles, published in the "Journal De l'ecole Polytechnique," a translation of which is given in the first volume of the Repository.

Next, to find HO the distance of the centres. Since the arcs OQ and HG are perpendicular to AC, they will meet, when produced, in the pole E of the great circle AC, and EQ, EG will be quadrants, and EH and EO the complements of HC and OQ; also GQ, which is equal to $\frac{c-a}{2}$, will be the measure of the angle HEO. Therefore, by trigonometry,

$$\cos \mathrm{HO} = \cos \mathrm{HE} \cos \mathrm{OE} + \sin \mathrm{HE} \sin \mathrm{OE} \cos\frac{c-a}{2}$$

$$= \sin \mathrm{HG}\ \sin \mathrm{R} + \cos \mathrm{HG} \cos r \cos\frac{c-a}{2}.$$

Now to find sin. and cos. of HG, we have

$$\cos HG = \frac{\cos R}{\cos \frac{b}{2}}$$

and therefore

$$\sin HG = \frac{1}{\cos \frac{b}{2}} \surd \left(\cos^2 \frac{b}{2} - \cos^2 R\right):$$

substituting these values, we have

$$\cos HO = \frac{\sin r \surd \left(\cos^2 \frac{b}{2} - \cos^2 R\right) + \cos r \cos R \cos \frac{c-a}{2}}{\cos \frac{b}{2}}.$$

All has been now found that is required by the question, but on considering the problem a little further, I have been led to the discovery of several properties of spherical triangles which have not, I believe, been noticed, and many of which are remarkable for the analogy they have to some well-known properties of plane triangles.

If we conceive the side a to be produced till it becomes equal to s, and a perpendicular erected to meet BO produced; this perpendicular will evidently be the radius of a circle, which will touch b and the two other sides a and c, when produced. If r' be put for the radius of this circle, and r'' and r''' for the radii of the circles described in a similar manner to touch a and c and the other two sides produced, we shall have

$$\tan r' = \tan OBP \sin s$$

$$= \sqrt{\frac{\sin s \sin (s-a) \sin (s-c)}{\sin (s-b)}} \qquad (1)$$

$$\tan r'' = \tan OAQ \sin s$$

$$= \sqrt{\frac{\sin s \sin (s-b) \sin (s-c)}{\sin (s-a)}} \qquad (2)$$

$$\tan r''' = \tan OCQ \sin s$$

$$= \sqrt{\frac{\sin s \sin (s-b) \sin (s-a)}{\sin (s-c)}} \qquad (3)$$

multiply each of these equations by equation (a) then,

$$\tan r \tan r' = \sin (s-a) \sin (s-c)$$

$$= \sin AQ \sin QC \qquad (4)$$

$$\tan r \tan r'' = \sin (s - b) \sin (s - c)$$
$$= \sin \mathrm{BP} \sin \mathrm{PC} \qquad (5)$$
$$\tan r \tan r''' = \sin (s - b) \sin (s - a)$$
$$= \sin \mathrm{BR} \sin \mathrm{AR}. \qquad (6)$$

Again, by multiplying equations (1) (2) (3) two and two, we have

$$\tan r' \tan r'' = \sin s \sin (s - c) \qquad (7)$$
$$\tan r' \tan r''' = \sin s \sin (s - a) \qquad (8)$$
$$\tan r'' \tan r''' = \sin s \sin (s - b) \qquad (9)$$

and, by addition,

$$\tan r' \tan r'' + \tan r' \tan r''' + \tan r'' \tan r'''$$
$$= \sin s \left\{ \sin (s - c) + \sin (s - a) + \sin (s - b) \right\}. \qquad (10)$$

This last equation suggests a pretty enough theorem relating to plane triangles, which I do not recollect to have seen noticed, namely, that the sum of the rectangles of the radii of the circles described to touch the three sides of a triangle externally, taken two and two, is equal to the square of half the perimeter of the triangle.

Again, by multiplying together equations (1) (2) (3) and recollecting that $f = 2 \sqrt{\left\{ \sin s \sin (s - a) \sin (s - b) \sin (s - c) \right\}}$, we have

$$\tan r' \tan r'' \tan r''' = \sin s \times \frac{f}{2},$$

and multiplying equation (a) by $\sin^2 s$,

$$\tan r \sin^2 s = \sin s \times \frac{f}{2};$$

therefore

$$\tan r' \tan r'' \tan r''' = \tan r \sin^2 s. \qquad (11)$$

Also,

$$\tan r \tan r' \tan r'' \tan r''' = \tan r^2 \sin^2 s. \qquad (12)$$

Let p be put for the perpendicular from B upon b, then

$$\sin p = \sin a \sin \mathrm{C},$$

but $$\sin \mathrm{C} = \frac{f}{\sin a \sin b},$$

therefore $$\sin p = \frac{f}{\sin b};$$

but from equation (α)

$$2 \tan r \sin s = f;$$

therefore, by substitution,

$$\sin b \sin p = 2 \tan r \sin s. \qquad (13)$$

By comparing equations (a) (δ) we have also

$$\tan r \tan \mathrm{R} = \frac{2 \tan \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2}}{\sin s}. \qquad (14)$$

Again, since $\sin \mathrm{B} = \frac{f}{\sin a \sin c}$,

and $2 \tan r \sin s = f$,

we have $$\tan \mathrm{R} = \frac{\sin a \sin c \sin \mathrm{B}}{2 \sin s}. \qquad (15)$$

And $$\tan \mathrm{R} = \frac{4 \cos \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2}}{\sin a \sin c \sin \mathrm{B}}. \qquad (16)$$

Also $$\tan \mathrm{BO} = \frac{\tan \mathrm{BP}}{\cos \mathrm{OBP}}$$

$$= \tan (s - b) \sqrt{\frac{\sin a \sin c}{\sin s \sin (s - b)}}. \qquad (17)$$

$$\mathrm{Tan\ CO} = \tan (s - c) \sqrt{\frac{\sin a \sin b}{\sin s \sin (s - c)}}. \qquad (18)$$

$$\mathrm{Tan\ AO} = \tan (s - a) \sqrt{\frac{\sin b \sin c}{\sin s \sin (s - a)}}. \qquad (19)$$

Let A be put to denote the area of the spherical triangle ABC, then (notes to Legendre's Geometry),

$$\sin \frac{\mathrm{A}}{4} = \frac{f}{4 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c},$$

therefore, from equation (δ),

$$\tan \mathrm{R} \sin \tfrac{1}{4}\mathrm{A} = \frac{\sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2}}{\cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}};$$

and, consequently,

$$\sin \tfrac{1}{4}\mathrm{A} = \frac{\tan \frac{a}{2} \tan \frac{b}{2} \tan \frac{c}{2}}{\tan \mathrm{R}}. \qquad (20)$$

II. QUESTION 412, *by* GEOMETRICUS.

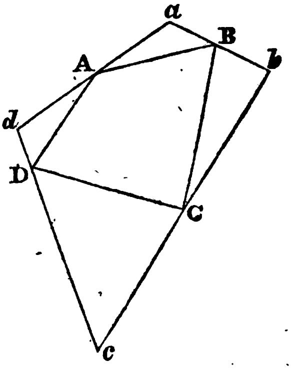
A mistake was made in transcribing this question for the press, by putting "inscribe" instead of circumscribe: It should have been

About a given polygon to circumscribe another of the same number of sides, having its angles respectively equal to given angles, and either its perimeter or area a given magnitude.

SOLUTION, *by M.* LHUILIER, *taken from the Annales de Mathématiques.*

As the process which I shall develope for the solution of each of these problems is exactly the same whatever may be the number of sides of the proposed polygon, and as the operations differ only in their length, and in the number of terms which compose the equation to which the process leads: I shall, for the sake of brevity, give only the developement for the quadrilateral.

Let ABCD be the proposed quadrilateral. It is required to circumscribe it with a quadrilateral *abcd*, the sides of which *ab*, *bc*, *cd*, *da* pass respectively through the angular points A, B, C, D of the first quadrilateral, when the angles *a*, *b*, *c*, *d* are given and either the perimeter or the area of the quadrilateral *abcd*.



Let the angles of the given polygon be designated by A, B, C and D respectively, and those of the required polygon by *a*, *b*, *c*, *d*. Also, let *x* denote one of the angles, as *a*AB, made by a side of the given polygon and the adjacent side of the polygon sought, then, by means of this angle and the angles of the polygons, we can express the mutual inclinations of the other corresponding sides of the two polygons.

We find, in effect, successively, putting unit for the measure of a right angle,

$$aAB = x,$$
$$bBC = a - B + x,$$
$$cCD = a - B + b - C + x,$$
$$dDA = a - B + b - C + c - D + x,$$

$$aBA = 2 - (a + x),$$
$$bCB = 2 - (a + b - B + x),$$
$$cDC = 2 - (a + b + c - B - C + x),$$
$$dAD = 2 - (a + b + c + d - B - C - D + x);$$

from which there results

$$\mathrm{A}a = \mathrm{AB} \cdot \frac{\sin(a+x)}{\sin a},$$

$$\mathrm{B}b = \mathrm{BC} \cdot \frac{\sin(a+b-\mathrm{B}+x)}{\sin b},$$

$$\mathrm{C}c = \mathrm{CD} \cdot \frac{\sin(a+b+c-\mathrm{B}-\mathrm{C}+x)}{\sin c},$$

$$\mathrm{D}d = \mathrm{DA} \cdot \frac{\sin(a+b+c+d-\mathrm{B}-\mathrm{C}-\mathrm{D}+x)}{\sin d},$$

$$a\mathrm{B} = \mathrm{AB} \cdot \frac{\sin x}{\sin a},$$

$$b\mathrm{C} = \mathrm{BC} \cdot \frac{\sin(a+b+x)}{\sin b},$$

$$c\mathrm{D} = \mathrm{CD} \cdot \frac{\sin(a+b-\mathrm{B}-\mathrm{C}+x)}{\sin c},$$

$$d\mathrm{A} = \mathrm{DA} \cdot \frac{\sin(a+b+c-\mathrm{B}-\mathrm{C}-\mathrm{D}+x)}{\sin d}.$$

Problem 1. When the perimeter of the required polygon is given.

From the preceding we have

$$\mathrm{A}a + a\mathrm{B} = \mathrm{AB} \cdot \frac{\sin x + \sin(a+x)}{\sin a},$$

$$\mathrm{B}b + b\mathrm{C} = \mathrm{BC} \cdot \frac{\sin(a-\mathrm{B}+x) + \sin(a+b-\mathrm{B}+x)}{\sin b},$$

$$\mathrm{C}c + c\mathrm{D} = \mathrm{CD} \cdot \frac{\sin(a+b-\mathrm{B}-\mathrm{C}+x) + \sin(a+b+c-\mathrm{B}-\mathrm{C}+x)}{\sin c},$$

$$\mathrm{D}d + d\mathrm{A} =$$

$$\mathrm{DA} \cdot \frac{\sin(a+b+c-\mathrm{B}-\mathrm{C}-\mathrm{D}+x) + \sin(a+b+c+d-\mathrm{B}-\mathrm{C}-\mathrm{D}+x)}{\sin d},$$

taking the sum of these quantities, and remarking that, in general,

$$\frac{\sin z + \sin(k+z)}{\sin k} = \operatorname{cosec} \tfrac{1}{2}k \sin(\tfrac{1}{2}k + z),$$

we find

$$ab + bc + cd + da =$$

$$\mathrm{AB} \cdot \operatorname{cosec} \cdot \tfrac{1}{2}a \sin \tfrac{1}{2}a + x),$$

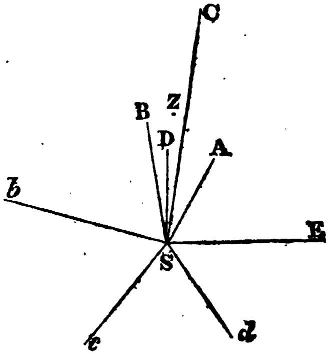
$$+ \mathrm{BC} \cdot \operatorname{cosec} \cdot \tfrac{1}{2}b \sin(a + \tfrac{1}{2}b - \mathrm{B} + x),$$

$$+ \mathrm{CD} \cdot \operatorname{cosec} \cdot \tfrac{1}{2}c . \sin(a + b + \tfrac{1}{2}c - \mathrm{B} - \mathrm{C} + x),$$

$$+ \mathrm{DA} \cdot \operatorname{cosec} \cdot \tfrac{1}{2}d \sin(a + b + c + \tfrac{1}{2}d - \mathrm{B} - \mathrm{C} - \mathrm{D} + x).$$

From this flows the following construction, founded on the properties of the centre of mean distances.

At a point S in the straight line SE, make the angles ESA, ESb, ESc, ESd, respectively equal to the angles $\frac{1}{2}a$, $a + \frac{1}{2}b$, $a + b + \frac{1}{2}c$, $a + b + c + \frac{1}{2}d$, by turning always in the same way. On the straight lines Sb, Sc, Sd make the angles bSB, cSC, dSD respectively equal to the angles B, B + C, B + C + D, by turning always the same way, but contrary to the first.



On the straight lines AS, SB, SC, SD let there be taken the lengths SA, SB, SC, SD respectively equal to AB . cosec $\frac{1}{2}a$, BC . cosec $\frac{1}{2}b$, CD . cosec $\frac{1}{2}c$, DA . cosec $\frac{1}{2}d$, and find the centre Z of the mean distances of the extremities A, B, C, D from these lines, and about Z as a centre, with a radius equal to a fourth part of the given perimeter, describe a circle.

From the point S let a tangent be drawn to this circle, and the angle formed by this tangent and the straight line SE is the angle x sought.

Remark. The given perimeter must not be greater than the quadruple of SZ. When it is less, the problem will admit of two solutions.

Problem 2. When the area of the required polygon is given.

From the formulae above, and the known expression for the area of a triangle in terms of two of the sides and the included angle,

$$4\mathrm{A}a\mathrm{B} = 2\mathrm{AB}^2 . \frac{\sin x \sin (a + x)}{\sin a},$$

$$4\mathrm{B}b\mathrm{C} = 2\mathrm{BC}^2 . \frac{\sin (a - \mathrm{B} + x) \sin (a + b - \mathrm{B} + x)}{\sin \mathrm{B}}$$

$$4\mathrm{C}c\mathrm{D} = 2\mathrm{CD}^2 . \frac{\sin (a + b - \mathrm{B} - \mathrm{C} + x) \sin (a + b + c - \mathrm{B} - \mathrm{C} + x)}{\sin c}$$

$$4\mathrm{D}d\mathrm{A} = 2\mathrm{DA}^2 . \frac{\sin (a + b + c - \mathrm{B} - \mathrm{C} + x) \sin (a + b + c + d - \mathrm{B} - \mathrm{C} - \mathrm{D} + x)}{\sin d}.$$

By adding these equations, member to member, and adding to the two members of the resulting equation quadruple the surface of the polygon ABCD, and remarking that in general

$$\frac{\sin z \sin (k+z)}{\sin k} = \frac{\cos k - \cos 2(\frac{1}{2}k+z)}{2 \sin k} = \frac{1}{2} \cot k - \frac{\cos 2(\frac{1}{2}k+z)}{2 \sin k},$$

we shall find

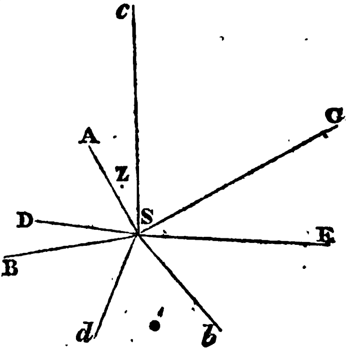
$$4abcd = 4\mathrm{ABCD} + \mathrm{AB}^2 . \cot a + \mathrm{BC}^2 . \cot b + \mathrm{CD}^2 . \cot c + \mathrm{DA}^2 . \cot d$$
$$- \mathrm{AB}^2 . \operatorname{cosec} a \cos 2 (\tfrac{1}{2}a + x)$$
$$- \mathrm{BC}^2 . \operatorname{cosec} b \cos 2 (a + \tfrac{1}{2}b - \mathrm{B} + x)$$
$$- \mathrm{CD}^2 . \operatorname{cosec} c \cos 2 (a + b + \tfrac{1}{2}c - \mathrm{B} - \mathrm{C} - x)$$
$$- \mathrm{DA}^2 . \operatorname{cosec} d \cos 2 (a + b + c + \tfrac{1}{2}d - \mathrm{B} - \mathrm{C} - \mathrm{D} + x).$$

From which flows the following construction, founded also on the properties of the centre of mean distances.

At the point S on the straight line SE, let the angles ESA, ESb, ESc, ESd be made respectively equal to the angles $2\frac{1}{2}a$, $2(a + \frac{1}{2}b)$, $2(a + b + \frac{1}{2}c)$, $2(a + b + c + \frac{1}{2}d)$, by turning always in the same way.

On the straight lines Sb, Sc, Sd, let the angles bSB, cSC, dSD, be made respectively equal to the angles 2B, 2(B + C), 2(B + C + D), by turning always in the same way, but contrary to the first.

From quadruple the excess of the area of the polygon sought above that of the polygon given, subtract the sum $\mathrm{AB}^2 . \cot a + \mathrm{BC}^2 . \cot b + \mathrm{CD}^2 . \cot c + \mathrm{DA}^2 . \cot d$, and let the remainder be equal to the rectangle of the two straight lines l and m.



Let the squares on the given sides AB, BC, CD, DA be converted into rectangles, having for one of their sides one of the straight lines as m; and let the other sides of these rectangles be α, β, γ, δ respectively.

On the straight lines SA, SB, SC, SD let there be taken from the point S the lengths SA, SB, SC, SD, respectively equal to $\alpha \operatorname{cosec} a$, $\beta \operatorname{cosec} b$, $\gamma \operatorname{cosec} c$, $\delta \operatorname{cosec} d$.

Find the centre Z of the mean distances of the points A, B, C, D, and from Z as a centre, with a radius equal to $\frac{1}{4}l$, let a circle be described.

From the point S let a tangent be drawn to this circle, and at the same point S, a perpendicular to this tangent; the angle formed by this perpendicular and the straight line SA is double of x the angle sought.

Remark. We draw from this construction, consequences, relative to the second problem, analogous to those which we deduced from the construction to the first problem.

III. QUESTION 413, *by* ANALYTICUS.

Two vessels A and B capable of containing a and b gallons are filled with mixtures of wine and water, in given proportions for each vessel; two equal measures, c gallons each, are filled at the same time, namely one from each vessel, and then emptied into the other. Supposing this operation to be repeated n times, what would then be the proportion of wine and water in each vessel?

Two Solutions taken from the Annales de Mathématiques.

FIRST SOLUTION, *by* M. LHUILIER.

Let x, x', x'', be the quantities of water which remain in the vessel A after any three consecutive operations, and y, y', y'', the corresponding quantities in the vessel B.

Then by the operation for changing the quantities of water in the two vessels from x and y to x' and y' there will be extracted from A a quantity of water equal to $\frac{c}{a}x$, and from B a quantity equal to $\frac{c}{b}y$; we have therefore

$$x' = x - \frac{c}{a}x + \frac{c}{b}y,$$

and similarly

$$x'' = x' - \frac{c}{a}x' + \frac{c}{b}y';$$

we have also

$$x + y = x' + y';$$

eliminating therefore y and y' from these three equations

$$x'' = \left(2 - \frac{c}{a} - \frac{c}{b}\right)x' - \left(1 - \frac{c}{a} - \frac{c}{b}\right)x;$$

therefore the successive quantities of water contained in the first vessel form a recurring series of the second order of which the scale of relation is

$$+\left(2 - \frac{c}{a} - \frac{c}{b}\right), \quad -\left(1 - \frac{c}{a} - \frac{c}{b}\right);$$

an equation of differences of the first order between the two variables X and z, of which the co-efficients are constant, and of which the integral is

$$X_z = a \cdot \frac{\alpha + \beta}{a + b} + C\left(1 - \frac{c}{a} - \frac{c}{b}\right)^z,^*$$

C being a constant arbitrary.

Or because $z = 0$ corresponds to $X_z = \alpha$, therefore

$$\alpha = a \cdot \frac{\alpha + \beta}{a + b} + C; \text{ whence } C = \frac{\alpha b - \beta a}{a + b},$$

and consequently

$$X_z = a \cdot \frac{\alpha + \beta}{a + b} + \frac{\alpha b - \beta a}{a + b}\left(1 - \frac{c}{a} - \frac{c}{b}\right)^z.$$

Hence if we denote simply by X and Y the quantities of water, and by X′ and Y′ the quantities of wine contained in the two vessels respectively after the nth operation, and by α' and β' the quantities of wine contained in the two vessels before the first operation, by observing that

$$X + X' = \alpha + \alpha' = a, \quad X + Y = \alpha + \beta,$$

$$Y + Y' = \beta + \beta' = b, \quad X' + Y' = \alpha' + \beta',$$

we find

$$X = a \cdot \frac{\alpha + \beta}{a + b} + \frac{\alpha b - \beta a}{a + b}\left(1 - \frac{c}{a} - \frac{c}{b}\right)^n,$$

$$X' = a \cdot \frac{\alpha' + \beta'}{a + b} + \frac{\alpha' b - \beta' a}{a + b}\left(1 - \frac{c}{a} - \frac{c}{b}\right)^n.$$

$$Y = b \cdot \frac{\alpha + \beta}{a + b} - \frac{\alpha b - \beta a}{a + b}\left(1 - \frac{c}{a} - \frac{c}{b}\right)^n,$$

$$Y' = b \cdot \frac{\alpha' + \beta'}{a + b} - \frac{\alpha' b - \beta' a}{a + b}\left(1 - \frac{c}{a} - \frac{c}{b}\right)^n.$$

Note by the Editor of the Annales.

From the equation

$$X_{z+1} = \left(1 - \frac{c}{a} - \frac{c}{b}\right) X_z + \frac{c(\alpha + \beta)}{b},$$

we deduce

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* See Traitè èlèmentaire de calcul diffèrentiel et de calcul intègral de M. Lacroix, 2me. ed. p. 573.

$$x_{z+2} = \left(1 - \frac{c}{a} - \frac{c}{b}\right) x_{z+1} + \frac{c(\alpha + \beta)}{b};$$

whence by subtracting and transposing

$$x_{z+2} = \left(2 - \frac{c}{a} - \frac{c}{b}\right) x_{z+1} - \left(1 - \frac{c}{a} - \frac{c}{b}\right) x_z;$$

an equation of the second order, from which may be deduced that of M. Lhuilier.

For integrating put $x_z = p^z$, whence $x_{z+1} = p^{z+1}$, $x_{z+2} = p^{z+2}$, which give by substitution and dividing by p^z,

$$p^2 - \left(2 - \frac{c}{a} - \frac{c}{b}\right) p + \left(1 - \frac{c}{a} - \frac{c}{b}\right) = 0;$$

whence

$$p = 1 \text{ and } p = 1 - \frac{c}{a} - \frac{c}{b};$$

therefore

$$x_z = M + N \left(1 - \frac{c}{a} - \frac{c}{b}\right)^z.$$

The constants M and N are determined by an equation of the first degree, from the initial state of the mixture.

IV. QUESTION 414, *by* GALLICUS.

Find the position of a plane so that if a given triangle be projected orthographically upon it, the projection may be a triangle similar to a given triangle.

FIRST SOLUTION, *by Mr*, LOWRY.

Let a, b and c denote the sides of the triangle which is to be projected, a', b' and c' the sides of the triangle to which the projection is to be similar, α and β the given angles made by the sides a, b and a' b'; φ, θ and ψ the angles that the sides a, b and c make with their respective projections. Then the sides of the projected triangle will be expressed by $a \cos \varphi$, $b \cos \theta$ and $c \cos \psi$; and we shall have

$$a' : b' :: a \cos \phi : b \cos \theta$$

$$\text{or} \quad \cos \theta = \frac{ab'}{a'b} \cos \phi. \qquad (1)$$

In like manner

$$\cos \psi = \frac{ac'}{a'c} \cos \phi; \qquad (2)$$

and for a third equation we have

$$a \sin \phi - b \sin \theta = c \sin \psi. \qquad (3)$$

By squaring this last and transposing,

$$a^2 \sin^2 \phi + b^2 \sin^2 \theta - c^2 \sin^2 \psi = 2ab \sin \phi \sin \theta,$$

and putting for $\sin^2 \phi$, $\sin^2 \theta$ and $\sin^2 \psi$ their values in terms of the cosines found in equations (1, 2) we get, after dividing by $2ab$,

$$\frac{a^2 + b^2 - c^2}{2ab} - \frac{ab'}{a'b}\left(\frac{a'^2 + b'^2 - c'^2}{2a'b'}\right) \cos^2 \phi = \sin \phi \sin \theta$$

$$\text{or} \quad \cos \alpha - \frac{ab'}{a'b} \cos \beta \cos^2 \phi = \sin \phi \sin \theta$$

by putting for $\frac{a^2 + b^2 - c^2}{2ab}$ and $\frac{a'^2 + b'^2 - c'^2}{2a'b'}$ their equals $\cos \alpha$ and $\cos \beta$.

Again by squaring this last equation, substituting cosines for sines, and making proper reductions, we get

$$\cos^4 \phi - \frac{a'^2b^2 + a^2b'^2 - 2aba'b' \cos \alpha \cos \beta}{a^2b'^2 \sin^2 \beta} \cos^2 \phi = -\frac{a'^2b^2 \sin^2 \alpha}{a^2b'^2 \sin^2 \beta}.$$

To abridge the expressions a little, put

$$2d^2 = a'^2b^2 + a^2b'^2 - 2aba'b' \cos \alpha \cos \beta,$$

then, by completing the square,

$$\cos^2 \phi = \frac{d^2 \pm \sqrt{(d^4 - a'^2b^2a^2b'^2 \sin^2 \alpha \sin^2 \beta)}}{a^2b'^2 \sin^2 \beta}$$

or

$$\cos \phi = \pm \frac{\sqrt{\left\{d^2 \pm \sqrt{(d^4 - a'^2b^2a^2b'^2 \sin^2 \alpha \sin^2 \beta)}\right\}}}{ab' \sin \beta}$$

and therefore

$$\cos \theta = \pm \frac{\sqrt{\left\{d^2 \pm \sqrt{(d^4 - a'^2b^2a^2b'^2 \sin^2 \alpha \sin^2 \beta)}\right\}}}{a'b \sin \beta}.$$

These expressions may be further simplified by considering that $a\,b \sin\alpha$ and $a'\,b' \sin\beta$ are the double areas of the given triangles; wherefore if A and A' be put for those quantities we shall have

$$\cos\phi = \pm \frac{\sqrt{\left\{d^2 \pm \sqrt{(d^4 - A^2A'^2)}\right\}}}{ab' \sin\beta}$$

and

$$\cos\theta = \pm \frac{\sqrt{\left\{d^2 \pm \sqrt{(d^4 - A^2A'^2)}\right\}}}{a'b \sin\beta}.$$

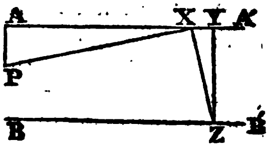
The angles ϕ and θ being found, the relative position of the given triangle and the plane of projection is also determined.

Second Solution, *by* M. Lhuilier, *from the Annales de Mathématiques.*

M. Lhuilier begins his solution by resolving the problems contained in the two following lemmas.

Lemma 1. To find two straight lines of which we know the rectangle and the difference of the squares.

Lemma 2. Two parallel straight lines AA′, BB′ being given in position, and also a point P given on the same plane with these lines. It is required to find a point X in one of them as AA′, such that joining PX and drawing XZ perpendicular to PX to terminate in the other parallel in Z, the difference of the squares of XZ and PX may be a constant magnitude.



The first of these problems must be well known to our readers and we shall leave out M. L—'s solution. The second problem obviously reduces to the first by drawing PA and ZY perpendicular to AA′, for then the rectangle and difference of the squares of AX and YX are given.

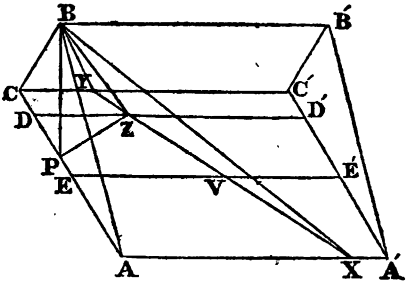
M. L. then proceeds with the following problem.

To cut a given triangular prism by a plane in such a manner that the section shall be given in species.

Let B be a given point in one of the sides BB′ of a triangular prism of which the other two sides are AA′ CC′. It is required to cut this prism by a plane passing through B so that the section may be given in species.

Let BXY be the section required.

Analysis. From the point B on the plane of the face opposite, draw the perpendicular BP. From the point P, in the common section of this face with the plane sought, draw the perpendicular PZ and join BZ. The line BZ is the height of the section, taking XY for the base and B for the vertex. The triangle BXY being given in species, the ratio XZ to ZY is known, and therefore the point Z is always in a straight line given by position parallel to AA′ and CC′, and on the plane of these straight lines; let this be the straight line DD′. The ratio of XZ to BZ is also given, and therefore if on the straight line XZ we lay off ZV equal to BZ, the point V will also be found in a straight line given by position, parallel to DD′ and in the plane of AA′ and CC′. Let EE′ be this straight line.



This being done, the difference of the squares of BZ and PZ is equal to the square of the given line BP, the difference of the spuares of ZV and PZ is equal to the square of BZ, and the lines PZ, ZN are perpendicular to one another, therefore these lines are determined by lemma 2.

Whence this construction.

From the point B, draw, upon the face opposite, the perpendicular BP. On this face let two straight lines be determined (parallel to the sides of the prism,) such that, taking any straight line on the plane of this face, the parts of this line comprized between the first parallel and the two sides, may be in the given ratio of the segments made on the base of the section by the perpendicular let fall from the vertex upon the base; and that the part of the same straight line, comprized between the two parallels, may be to the part of this line, comprized between the first and one of the sides, in the given ratio of the height of the same triangle to the corresponding segment of the base.

Let DD′ and EE′ be these two parallels. Determine in the first by lemma (2) a point Z such that, drawing from this point two straight lines perpendicular to one another, the one ZP, terminating at P, the foot of the perpendicular BP, and the other ZV, terminating in V upon EE′, so that the difference of the squares of these two lines may be equal to the square of BP. The section XBZ which passes through the point B and through the straight line ZV is the section sought.

Application to the proposed problem.

Let the projection given in species be taken for the base of a right prism; let this prism be cut by a plane in such a manner that the section shall be similar to the given triangle. The base of the prism and the section are to one another as the projection required is to the given triangle.

Cor. A parallelogram being proposed, we may project it orthographically in such a manner that its projection shall be a parallelogram given in species. In particular, we may project a parallelogram orthographically so that its projection shall be a square.

We may find immediately the angles that the sides BX and BY make with the side BB′ and consequently the inclination of the plane of the given triangle and that on which it is projected.

Let the sides BA and BC of the section perpendicular to the sides adjacent to the point B, be designated by a and c, the angle which they include by β, the angles B′BX and B′BY by x and y, the angle B of the triangle XBY by b, and lastly the ratio of the sides BX and BY by that of a to γ, we have

$$a = \mathrm{BX} \sin x, \qquad c = \mathrm{BY} \sin y$$

hence

$$a : c = a \sin x : \gamma \sin y, \text{ whence } c \sin x = \gamma \sin y;$$

therefore

$$\sin y = \frac{c}{\gamma} \sin x, \quad \cos y = \sqrt{(1 - \frac{c^2}{\gamma^2} \sin^2 x)}.$$

Or, in the solid angle formed at B, by the angles B′BX, B′BY, XBY, we have

$$\cos \beta = \frac{\cos b - \cos x \cos y}{\sin x \sin y}.$$

Putting for sin y and cos y their values, and reducing, we find

$$c \cos \beta \sin^2 x = \gamma \cos b - \cos x \sqrt{(\gamma^2 - c^2 \sin^2 x)};$$

disengaging this equation from its irrationality, and putting for $\cos^2 x$ its value $1 - \sin^2 x$, it becomes after proper reduction,

$$c^2 \sin^2 \beta \sin^4 x - (c^2 - 2c\gamma \cos b \cos \beta + \gamma^2) \sin^2 x + \gamma^2 \sin^2 b = 0;$$

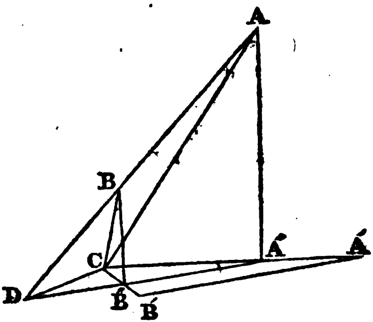
we obtain from the same

$$\gamma^4 \sin^2 \beta \sin^4 y - \gamma^2 (c^2 - 2c\gamma \cos b \cos \beta + \gamma^2) \sin^2 y + c^2 \gamma^2 \sin^2 b = 0.$$

When we have determined the angles x and y, we have determined the ratio of the dimensions of the triangle to be projected to those of its projection, and therefore we have determined the ratio of the surfaces of this triangle and its projection: but this ratio is that of radius to the cosine of the angle of inclination of the two planes, therefore this inclination is also known.

THIRD SOLUTION, *by* M. TEDENAT, *(from the Annales de Mathématiques.)*

Let ACB be the triangle to be projected, and suppose, which is permitted, that the plane of projection passes through the point C; let CD be the intersection of the plane of this projection with the plane of the triangle ACB; and from the points A, B, let fall, on the plane of projection the perpendiculars AA$''$, BB$''$, and join CA$''$, CB$''$, A$''$ B$''$, the triangle A$''$ C B$''$ is the projection of the triangle ACB, and the prolongations of the straight lines AB, A$''$B$'$ meet in the same point D in the straight line which is the intersection of the planes of the two triangles. Prolong the lines CA$''$, CB$''$ to A$'$ and B$'$ till CA$'$ and CB$'$ are respectively equal to the two sides about the angle equal to C in the triangle given in species, to which the projection of ABC is to be similar. Joining A$'$B$'$, this line is parallel to A$''$B$''$, and A$'$CB$'$ is the triangle given in species.



The triangles ACB, A$'$CB$'$ being given, let us suppose,

$$\text{CA} = a,\ \text{CB} = b,\ \text{angle ACB} = \gamma,\ \text{AB} = c$$

$$\text{CA}' = a',\ \text{CB}' = b',\ \text{angle A}'\text{CB}' = \gamma',\ \text{A}'\text{B}' = c',$$

designating by λ the unknown ratio between the homologous sides of the two triangles A$'$CB$'$, A$''$CB$''$, we have

$$\text{CA}'' = \lambda a',\ \text{CB}'' = \lambda b',\ \text{A}''\text{B}'' = \lambda c'$$

we have also

$$\text{area of ACB} = \tfrac{1}{2}ab\sin\lambda,\ \text{area of A}''\text{CB}'' = \tfrac{1}{2}\lambda^2 a'b'\sin\gamma'.$$

If therefore we designate by θ the inclination of the two planes, we have

$$\lambda^2 a'b' \sin\gamma' = ab \sin\gamma \cos\theta. \qquad (1)$$

Making $AA'' = x$, $BB'' = y$, the right-angled triangles $AA''C$, $BB''C$ and the quadrilateral $AA''B''B$ give

$$x^2 = a^2 - \lambda^2 a'^2, \qquad y^2 = b^2 - \lambda^2 b'^2,$$

$$(x-y)^2 = c^2 - \lambda^2 c'^2;$$

the last equation being subtracted from the sum of the other two, we have

$$2xy = a^2 + b^2 - c^2 - \lambda^2 (a'^2 + b'^2 - c'^2) = 2ab \cos\gamma - 2\lambda^2 a'b' \cos\gamma',$$

and dividing by 2 and squaring

$$x^2y^2 = a^2b^2\cos^2\gamma - 2\lambda^2 aba'b' \cos\gamma \cos\gamma' + \lambda^4 a'^2b'^2 \cos^2\gamma';$$

equating this value of x^2y^2 with that which results from the multiplication of the two first equations, and changing cosines and sines, we find

$$\lambda^4 a'^2 b'^2 \sin^2\gamma - (a^2b'^2 - 2aa'bb' \cos\gamma \cos\gamma' + a'^2b^2)\lambda^2 + a^2b^2\sin^2\gamma = 0,$$

and lastly, substituting for λ^2 its value given by equation (1), we have

$$a^2a'b^2b' \sin\gamma \sin\gamma' \cos^2\theta - ab(a^2b'^2 - 2aa'bb' \cos\gamma \cos\gamma'$$
$$+ a'^2b^2) \cos\theta + a^2a'b^2b' \sin\gamma \sin\gamma' = 0.$$

This equation being resolved, gives the value of $\cos\theta$[*], from whence we deduce that of λ by means of equation (1), we then have x and y by the equations

$$x^2 = a^2 - \lambda^2 a'^2, \quad y^2 = b^2 - \lambda^2 b'^2,$$

and we then know the angle BCB''; and this angle being determined, we have

$$\sin BCD = \frac{\sin BCB''}{\sin\theta},$$

and lastly,

$$\tan B''CD = \tan BCD \cos\theta,$$

we can then, without trouble, construct the respective situations of the two triangles ACB and $A''CB''$.

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[*] Note by the Editor of the Annales.

Suppose, for abridging,

$$a^2b'^2 - 2aa'bb' \cos(\gamma + \gamma') + a'^2b^2 = M^2,$$

$$a^2b'^2 - 2aa'bb' \cos(\gamma - \gamma') + a'^2b^2 = N^2;$$

whence

$$a^2b'^2 - 2aa'bb' \cos\gamma \cos\gamma' + a'^2b^2 = \tfrac{1}{2}(M^2 + N^2);$$

$$2aa'bb' \sin\gamma \cos\gamma = \tfrac{1}{4}(M^2 - N^2);$$

the values of cos θ take this very simple form

$$\cos\theta = \frac{(M^2 \pm N^2)^2}{M^2 - N^4} = \frac{M \pm N}{M \mp N};$$

or, as the adoption of the upper signs leads to an absurdity, $\cos\theta > 1$, we may write simply

$$\cos\theta = \frac{M - N}{M + N};$$

and which furnishes this very remarkable construction:

"Find a mean proportional between CA and CB′ and another "between CB and CA′; make these lines the two sides of two "triangles, of which the included angle is in the one the "sum, and in the other the difference of the angles ACB and "A′CB′; then if we construct a right-angled triangle of which "the hypothenuse is the sum, and one of the sides about the "right angle the difference, of the third sides of these triangles, "the angle opposite to the other side about the right angle, "is the measure of the inclination of the two planes."

This question was also answered by NOV-ANGLUS.

V. QUESTION 415, *by* MECHANICUS.

The chord of an arc described by the lower extremity of a simple pendulum being given in length, it is required to determine the length of the pendulum when its vibrations through this arc are performed in the least time.

SOLUTION, *by* M. J. F. FRANCAIS, *(from the Annales de Mathématiques.)*

Let $2a$ be the length of the given chord, 2α the amplitude of the corresponding oscillation, r the unknown length of the pendulum, and θ the angle that its direction makes with the vertical at any epoch t. Supposing the initial velocity to be nothing, and designating gravity by g, it is known that we have

$$dt = \sqrt{\left(\frac{r}{2g}\right)} \cdot \frac{d\,\theta}{\sqrt{(\cos\theta - \cos\alpha)}}; \qquad (1)$$

we have also

$$r \sin\alpha = a; \qquad (2)$$

by means of which, if we eliminate r from (1), we have

$$dt = \sqrt{\left(\frac{a}{2g}\right)} \cdot \frac{d\,\theta}{\sqrt{\left\{\sin\alpha\,(\cos\theta - \cos\alpha)\right\}}};$$

Integrating between $\theta = \alpha$ and $\theta = 0$, and designating by $2\mathrm{T}$ the time of a whole oscillation we find*,

$$2\mathrm{T} = \varpi \sqrt{\left(\frac{a}{g}\right)} \cdot \frac{1}{\sin\alpha} \cdot \left(1 + \mathrm{A}_1 \sin^2 \tfrac{1}{2}\alpha + \mathrm{A}_2 \sin^4 \tfrac{1}{2}\alpha + \mathrm{A}_3 \sin^6 \tfrac{1}{2}\alpha + \right); \qquad (4)$$

the coefficients A_1, A_2, A_3, ... &c. being given by the following law:

$$\mathrm{A}_n = \left\{\frac{1 \cdot 3 \cdot 5 \ldots\ldots 2n - 1}{2 \cdot 4 \cdot 6 \ldots\ldots 2n}\right\}^2. \qquad (5)$$

By considering T as a function of α, differentiating equation (4) under this point of view, and equating $\frac{d\mathrm{T}}{d\alpha}$ to zero, we find, after proper reduction,

$$1 = \mathrm{B}_1 \sin^2 \tfrac{1}{2}\alpha + \mathrm{B}_2 \sin^4 \tfrac{1}{2}\alpha + \mathrm{B}_3 \sin^6 \tfrac{1}{2}\alpha + \ldots \qquad (6)$$

an equation in which the coefficients B_1, B_2, B_3, &c. are derived by the following law:

$$\mathrm{B}_n = \frac{4n^2 + 8n - 1}{4n^2 - 4n + 1} \cdot \mathrm{A}_n. \qquad (7)$$

Equation (6) is not susceptible of exact or direct solution, but by reversion of series we find $\sin \frac{1}{2}\alpha = 0{\cdot}338255$, whence $\alpha = 17^\circ\, 7'\, 33''$, and $r = \frac{a}{\sin\alpha} = 1{\cdot}356823\, a$; equation (4) gives also

$$2\mathrm{T} = \varpi \sqrt{\left(\frac{a}{g}\right)} \cdot 1{\cdot}126105 = \sqrt{\frac{a}{0^{m},986547}}.$$

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* The details of the integration may be seen at page 415, vol. 1, of M. Poisson's *Traité de Mecanique*.

But is this value of $2\mathrm{T}$ really a minimum?

For answering this question we remark first, that if we make $\alpha = 0$ or $\alpha = 180°$, we find equally $2\mathrm{T} = \frac{0}{0} = \infty$; so that the value in question is found comprized between two *maxima*, and this is the character of a true minimum; but it is only by calculating the particular values, that we can be assured, with certainty, that their does not exist others between these limits.

By supposing successively $\alpha = 71°$ and $\alpha = 71° \; 15'$, we find $2\mathrm{T} = \varpi \sqrt{\left(\frac{a}{g}\right)} \cdot 1{\cdot}136394$, and $2\mathrm{T} = \varpi \sqrt{\left(\frac{a}{g}\right)} \cdot 1{\cdot}136376$; whence we see that the value found above is less than either of these two between which it is comprized.

VI. QUESTION 416, *by* MECHANICUS.

If the point of suspension of a simple pendulum, instead of being fixed, be constrained to move along a straight horizontal line with a given uniform velocity, it is required to assign the nature of the curve described by its lower extremity, and determine the other circumstances of its motion?

SOLUTION, *by* M. DUBUAT, *(from the Annales de Mathématiques.)*

The co-ordinates of the moveable point of suspension being x', y' and those of the lower extremity x, y, the differential equations of condition are, by putting b for the constant velocity of the point of suspension,

$$(x' - x)(\mathrm{d}x' - \mathrm{d}x) + (y' - y)(\mathrm{d}y' - \mathrm{d}y) = 0,$$

$$\mathrm{d}y' = 0, \quad \mathrm{d}x' = b\,\mathrm{d}t.$$

We have also the equations

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \mu\,(x' - x), \quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \mu\,(y' - y) - g,$$

which are those of the motion of the pendulum; its length and its mass being each taken $= 1$, its gravity $= g$, and the indeterminate μ being the tension of its rod. The elimination of μ gives

$$(y' - y)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = (x' - x)\left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - g\right).$$

Let $x' - x = \sin\varphi$, $y' - y = \cos\varphi$; if we substitute these values in $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, the preceding equation becomes

$$\frac{d^2\varphi}{dt^2} = -g\sin\varphi,$$

because $\frac{dy'}{dt} = 0$ and $\frac{d^2x'}{dt^2} = 0$,

the integral of which is

$$\left(\frac{d\varphi}{dt}\right)^2 = 2g(c + \cos\varphi).$$

The constant c is determined by the condition that at the beginning of the motion, the pendulum is in a vertical situation, and this gives $x' - x = 0$, $\sin\varphi = 0$, $\cos\varphi = 1$, and consequently $\frac{dx' - dx}{dt} = \frac{d\varphi}{dt} = b - \frac{dx}{dt}$. The angular velocity at the origin of the motion, is therefore, equal to the given velocity b, less the velocity $\frac{dx}{dt}$ of the pendulum according to the axis of x, but this being nothing, as it is easy to prove from the equations of condition, it results that the value of $\frac{d\varphi}{dt}$ at the beginning of the motion is reduced to b.

Substituting this value in the above integral, and making $\cos\varphi = 1$, we find $c = \frac{b^2}{2g} - 1$, which gives for the differential equation between φ and t.

$$d\varphi = dt\sqrt{2g}\left\{\frac{b^2}{2g} - 1 + \cos\varphi\right\}^{\frac{1}{2}},$$

an equation which is also that of the motion of the simple pendulum with a fixed centre, of which the length is 1, and its velocity at the lowest point $\frac{b^2}{2g}$.

Now let $\varphi = F(t)$ be the expression of the angular velocity in a function of the time, we have, because $x' = bt$ and $y' = 1$,

$$x = bt - \sin\left\{F(t)\right\}, \qquad y = 1 - \cos\left\{F(t)\right\}.$$

Such is the general solution; and we shall now notice some particular cases.

1st. If $b = 0$ the radical is imaginary, and there is not any angular velocity or the pendulum remains at rest. But if b, instead of being 0, be very little with regard to g, the variable cosine φ is always less then unity, so that the quantity under the radical is positive; sin φ is therefore always very little and the abscissa x differs very little from the abscissa x', and the ordinate y very little from unity. The pendulum therefore rests very near the vertical.

2nd. If the velocity b is such that $\frac{b^2}{2g} = 2$, the differential equation between dφ and dt is integrable, and we have the expression of x and y in a function of the time under a finite form without transcendents.

3rd. Lastly, if the vertical amplitude of the oscillation is equal to radius or if $b^2 = 2g$ the equation between dφ and dt is of the very simple form $d\varphi = dt\sqrt{(2g \cos \varphi)}$; but this is not integrable.

NOV-ANGLUS *answered this question.*

VII. QUESTION 417, *by* ASTRONOMICUS.

Knowing the declination of a star, and the time of its rising or setting for any determinate place on the globe; it is required to determine the correction which must be made to give the time of rising or setting for any other place near to the former.

SOLUTION, *by* M. BENJAMIN VALZ.
(From the Annales de Mathématiques.)

Let Z be the zenith distance of the star

D its declination $\begin{cases}\text{north } + \\ \text{south } -\end{cases}$

P the height of the pole or its latitude $\begin{cases}\text{north } + \\ \text{south } -\end{cases}$

H the horary angle $\begin{cases}\text{west } + \\ \text{east } -\end{cases}$

In the spherical triangle formed by the pole, the zenith and the star, we have by a known formula,

$$\cos Z = \sin D \sin P + \cos P \cos D \cos H; \qquad (1)$$

whence we draw

$$\cos H = - \tan D \tan P + \frac{\cos Z}{\cos D \cos P}. \qquad (2)$$

For finding the change that the variation of P may occasion in the value of H, we must differentiate this equation relative to those quantities considered as the only variables, this gives

$$dH = \frac{dP}{\sin H \cos^2 P}\left\{\tan D - \frac{\cos Z \sin P}{\cos D}\right\};$$

but, as we propose to make use of this formula for the epoch only of the apparent rising and setting, an epoch in which Z differs very little from 90°, cos Z will differ very little from zero, we may therefore be permitted to suppress the second term of the binomial and consequently write simply

$$dH = \frac{dP \tan D}{\sin H \cos^2 P}. \qquad (3)$$

For simplifying the expression, and avoiding at the same time the trouble of finding the declination, we may substitute for tan D its value deduced from equation (1), supposing cos Z to be left out; that is to say,

$$\tan D = -\frac{\cos H}{\tan P}; \qquad (4)$$

we then have

$$dH = -\frac{dP}{\sin P \cos P \tan H} = -\frac{2dP}{\sin 2P \tan H}. \qquad (5)$$

Such is therefore the correction that must be applied to the rising and setting at any given place, to give the proper rising and setting at another place near to the former. However simple this calculation may appear, it will still be found troublesome to have to repeat the operation each time; it will therefore be more convenient to form a table of corrections, having for argument the declination; but the preceding formula is not the most commodius for this purpose. A more simple way will be to make use of two rectilineal triangles, that is to say, by neglecting the last term in formula (2). We then have the two equations,

$$\cos H = -\tan D \tan P, \quad \cos H' = -\tan D \tan P'; \qquad (6)$$

from whence we draw, by the elimination of D,

$$\cos H' = \cos H \,.\, \frac{\tan P'}{\tan P}; \qquad (7)$$

a formula from which it is easy to draw the correction $H' - H$ with facility. To find the alteration which may be occasioned by the neglected term, we obtain by differentiating equation (1) relative to H and Z, considered as the only variables,

$$dH = \frac{dZ \sin Z}{\cos D \cos P \sin H};$$

or more simply, because Z approaches very near to 90°,

$$dH = \frac{dZ}{\cos D \cos P \sin H}.$$

We have similarly

$$dH' = \frac{dZ}{\cos D \cos P' \sin H'}.$$

and $dH' - dH$ is the error of the preceding result.

Applying these formulae to the horizon of Montpellier, which, by its distance from Paris, gives for France, one of the greatest reductions, we shall see that even in this case the part which we have neglected has very little influence on the correction found.

P, for Paris ...... $= 48° \; 50' \; 14''$; log. tan. 0·05835

P', for Montpellier $= 43 \; 36 \; 16$; log. tan. 9·97883

$$\log. \frac{\tan P'}{\tan P} = 9{\cdot}92048$$

$$\cos H = -\tan D \tan P$$

$$\cos H' = \cos H . \frac{\tan P'}{\tan P},$$

tan P ...... = 10·05835	tan P .... = 10·05835
tan. decl. sun = 9·63761 23° 28′	tan decl. moon = 9·73927 28° 45′
− cos H = 9·69596 119° 46′	− cos H = 9·79762 128° 52′
$\frac{\tan P'}{\tan P}$ = 9·92048	$\frac{\tan P'}{\tan P}$ = 9·92048
− cos H′ = 9·61644 114° 25′	− cos H′ = 9·71810 121° 30
H′ − H = 5° 21	H′ − H = 7° 22
$\frac{1}{15}$ = 0ʰ 21ᵐ, 5	$\frac{1}{15}$ = 0ʰ 29ᵐ, 5

$$dH = \frac{dZ}{\cos D \cos P \sin H}, \quad dH' = \frac{dZ}{\cos D \cos P' \sin H'}.$$

dZ = (33′) = 1·51851	1·51851	dZ = −24′ = 1·38021	1·38021
C . cos D = 0·03749	0·03749	C . cos D = 0·05714	0·05714
C . cos P = 0·18158	C . cos P′ = 0·14019	C . cos P = 0·18158	C . cos P′ = 0·14019
C . sin H = 0·06145	C . sin H′ = 0·04069	C . sin H = 0·10868	C . sin H′ = 0·06923
dH = 62′,95 = 1·79903	dH′ = 54′,56 = 1·73688	dH = −53′,41 = 1·72761	dH′ = 44′,33 = 1·64677

dH′ − dH = − 0° 8′, 40	dH′ − dH = + 0° 9′, 08
$\frac{1}{15}$ = − 0ʰ, 0ᵐ, 56.	$\frac{1}{15}$ = + 0ʰ 0ᵐ, 61.

For the sun dz is positive, and equal to the horizontal refraction (33′); but for the moon this term becomes negative and equal to (24′), the difference between the refraction and the mean horizontal parallax (57′). We have chosen the greatest declinations of the sun and of the moon, because the corrections are then the greatest, and we then see that (even in this extreme case) the term neglected alters the results only by half a minute of time. It will be proper however to take in this term when the declination exceeds 30°. This formula will be found insufficient if the declination be nearly equal to the complement of the height of the pole. In this case we must have recourse to finite differences, if it is worth the trouble, but the direct calculation appears then to be the shorter.

We shall now calculate the change of parallaxes and of declination and of right ascension, which may have taken place between the two settings and the two risings.

It is easy to be assured that the two first give a difference absolutely insensible. The change of declination, for the moon, has an influence of only 0′·05 of time and may therefore be neglected. We shall now examine that of the right ascension. Let L be the hour of setting of the moon, c the correction to be applied, and d the difference of the terrestrial meridians, we have,

$$\text{L} - [(\text{right ascen. moon} - \text{right ascen. sun}) \text{ to epoch L}] = \text{H},$$

$$\text{L} + c - [(\text{right ascen. moon} - \text{right ascen. sun}) \text{ to epoch } (\text{L} + c - d)] = \text{H}',$$

$$c = \text{H}' - \text{H} + \text{d}\,(\text{right ascen. moon} - \text{right ascen. sun})\,(c - d).$$

For reducing these formulae to tables, it will be best to employ the mean motion of the moon relative to the ecliptic, by considering this circle as the mean plane of motion of this star with respect to the equator; and making again the reduction to this last circle. For effecting this, the right-angled spherical triangle gives, by making ε = the obliquity of the ecliptic and l = the longitude, the two equations

$$\tan(\text{right ascen.}) = \cos\varepsilon \tan l, \quad \cos l = \cos(\text{right ascen.}) \cos \text{D}.$$

Differentiating the first and substituting again the value of cos (right ascen.) taken in the second, we have

$$\text{d}\,(\text{right ascen.}) = \frac{\text{d}l \,.\, \cos\varepsilon \cos^2(\text{right ascen.})}{\cos^2 l} = \frac{\text{d}l \cos\varepsilon}{\cos^2 \text{D}}.$$

Or, reckoning simply the mean motions on the equator only, we have

$$\text{mean diurnal motion} \begin{cases} \text{of the moon} = 13°,176 \\ \text{of the sun} = \underline{\ 0,986\ } \end{cases}$$

difference or relative motion of the moon 12°, 190 that which is made in a minute of time 0′, 508.

$$\text{Therefore} \quad c = d\text{H} + 0', 508 \, \frac{c-d}{15}.$$

We may easily see that these corrections are not the same for the south declination as for the north, the greatest difference being about two minutes of time.

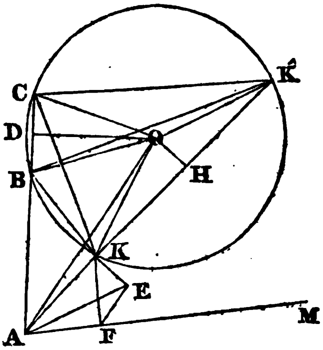
NOV-ANGLUS *also sent a solution.*

VIII. QUESTION 418, *by* EDINBURGENSIS.

An obelisk of a given height, stands on an inclined plane given by position: What is the *locus* of that point on the plane at which a statue of a given height on the top of the obelisk subtends a given angle?

FIRST SOLUTION, *by Mr.* LOWRY.

Let AB represent the obelisk, BC the statue at the top of it, and AKK′ a straight line drawn from the foot of the obelisk, in any direction, on the given inclined plane: And first, let it be required to find the points K and K′ in the line AKK′ where the statue would be seen under a given angle α. If we conceive a segment of a circle to be described on BC to contain the given angle, it is evident that this circle will intersect AK′ in the points required. Let O be its centre, and join OC, OK, OK′, AO, BK, CK, and draw OD and OH perpendicular to CB and AK′; then BC and KK′ will evidently be bisected in D and H.



Put $a = \text{CD} =$ half the height of the statue;

$b = \text{AB}$, the height of the obelisk;

then $\text{CO} = \text{OK} = \text{OK}' = \dfrac{a}{\sin\alpha}$,

$\text{OD} = a \cot \alpha$

$$\text{AO} = \sqrt{\left\{(a+b)^2 + a^2 \cot^2 \alpha\right\}}$$

$$\tan \text{CAO} = \frac{a}{a+b} \cot \alpha.$$

For simplyfying the expressions which are to follow, put $r = \text{CO}$, $r' = \text{AO}$, $\beta = \angle\ \text{CAO}$ or angle whose tangent is $\frac{a}{a+b} \cot \alpha$, and $\varphi =$ the angle CAK; then,

$$\text{OAK} = \varphi - \beta$$

$$\text{OH} = r' \sin(\varphi - \beta)$$

$$\text{AH} = r' \cos(\varphi - \beta)$$

$$\text{KH} = \text{HK}' = \sqrt{\left\{r^2 - r'^2 \sin^2(\varphi - \beta)\right\}}$$

and, therefore, since

$$\text{AK} = \text{AH} - \text{HK}$$

and

$$\text{AK}' = \text{AH} + \text{HK},$$

we have

$$\text{AK} = r' \cos(\varphi - \beta) - \sqrt{\left\{r^2 - r'^2 \sin^2(\varphi - \beta)\right\}} \quad (1)$$

$$\text{AK}' = r' \cos(\varphi - \beta) + \sqrt{\left\{r^2 - r'^2 \sin^2(\varphi - \beta)\right\}}. \quad (2)$$

When the line AK′ touches the circle, the part under the radical vanishes, and we have

$$\text{AK} = \text{AK}' = r' \cos(\varphi - \beta).$$

It is evident from the above equations (1, 2) that there are two distinct curves that will answer the question, and that the distance between them, reckoned on AK′, is

$$= 2\sqrt{\left\{r^2 - r'^2 \sin^2(\varphi - \beta)\right\}}.$$

From these expressions we can also find as many points in the curves as we please, for if we draw any line AK on the plane, and measure the angle CAK, we shall obtain, by calculation, the values of AK and AK′, corresponding to that position of AK′.

Or we may easily determine the values of these lines by construction, by drawing AK′ to intersect the circle and make with AC the proper angle. A distinct notion of the forms of the curves may be obtained by supposing the circular segment CBKK′ to revolve round the line BC; for the intersection of this circle with the plane will evidently trace out the two curves on that plane. When the circle touches the plane at the lowest point, the curves will touch each other at that point; but if the given angle be so large that the circle ceases to touch the plane, before it comes to the lowest point, the curves will meet, or rather approach indefinitely near to each other in two points, and there is then no part of the plane below these points, where the statue can be seen under the given angle.

In the case when the plane is horizontal, φ is a right angle, and the values of AK and AK′ in equations (1) (2) become

$$r' \sin\beta - \sqrt{(r^2 - r'^2 \cos^2\beta)} \qquad (3)$$

and

$$r' \sin\beta + \sqrt{(r^2 - r'^2 \cos^2\beta)} \qquad (4)$$

which, being constant quantities, indicate that the curves are in this case circles; and the distance between them is then $2\sqrt{(r^2 - r'^2 \cos^2\beta)}$. When this distance vanishes or

$$\cos\beta = \frac{r}{r'} = \frac{a}{\sin\alpha \sqrt{\left\{(a+b)^2 + a^2 \cot^2\beta\right\}}},$$ the

circles coincide, and the statue is then seen under the greatest angle possible.

So far we have only obtained the equation of the curve in terms of AK′ and the angle which that line makes with the vertical AC; but to find the equation in terms of AK′ and an angle in the given plane, let AM be the intersection of the given plane and the horizontal plane that passes through A. Let KE be perpendicular to the latter plane, and KF prependicular to AM; then EF will also be perpendicular to AM, and the angle EFK will be the inclination of the given plane to the horizon. Put γ = this angle, θ = angle KAF and z = AK; then, to find the relation between z and θ, we have,

$$z \sin\theta = \text{KF} = \frac{\text{KE}}{\sin\gamma} = \frac{z\cos\varphi}{\sin\gamma},$$

whence

$$\cos\varphi = \sin\gamma \sin\theta$$

$$\sin\varphi = \sqrt{(1 - \sin^2\gamma \sin^2\theta)}$$

$$\cos(\varphi - \beta) = \sin\gamma \sin\theta \cos\beta + \sin\beta \sqrt{(1 - \sin^2\gamma \sin^2\theta)}$$
$$\sin(\varphi - \beta) = \cos\beta \sqrt{(1 - \sin^2\gamma \sin^2\theta)} - \sin\gamma \sin\theta \sin\beta$$

and substituting these values in equations (1), (2), we get,

$$z = r' \left\{ \sin\gamma \cos\beta \sin\theta + \sin\beta \sqrt{(1 - \sin^2\gamma \sin^2\theta)} \right\} \quad (5)$$
$$\mp \sqrt{\left\{ r^2 - r'^2 \left[\cos\beta \sqrt{(1 - \sin^2\gamma \sin^2\theta)} - \sin\gamma \sin\theta \sin\beta \right]^2 \right\}}$$

which equation exhibits the value of z in terms of θ and known quantities.

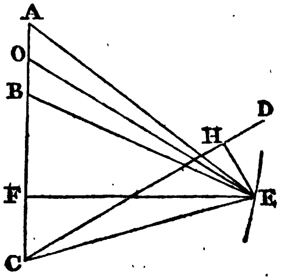
When $\theta = 0$, we have

$$z = r' \sin\beta \mp \sqrt{(r^2 - r'^2 \cos^2\beta)}$$

agreeing, as it ought to do, with the expressions (3) and (4); and these two values of z give the points where the curves cut the horizontal line AM. It is evident from equation (5) that the curve is a line of the fourth order.

SECOND SOLUTION, *by the Proposer.*

Let BC be the obelisk, AB the statue, DCE the inclined plane, E a point in the *locus*. The angle AEB is by hypothesis given in magnitude. Bisect AB in O; join EO, and draw EF perpendicular to AC. In the triangle AEB, we have



$$\text{AE} \times \text{EB} \times \sin \text{E} = \text{twice area} =$$
$$2\text{AO} \times \text{EF}:$$

But $2\,\text{AE} \times \text{EB} \times \cos \text{E} = \text{AE}^2 + \text{EB}^2 - \text{AB}^2 = 2(\text{OE}^2 - \text{AO}^2)$.

Hence, dividing the sides of the former equation by the sides of the latter, each by each, we obtain

$$\tan \text{E} = \frac{2\text{AO} \times \text{EF}}{\text{OE}^2 - \text{AO}^2};$$

$$\text{or } \tan \text{E}\,(\text{OE}^2 - \text{AO}^2) = 2\text{AO} \times \text{EF}.$$

Put the given lines CO $= h$, and AO $= a$; the variable line CE $= r$; the given angles AEB $= \varepsilon$, and ACD $= \alpha$, (ACD being supposed the angle which AC makes with the given plane DCE,)—. Also put the variable angles ACE $= \theta$, and DCE $= \varphi$. Then we have

$$\text{OE}^2 = h^2 + r^2 - 2hr \cos\ ,$$
$$\text{OE}^2 - \text{OA}^2 = h^2 - a^2 - 2hr\cos\theta + r^2,$$
$$\text{and CF} = r\sin\theta.$$

Therefore, by substitution, we obtain

$$\tan \varepsilon\,(h^2 - a^2 + r^2 - 2rh \cos \theta) = 2ar \sin \theta:$$

But by spherical trigonometry $\cos \theta = \cos \alpha \cos \varphi$,
and therefore, $\sin \theta = \sqrt{(1 - \cos^2 \alpha \cos^2 \varphi)}$.

Hence, substituting, and squaring both sides of the equation, we have

$$\tan^2 \varepsilon\,(h^2 - a^2 + r^2 - 2hr \cos \alpha \cos \varphi)^2 = 4a^2r^2(1 - \cos^2 \alpha \cos^2 \varphi).$$

Draw EH perpendicular to CD; make CH $= x$, EH $= y$, then

$$r^2 = x^2 + y^2, \text{ and } r \cos \varphi = x.$$

The preceding equation therefore becomes,

$$\tan^2 \varepsilon\,(h^2 - a^2 + x^2 + y^2 - 2hx \cos \alpha)^2 = 4a^2(x^2 + y^2 - x^2 \cos^2 \alpha).$$

If the plane be supposed horizontal, so that the angle α becomes a right angle, then $\cos \alpha = 0$, and the equation becomes

$$\tan^2 \varepsilon\,(h^2 - a^2 + x^2 + y^2)^2 = 4a^2 (x^2 + y^2);$$

or, putting r^2 for $x^2 + y^2$, and resolving the equation, we find

$$r = \frac{a}{\tan \varepsilon} \pm \sqrt{\left\{\frac{a^2}{\tan^2 \varepsilon} - (h^2 - a^2)\right\}}.$$

Hence it appears that when α is a right angle, the *locus* required is a circle whose radius is determined by the above formula.

NOV-ANGLUS *likewise sent a solution to this question.*

IX. QUESTION 419, *by* PALABA.

The distance of the centre of gravity of the surface of a certain solid from the vertex is equal to half the abscissa: determine the nature of the curve by the revolution of which round its axis the surface was generated.

FIRST SOLUTION, *by* PALABA, *the Proposer.*

By the question $\frac{\int yx\dot{z}}{\int y\dot{z}} = \frac{x}{2}$; x and y being the co-ordinates and z the corresponding arc.

$$\therefore \int yx\dot{z} = \frac{x}{2}\int y\dot{z}; \text{ take the fluxion,}$$

$$\therefore yx\dot{z} = \frac{\dot{x}}{2}\int y\dot{z} + \frac{yx\dot{z}}{2} \therefore \frac{yx\dot{z}}{2} = \frac{\dot{x}}{2}\int y\dot{z} \therefore yx\dot{z} = \dot{x}\int y\dot{z};$$

take the fluxion again, making $\dot{x}$ constant

$$\therefore\ yx\ddot{z} + y\dot{x}\dot{z} + \dot{y}x\dot{z} = y\dot{x}\dot{z}$$

$$\therefore\ yx\ddot{z} + \dot{y}x\dot{z} = 0, \text{ or } y\ddot{z} + \dot{y}\dot{z} = 0,\ \therefore\ y\dot{z} = c\dot{x},$$

$$\text{or } y\sqrt{(\dot{x}^2 + \dot{y}^2)} = c\dot{z}\ \therefore\ y^2\dot{x}^2 + y^2\dot{y}^2 = c^2\dot{x}^2$$

$$\therefore\ y^2\dot{y}^2 = (c^2 - y^2)\dot{x}^2\ \therefore\ \dot{x} = \frac{y\dot{y}}{\sqrt{(c^2 - y^2)}}\ \therefore\ x = c' - \sqrt{(c^2 - y^2)}$$

$$\therefore\ (c' - x)^2 = c^2 - y^2 \text{ and } y^2 + (c' - x)^2 = c^2;$$

and therefore the curve is a circle.

SECOND SOLUTION, *by* NOV-ANGLUS.

Let the question be generalized by supposing that the centre of gravity is required to divide the axis in any constant ratio. The general expression for the distance from the vertex, of the centre of gravity of a superficies is $\frac{\int xy\dot{z}}{\int y\dot{z}}$. An obvious method of making $\frac{\int xy\dot{z}}{x\int y\dot{z}}$ constant is to assume $y\dot{z} = ax^n\dot{x}$. By substituting and taking the fluents, (which need no correction if the superficies and abscissa be supposed to begin together,) $\frac{\int xy\dot{z}}{x\int y\dot{z}} = \frac{n+1}{n+2}$, and the distance of the centre of gravity from the vertex is $= \frac{n+1}{n+2}x$. In order therefore to make this distance to the whole abscissa in any given ratio $m : 1$, take $n = \frac{2m-1}{1-m}$, and the differential equation of the curve will be $y\sqrt{(\dot{x}^2 + \dot{y}^2)} = ax^n\dot{x}$. In several cases the relation of the fluents may be expressed in finite terms.

1. When $n = 1$, the equation becomes $y\sqrt{(\dot{x}^2 + \dot{y}^2)} = ax\dot{x}$, and belongs to a *triangle*, in which the ordinate vanishes with the abscissa, and in which a is = secant × tangent of the angle made by the base and hypothenuse. The generated superficies is that of a *cone* and the distance of the centre of gravity $= \frac{1+1}{1+2}x = \frac{2}{3}x$.

2. When $n = 0$, $\frac{n+1}{n+2}$ becomes $= \frac{1}{2}$, and is the case proposed. Here the differential equation is $y\sqrt{(\dot{x}^2 + \dot{y}^2)} = a\dot{x}$,

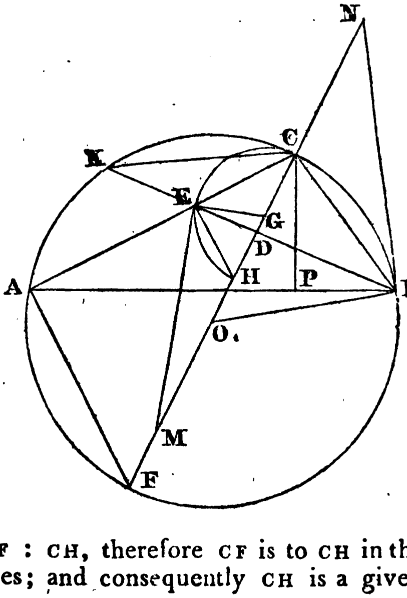
and belongs to a *circular arc* to radius $= a$. But as there is nothing which requires y to vanish when x and the superficies become $= 0$, the superficies required may be either a *whole sphere*, or *any part* of one contained between two parallel planes.

X. QUESTION 420, *by* PROTEUS.

It is required to inscribe a triangle in a given circle, such, that the ratio of two of its sides shall be given, and the rectangle of the segments of the other side, made by a perpendicular, from the opposite angle, shall be the greatest possible.

SOLUTION, *by Mr.* LOWRY.

Let ACB be the triangle required, CF the diameter of the circumscribing circle and O its centre. Draw BD perpendicular to CF, and produce it to meet AC in E, and the circle in K: join CK, AF and BO, and draw CP perpendicular to AB, and EH parallel to AF.



Because BDK is perpendicular to CF, the angles CBE, CAB are equal to one another, each being equal to the angle CKB; wherefore the triangles ACB and BCE are similar, and AC : CB :: CB : EC, or AC is to EC in the duplicate ratio of the sides; but AC : EC :: CF : CH, therefore CF is to CH in the duplicate ratio of the sides; and consequently CH is a given line.

Again, the right-angled triangles APC, BDC, and also BPC, EDC are similar; wherefore, by alternation,

$$AP : BD :: AC : BC$$

$$PB : DE :: BC : EC$$

and $$AP.PB : BD.DE :: AC : EC,$$

a given ratio; therefore when AP.PB is a maximum, BD.DE is also a maximum. Moreover, because HEC is a right angle, the point E will be in a semicircle described on the given line CH; we have therefore only to draw BE at right angles to CF, so that the points B, E may be in the semicircles CEH, CBF, and the rectangle ED.DB a maximum.

At E and B draw the tangents EM and BN to meet FN in M and N; then ED.DB will be a maximum when MN is bisected in D, for it will evidently be greater than the rectangle of the segments of any other line parallel to EB and terminating in the tangents, therefore, much more will it be greater than the rectangle of the segments terminating in the circles which are wholly included between the tangents.

Let G be the centre of the circle CEH and join EG. By the properties of the circle and right-angled triangle

$$DG \cdot DM = ED^2 = CD \cdot DH,$$

$$DO \cdot DN = DB^2 = CD \cdot DF,$$

therefore, because $DN = DM$,

$$DG \cdot DF = DO \cdot DH.$$

Whence this construction:

Take CF to CH in the duplicate ratio of the sides, and upon CF and CH as diameters describe the circles AFBC and CEH. Bisect CF and CH in O and G, and, by *Prob. IV. of Lawson's Determinate Section*, find the point D, so that $DG \cdot DF = DO \cdot DH$; through D draw DB perpendicular to CF; join BC, and draw CEA to meet the circumscribing circle in A, then ACB is the triangle required, as is evident from the Analysis.

It may be proper to remark that this solution is not materially different in principle, from some other solutions that have been given to this question in another place.

Expressions for the sides and perpendicular of the required triangle, in terms of the given quantities, may easily be obtained from the preceding analysis.

For if d be put $=$ CF the given diameter, $r =$ CH, a to b the given ratio of AC to CB, then

$$DG = CD - \frac{r}{2}$$

$$DO = \frac{d}{2} - CD$$

$$DH = r - CD$$

$$DF = d - CD$$

therefore, because $DG \cdot DF = DO \cdot DH,$

This is a very elegant and remarkable property of the triangle required, and from it we can easily deduce the values of the angles.

$$\tan^2 \text{B} = \frac{\sin^2 \text{B}}{1 - \sin^2 \text{B}}$$

$$\tan^2 \text{A} = \frac{\sin^2 \text{A}}{1 - \sin^2 \text{A}}$$

$$\text{therefore} \quad \frac{\sin^2 \text{B}}{1 - \sin^2 \text{B}} + \frac{\sin^2 \text{A}}{1 - \sin^2 \text{A}} = 2$$

and, by reduction,

$$3 \sin^2 \text{B} + 3 \sin^2 \text{A} - 4 \sin^2 \text{B} \sin^2 \text{A} = 2.$$

But $\sin \text{A} = \frac{b}{a} \sin \text{B}$, therefore by substitution, the above equation becomes

$$3 \cdot \frac{a^2 + b^2}{b^2} \cdot \sin^2 \text{A} - 4 \cdot \frac{a^2}{b^2} \cdot \sin^4 \text{A} = 2,$$

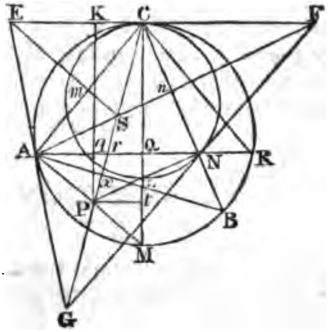
and by completing the square.

$$\sin \text{A} = \sqrt{\frac{3(a^2 + b^2) + \sqrt{(9a^4 - 14a^2b^2 + 9b^4)}}{8a^2}}$$

whence

$$\sin \text{B} = \sqrt{\frac{3(a^2 + b^2) + \sqrt{(9a^4 - 14a^2b^2 + 9b^4)}}{8b^2}}.$$

SECOND SOLUTION, *by the Proposer.*



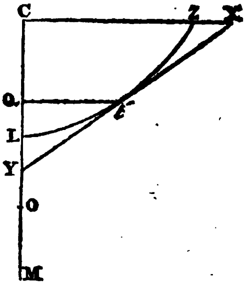
Let ACN represent the required triangle, then CP, the diameter of the circumscribing circle is given by the question. Let AC and CN be the two sides which have the given ratio; then per similar triangles CAP, CNL; CA : CN :: CP : CL a given ratio by the question: Wherefore CL is given, and by art. 3, page 117, part 2 of the 4th vol. of the Repository, CL × CM = CP², whence CM becomes known; and therefore C, L and M are three given points in the same straight line.

Now CP being a diameter of the circumscribing circle, as before adverted to, its periphery will pass through the points

C, A, P, t, N, because CAP, CtP, and CNP are right angles. Therefore by the known property of the circle $CQ \times Qt = AQ \times QN =$ the rectangle of the segments of the base AQ and QN.

But by art. 10, page 119, part ii. vol. iv. of the Repository, $Qt^2 = QL \times QM$.

Now take Qt' at right angles to CQ, and such, that $Qt'^2 = QL \times QM$, then the locus of t' will be an equilateral hyperbola whose principal axis is LM (see Emerson's Conic Sections, prop. 6, Hyperbola). To the principal transverse axis LM describe the equilateral hyperbola Lt'Z; and suppose the line Yt'X drawn to touch the curve in t', and terminate in the lines CM, CZ in Y and X. When the rectangle $CQ \times Qt = CQ \times Qt' = AQ \times QN$ is a maximum, the line YX will be bisected at the point of contact t', and therefore $CQ = QY$, (see the scholium to theorem 8, Simpson on the max. et min.



Let O be the middle of LM, or the centre of the hyperbola, then by prop. 15, Emerson on the hyperbola, $OY \times OQ = OL^2$;

$$\text{but } YQ = \frac{YC}{2} = \frac{OC - OY}{2},$$

$$OQ = OY + YQ = OY + \frac{OC - OY}{2} = \frac{OY + OC}{2};$$

$$\text{therefore} \qquad OY \times OQ = OY \times \frac{OY + OC}{2} = OL^2,$$

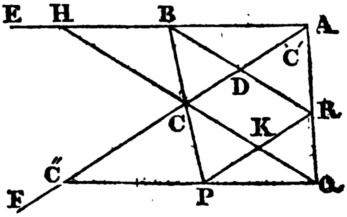
or, $OY \times (OY + OC) = 2 \times OL^2$, which determines the point Y; and Q is the middle of YC, and therefore CQ or the perpendicular of the triangle ACN becomes known, whence the problem may be easily constructed.

XI. QUESTION 421, *by* Scandinianus.

Suppose two straight lines and a point without them to be given by position; two points, given by position, may be found, such, that if any two parallels be drawn through them, to meet the lines given by position, one of the diagonals of the trapezium thus formed shall pass through the other given point.

SOLUTION, *by Mr.* LOWRY.

Let AE and AF be the straight lines given by position, P the given point, Q and R the points which are to be found, such, that if any two parallels QC, RB be drawn through them to meet AF and AE in the points C, H, B and D, the diagonal BC of the trapezium CHBD, thus formed, shall pass through the given point P.



To determine the position of the points Q, R, let us consider the extreme cases, namely, when B and C coincide with A, and when they are at an infinite distance from that point.

When RB coincides with RA, QC must coincide with QA, for if it meet AF in any other point C', the points A, C' and P would not be in a straight line, which is contrary to the hypothesis. The points A, R and Q must therefore be in a straight line. Again, when RB is parallel to AE, let QC, which is then also parallel to AE, meet AF in C''; the point B being then at an infinite distance from A, the line $C''P$, must, to agree with the hypothesis, be parallel to AE; therefore the points C'', P and Q are in a straight line, or PQ is parallel to AE. In the same manner, by supposing QC to be parallel to AF, it appears that PR is parallel to AF. Therefore, if through the given point P, two straight lines be drawn parallel to the given lines, and any straight line be drawn through A, to meet them in R and Q, these points will satisfy the conditions of the question. These points, however, have been found by tracing the law of continuity only, in the extreme cases when AB and AC were nothing, or infinite. It remains, therefore, to prove that the points B, C and P are in a straight line for every position of the parallels QC, RB,

Let QC meet PR in K; then by reason of the parallels BR, HQ,

$$BR : DR = CK :: HQ : CQ,$$

but (Euc. vi. 2.) $HQ : CQ :: AC'' : CC'' :: RP : PK;$

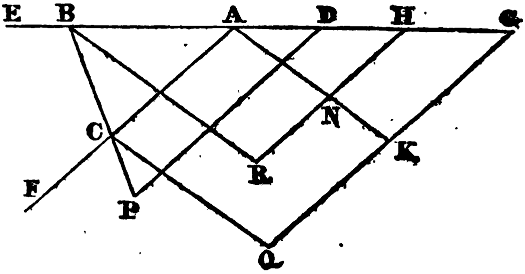
therefore, by equality of ratios,

$$BR : CK :: RP : KP:$$

consequently the points B, C and P are in a straight line.

This porism may be supposed to have been discovered in resolving the following problem, viz.

"Q and R are two given points, and AE, AF two straight lines given by position. It is required to draw through Q and R, two parallels, QC and RB, to terminate in the given lines in C and B, so that the line BC may pass through a given point P." This problem may be resolved either geometrically or algebraically, and it will be found, in both ways, that in general, it admits of two solutions, but that when the given points have a certain position, it becomes indeterminate or admits of an indefinite number of solutions. I shall give the algebraical investigation here, in preference to the geometrical, because there are not many examples in any of our books in which algebra is applied to the discovery of porisms.



From the points P, R and Q draw PD, RH, QG parallel to AF, meeting AE produced in D, H and G; also draw AK parallel to QC or RB, meeting RH in N.

Put $\text{AD} = \alpha$, $\text{PD} = \beta$.

$\text{AH} = a$, $\text{RH} = b$,

$\text{AG} = a'$, $\text{QG} = b'$,

and $\text{GK} = x$;

then because of the parallels and the similar triangles ANH, AKG;

$$\text{AC} = b' - x,\ \text{HN} = \frac{ax}{a'},$$

and
$$\text{RN} = b - \frac{ax}{a'} = \frac{a'b - ax}{a'}.$$

Again, because $\text{NH} : \text{AH} :: \text{RN} : \text{AB}$,

or
$$\frac{ax}{a'} : a :: \frac{a'b - ax}{a'} : \text{AB};$$

$$\text{AB is} = \frac{a'b - ax}{x}.$$

But, the points B, C, and P being in a straight line, the triangles BAC, BDP are equiangular; therefore

$$\frac{a'b - ax}{x} : b' - x :: \frac{a'b - ax}{x} + \alpha : \beta,$$

or $$(a'b - ax)\,\beta = (b' - x)\,(a'b - ax + \alpha x),$$

and, by reduction,

$$(a - \alpha)\,x^2 + (b'\alpha - a'b - b'a + a\beta)\,x + ba'b' - \beta ba' = 0.$$

This equation being resolved, we shall have two values of x that will answer the question. But to find, whether there is a porismatic case or not, or, under what circumstances the problem becomes indeterminate, we must consider that in that case x will be indeterminate. Now, x will evidently be indeterminate in the above equation, when the following conditions are fulfilled, viz.

$$a - \alpha = 0,$$

$$b'\alpha - a'b - b'a + a\beta = 0.$$

$$ba'b' - \beta ba' = 0.$$

From the first condition, we have $a = \alpha$ or AD = AH, which shews that the point R is in the straight line PD. From the third we have $b' = \beta$ or QG = PD, which shews that the point Q is in the straight line QP parallel to AE. Again, from the second condition, putting $a = \alpha$ and $b' = \beta$, we have $b'a = a'b$, or AH : HR :: AG : GQ, which shews that the points A, R and Q are in a straight line.

SECOND SOLUTION, *by* M. SCHUMACHER, *of the University of Copenhagen, (from the Annales de Mathématiques).*

Let C and C′ be two sides of the angle, S its summit, O the given point, P and P′ the two points sought, D and D′ the two parallels drawn respectively through these points, and lastly K and K′ their respective intersections with C and C′, it is required to determine the points P and P′ such, that in whatever common direction the two parallels be drawn, the point O may be always in a straight line with the points K and K′.

Let the summit S of the given angle be taken for the origin of the co-ordinates, its side C for the axis of x and its side C′ for the axis of y.

Designate the ordinates

of the given point O by a, β,
of the point sought P by x', y',
of the point sought P′ by x'', y''.

The equations of the two arbitrary parallels D and D′, drawn through P and P′ are of the form

$$y - y' = \text{N}\,(x - x') \qquad y - y'' = \text{N}\,(x - x'')$$

where N remains indeterminate.

We find from this, for the co-ordinates

of K .... $\text{N}x' - y', \quad 0;$

of K' .... $0, \quad y'' - \text{N}x''.$

The equation of condition, for three points (x', y'), (x'', y''), (x''', y''') in a straight line being,

$$x'(y'' - y''') + x''(y''' - y') + x'''(y' - y'') = 0,$$

it gives when applied to the points O, K, K'

$$\alpha\,(\text{N}x'' - y'') + \frac{\text{N}x' - y'}{\text{N}}\left\{(y'' - \text{N}x'') - \beta\right\} = 0$$

whence

$$x''(x - \alpha)\,\text{N}^2 - \left\{y''(x' - \alpha) + x''y' - \beta x'\right\}\text{N} + y'(y'' - \beta) = 0$$

an equation, which, by reason of N being indeterminate, may be parted into the three following,

$$x''(x' - \alpha) = 0, \quad y'(y'' - \beta) = 0,$$

$$y''(x' - \alpha) + x''y' - \beta x' = 0;$$

the problem is therefore indeterminate, since it furnishes three equations only between the four co-ordinates x', y', x'', y'', of the two points sought P and P',

The two first equations cannot be satisfied, unless by one of the four systems of value

$$\begin{cases} x'' = 0, \\ y' = 0, \end{cases} \quad \begin{cases} x'' = 0, \\ y'' = \beta, \end{cases} \quad \begin{cases} x' = \alpha \\ y' = 0 \end{cases} \quad \begin{cases} x' = \alpha \\ y'' = \beta. \end{cases}$$

Of these four systems, it is only the first and the last that can agree with the third equation.

The first, which expresses that the point P is on the axis of x and the point P' on the axis of y, changes the third equation to $x'(y'' - \beta) - \alpha y'' = 0$; which shews that the two points P and P' are in a straight line with the point O; also, in this manner, the points sought are the intersections of the two sides of the angle with a straight line drawn in any manner whatever, through the given point.

In the last system, which expresses that the points sought are on the parallels to the two axes which pass through o, it reduces the third equation to

$$x''y' - \beta x' = 0, \quad \text{or} \quad x''x' - y''x' = 0,$$

which shews that the points P and P′ are in a straight line with the origin S; and shews that in this second case, the points sought are the intersections of a straight line drawn, in any manner, through the summit of the given angle with the lines drawn through the given point parallel to the sides of that angle.

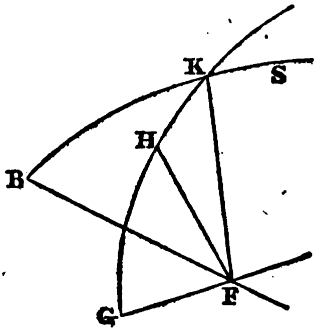
It was answered by NOV-ANGLUS.

XII. QUESTION 422, *by* LAPUTIENSIS.

It is required to draw two right lines, from the focus of a given conic section, to terminate in the curve, so that these lines shall have a given ratio to each other, and shall also include a given angle?

FIRST SOLUTION, *by Mr.* LOWRY.

Let F be the focus of the given conic section GHK, FH and FK the lines to be drawn, having the ratio of m to n, and containing a given angle. Put a = the semi-transverse, e = the excentricity, c = the given angle, θ = the angle GFH; then, by the property of the curve,



$$\text{FH} = \frac{a^2 - e^2}{a + e\cos\theta}$$

$$\text{FK} = \frac{a^2 - e^2}{a + e\cos(\theta + c)}$$

therefore

$$\frac{1}{a + e\cos\theta} : \frac{1}{a + e\cos(\theta + c)} :: m : n,$$

or $$ma + me\cos\theta = na + ne\cos(\theta + c);$$

but $$\cos(\theta + c) = \cos\theta\cos c - \sin\theta\sin c;$$

therefore

$$ma + me \cos \theta = na + ne \cos \theta \cos c - ne \sin \theta \sin c,$$

or putting, for the present,

$$b = \frac{a}{e} \cdot \frac{n - m}{n \sin c}$$

$$d = \frac{n \cos c - m}{n \sin c}$$

we have

$$b + d \cos \theta = \sin \theta,$$

or $$b^2 + 2bd \cos \theta + d^2 \cos^2 \theta = 1 - \cos^2 \theta;$$

therefore

$$\cos^2 \theta + \frac{2bd}{d^2 + 1} \cos \theta = \frac{1 - b^2}{d^2 - 1},$$

and $$\cos \theta = bd \pm \sqrt{\frac{d^2 + 1 - b^2}{d^2 + 1}};$$

or, if we now restore the values of b and d, we shall have $\cos \theta$ in terms of the given quantities a, e, m, n and c.

The problem may be resolved in the same manner when F is any other point than the focus, and GHK any other curve; and also when the lines FH and FK are required to terminate in different curves: For we shall still have

$$\text{FH} = \text{some function of } \theta = f(\theta),$$

and for the same curve,

$$\text{FK} = \text{the same function of } (\theta + c) = f(\theta + c),$$

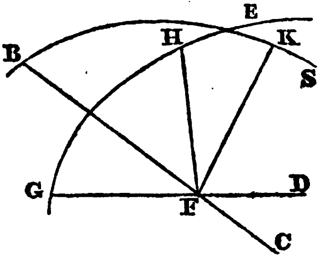
or for a different curve,

$$\text{FK} = \text{F}(\theta + c)$$

therefore $$f(\theta) : f(\theta + c) \text{ or } \text{F}(\theta + c) :: m : n;$$

The question may also be answered in a different manner by determining the *locus* of one of the points H or K when the other is in the given curve.

It is well-known that if from a given point two straight lines be drawn to contain a given angle and have a given ratio; and one of them terminate in a straight line given by position, the other will also terminate in a straight line given by position; or if one of them terminate in a circle given in magnitude and position; the other will also terminate in a circle given in magnitude and position. These are two of the well-known *Loci* of Apollonius, and it is easy to perceive that the proposition is general, or that, if one of the lines terminate in a curve of any kind given by position, the other will terminate in a *similar* curve which will also be given by position.



Let F be a given point, GHE a curve of any kind given by position, FH and FK lines containing a given angle and having a given ratio; if FH terminate in the curve GHE, FK will terminate in a similar curve. Through F draw any line GFD to meet the curve in G, and draw FB to make the angle GFB equal to the given angle HFK; take FG to FB in the given ratio of FH to FK, and through B describe a curve BKS similar to GHE and similarly situated with respect to FB as GHE is with respect to GF; this curve will be the locus of the point K: For the angle GFB being equal to HFK by construction, the angle GFH is equal to the angle BFK; therefore FH and FK, wherever drawn, make equal angles with the respective axes FG and FB, and, therefore, by the nature of similar curves,

$$FH : FG :: FK : FB,$$

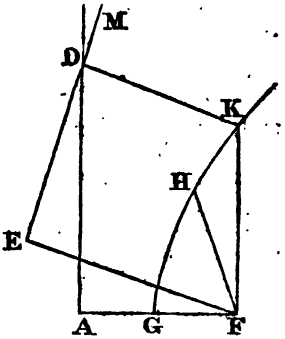
or, by alternation,

$$FH : FK :: FG : FB,$$ the given ratio.

The application of this proposition to the question is obvious; for if FB be drawn to make the angle GFB equal to the given angle, and FG to FB be taken in the given ratio, and a conic section similar to the given one, and having F for the focus, be described through B, it will intersect GHK in K, one of the required points; then if FH be drawn to make the given angle with FK, FH will have to FK the same ratio as GF to BF, the given ratio by construction.

The point of intersection of the curves, when they are conic sections, may be found without actually describing the curve BKS; for if perpendiculars be drawn from K upon the directrices of the similar curves, it is evident, since the determining ratio is the same and the point K common to both curves, that those perpendiculars are equal, and therefore the line drawn from K to the intersection of the directrices will bisect the angle made by them at that point. This consideration suggests the following very simple construction:

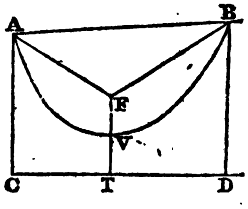
Let AD be the directrix of the given conic section, draw FE to make the angle AFE equal to the given one, and take AF to FE in the given ratio; draw EM perpendicular to FE meeting AD in D; draw DK to bisect the angle MDA, and K is the point required.



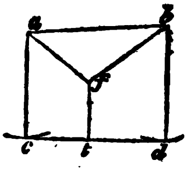
From what has been done, it is obvious how we may resolve the problem; when two conic sections, or two curves of any kind, and a point are given by position; and it is required to draw from the given point two straight lines to terminate in the curves, so as to contain a given angle and have a given ratio. From the preceding proposition we are also led to a more general one. namely, if from two given points two straight lines be drawn to contain a given angle and have a given ratio, and one of them terminate in a curve of any kind given by position, the other will terminate in a similar curve which is also given by position.

SECOND SOLUTION, *by Mr.* CUNLIFFE.

Let AVB be the given conic section; F its focus; V, its principal vertex; CTD its directrix cutting FV at right angles in T. Form a triangle *afb*, having the sides *fa*, *fb* in the given ratio of the lines from the focus to the curve, and the angle *afb*, equal to the given angle included by those lines, and join *ab*; then the triangle *afb* will be manifestly given in species.



Take *fb* to *bd* in the given ratio of FV to VT; also take *fa* to *ac* in the same given ratio. Then with the centres *b* and *a*, and respective radii *bd*, *ac* describe two circular arcs, and draw *cd* to touch these circles in *d* and *c*: also draw *ft* perpendicular to *cd*.



Now draw FB meeting the curve in B, and making $\angle$ TFB $=$ $\angle$ *tfb*: also draw FA meeting the curve in A, and making $\angle$ TFA $=$ $\angle$ *tfa*, and the thing is done.

Demonstration. Draw AC and BD, at right-angles to the directrix CTD. Then, by the construction and the property of the directrix,

FB : BD :: FA : AC :: FV : VT :: *fb* : *bd* :: *fa* : *ac*;

and, by alternation,

FB : FA :: BD : AC :: FV : *fb* :: VT : *bd* :: *fb* : *fa* :: *bd* : *ac*,

and therefore FB : FA :: *fb* : *fa* the given ratio by the construction.

Therefore the two lines FA, FB, drawn from the focus F, of the given conic section, and terminating in the curve, include the given angle *afb*, and are in the given ratio of *fa* to *fb*.

THIRD SOLUTION, *by Mr.* ANDREW JOHNSTON, *(Rector of the Academy at Montrose.)*

Let the straight lines be x and x', let $d =$ distance of the focus from the directrix, and let x the nearest to d, make with d the angle z. Let the angle contained by the lines $= a$, their ratio as $1 : m$, and the determining ratio of the curve as $1 : n$.

Then $$x = \frac{d}{n + \cos z}, \qquad x' = \frac{d}{n + \cos (a + z)}.$$

But $x' = mx$, therefore $$\frac{d}{n + \cos (a + z)} = \frac{md}{n + \cos z},$$

or $n + \cos z = mn + m \cos (a + z)$, or $(m - 1) n = \cos z$ $- m \cos (a + z) = \cos z - m \cos a \cos z + m \sin a \sin z$.

Let $\cos a = c$, $\sin a = s$, and $\cos z = y$.

Then $(m - 1) n = y - mcy + ms \sqrt{1 - y^2}$.

Wherefore $$\sqrt{1 - y^2} = \frac{mc - 1}{ms} y + \frac{(m - 1) n}{ms}.$$

Let $\dfrac{mc - 1}{ms} = \text{A}$, $\dfrac{(m - 1) n}{ms} = \text{B}$.

Then $\sqrt{1 - y^2} = \text{A}y + \text{B}$, $1 - y^2 = \text{A}^2 y^2 + 2\text{AB}y + \text{B}^2$:

whence $(\text{A}^2 + 1) y^2 + 2\text{AB}y = 1 - \text{B}^2$, and

$$y = \frac{-\text{AB} \pm \sqrt{1 + \text{A}^2 - \text{B}^2}}{1 + \text{A}^2},$$ or, substituting,

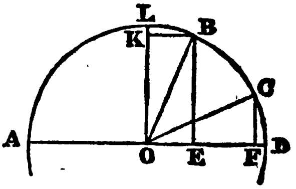
$$y = \frac{-n(m-1)(mc-1) \pm ms\sqrt{1-2mc+m^2-n^2(m-1)^2}}{1-2mc+m^2}.$$

Mr. Johnston also answered Questions 418, 419, *and* 420.

XIII. QUESTION 423, *by* C. B.

In a circle of the nth order, whose equation is $x^{2n} + y^{2n} = a^{2n}$, any point C being taken, and the abscissa and ordinate OF and CF being drawn; if the triangle COF is placed in the situation OEB, then the point B shall be situated in the curve, and the sum of the two arcs, DB and DC, shall always be constant.

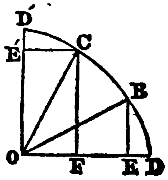
First Solution, *by* C. B. *the Proposer.*



Let OF $= x$, CF $= y$; then $y = (a^{2n} - x^{2n})^{\frac{1}{2n}}$. Now it is first to be shown that if OE $=$ CF, then BE the ordinate at E is equal to OF. Since BE $= (a^{2n} - \text{OE}^{2n})^{\frac{1}{2n}}$. Put for OE its value CF $= (a^{2n} - x^{2n})^{\frac{1}{2n}}$ then BE $= \left\{a^{2n} - (a^{2n} - x^{2n})^{\frac{1}{2n}}\right\}^{\frac{1}{2n}} = x$, which was to be proved.

Also since KB $=$ OE $=$ CF and KO $=$ BE $=$ OF, and B and C are two points in the curve it is symmetrical relative to the two axes LO and OD; therefore the arc LB is equal to the arc CD, and the sum of the two arcs BD $+$ CD is always equal to BD $+$ LB, which is constant.

Second Solution, *by* Nov-anglus.



Because x and y enter alike into the equation, it is obvious that when x is made $= y$, y must become $= x$: that is, if the absciss OE be set off $=$ the ordinate FC $=$ OE′, the corresponding ordinate EB must be equal to the absciss OF $=$ EC′. Hence if OFC be placed in the position OEB, B must be a point in the curve. Again, if E′C be supposed $=$ BE, the arc D′C must be equal to the arc DB; for if these arcs are generated by the motions of E′C and EB from D′ and D, E′C will be always equal to EB at equal distances from the centre

o. The arcs are therefore determined by precisely the same conditions, and must be equal. Since DC = DB, CD + DB = DD′ = a *constant quantity*.

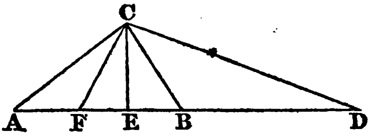
XIV. QUESTION 424, *by Mr.* CUNLIFFE, *R. M. C.*

Given the segments of the base, made by a perpendicular from the vertical angle, and the rectangle of the other two sides, to construct the plane triangle by a method purely geometrical.

FIRST SOLUTION, *by Mr.* LOWRY.

Suppose that ABC is the triangle required, in which are given the segments AE, EB of the base, and the rectangle of the sides AC . CB.

Make EF = EB, and join CF; then CF is = CB and the angle AFC = CBD. Draw CD to make the angle FCD = AFC or CBD, then the angle at D is evidently = ACF, and therefore the three triangles CBD, ACF and ACD are similar. From the first two we have



$$AC : AF :: CD : CB,$$

therefore $$AC \,.\, CB = AF \,.\, CD;$$

but AF is a given line, being equal to the difference of the given segments, therefore CD is a given line.

Again, from the triangles ACD and BCD,

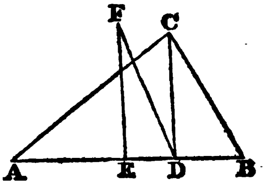
$$AD : CD :: CD : BD,$$

or $$AD \,.\, BD = CD^2.$$

Hence this construction: Having made AE and BE = the given segments, and taken EF = EB, find CD so that the rectangle AF . CD may be = to the given rectangle, and produce AB to D so that AD . DB may be = CD^2; from D, to the perpendicular EC, apply CD of the given length, join AC and BC and the triangle is constructed.

Because AD . BD = CD^2, AD : CD :: CD : BD, and since the angle at D is common to the two triangles ACD and BCD, they are similar; therefore the angle ACD = CBD = AFC, and consequently the triangles ACF and BCD are similar; therefore AC : AF :: CD : CB, or AC . CB = AF . CD = the given rectangle.

SECOND SOLUTION, *by* NOV-ANGLUS.



Let AB be the base and DC the perpendicular which is to pass through the vertex of the required triangle. From E the middle of AB raise the perpendicular EF, and make AB . EF equal to the given rectangle of the sides. Join DF; and with the centre A and radius AC, equal to a mean proportional between AB and DF + DE, intersect DC in C, join BC and ABC is the triangle required.

By construction

$$AC^2 = AB \cdot (DF + DE),$$

and by a well-known property,

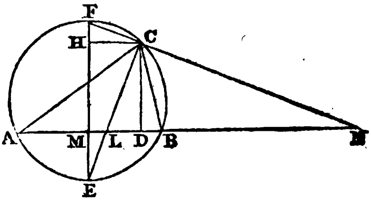
$$AC^2 - CB^2 = AB \cdot 2DE;$$

therefore, by subtraction,

$$CB^2 = AB \cdot (DF - DE).$$

Whence $AC \cdot CB$ is $= AB \times$ a mean proportional between $DF + DE$ and $DF - DE$. But $(DF + DE) \cdot (DF - DE) = EF^2$, therefore $AC \cdot CB = AB \cdot EF$.

THIRD SOLUTION, *by Mr.* CUNLIFFE.



Let ACB represent the required triangle circumscribed by a circle, EF being a diameter bisecting the base AB in M; draw EC cutting AB in L; also draw FC which continue till it meets AB produced in N; draw CH at right-angles to EF, and CD perpendicular to AB. The right-angled triangles ECF and LCN are similar, therefore $EF : HC :: LN : CD$, and hence $LN \times HC = LN \times MD = EF \times CD$. But it is well-known that $EF \times CD = AC \times BC$; therefore $LN \times MD = AC \times BC$ a given magnitude by the question. But MD being half the difference of the given segments AD and BD is given, wherefore LN becomes known.

Now by the property of the circle $EL \times LC = AL \times LB = (AM + ML) \times (AM - ML) = AM^2 - ML^2$. Again the angles EMN and ECN being right-angles a circle will pass through the points E, M, C, N, and therefore by the property of this circle $EL \times LC = ML \times LN$, and consequently $ML \times LN = AM^2 - ML^2$; add ML^2 to each side, and then we shall have $ML^2 + ML \times LN = ML \times (ML + LN) = AM^2$, which is given by the question, and from whence we determine ML; and hence LD becomes known, which renders the construction of the problem very easy.

The construction may be as follows:—

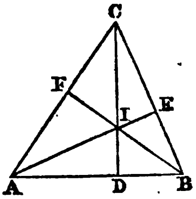
Having found LN, ML and LD by means of what has been deduced, upon LN as a diameter describe a semicircle, draw DE perpendicular to LN meeting the circumference in C. From the point M, which is now given, set off MA, MB each equal to half the given base, and join AC, BC; then ACB is the required triangle, as sufficiently appears from the foregoing analysis.

XV. QUESTION 425, *by Mr.* CUNLIFFE.

By some inadvertency this question has been improperly enunciated, it was intended to have been as follows:

If perpendiculars be drawn from the angles to the opposite sides of a plane triangle, the three rectangles of the segments of these perpendiculars, made at the point of their intersection will be equal to each other.

SOLUTION, *by the Proposer.*



ACB is a plane triangle AE, BF and CD perpendiculars from the angles upon the opposite sides; I the point of intersection of the perpendiculars: then by the question $AI \times IE = BI \times IF = CI \times ID$.

Demonstration. AFB and AEB are right-angles, therefore a circle whose diameter is AB will pass through F and E; and by the property of this circle $AI \times IE = BI \times IF$, and for a like reason, a circle whose diameter is AC will pass through E and D; and by the property of this circle $AI \times IE = CI \times ID$; and consequently $AI \times IE = BI \times IF = CI \times ID$. *Q.E.D.*

It would be easy to shew that the property enunciated in the question, could only have place in an equilateral triangle.

This question was likewise answered by NOV-ANGLUS.

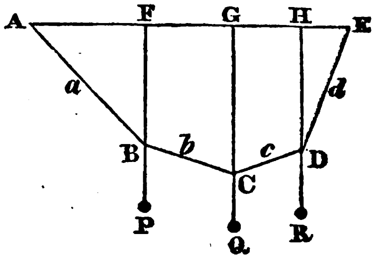
XVI. QUESTION 426, *by* LAPUTIENSIS.

Suppose two given weights, appended at given points, to a flexible line of a given length, considered without weight and fastened at the ends to two given points in the same horizontal line, and imagine a third given weight, appended to the line between the other two, but at liberty to slide freely thereon by means of a ring; it is required to determine the position of the ring, and the points in the line, to which the other two weights are appended, when at rest, in the abovementioned circumstances?

FIRST SOLUTION, *by Mr.* LOWRY.

Let A and E be the points in the horizontal line to which the ends of the string are fastened; B, C and D the points where the given weights P, Q and R are appended.

It is evident that the points A, B, C, D and E must be in the same vertical plane; and since the weight Q is at liberty to slide freely along the string till the whole system is in a state of equilibrium, the tensions of the strings BC, DC must be equal, and also the angles BCG, DCG made by these strings and the vertical line CG.



Let a, b, c and d denote the respective tensions of the strings AB, BC, CD and DE; then the point B is kept in equilibrio by the three forces a, b and P; the point C by the three forces b, c and Q; and the point D by the three forces c, d and R; we have, therefore, by the principles of statics,

$$P : b :: \sin ABC : \sin ABF,$$

$$b : Q :: \sin GCD : \sin BCD;$$

therefore

$$P : Q :: \sin ABC \sin GCD : \sin ABF \sin BCD.$$

And in like manner

$$R : Q :: \sin EDC \sin GCB : \sin EDH \sin BCD.$$

But $\qquad GCD = GCB$, and $BCD = 2GCB$;

therefore $\sin \mathrm{BCD} = 2 \cos \mathrm{GCB} . \sin \mathrm{GCB};$

and $\mathrm{P} : \mathrm{Q} :: \sin \mathrm{ABC} : 2 \sin \mathrm{ABF} \cos \mathrm{GCB},$

$\mathrm{R} : \mathrm{Q} :: \sin \mathrm{EDC} : 2 \sin \mathrm{EDH} \cos \mathrm{GCB}.$

Put $\varphi =$ the angle ABF, $\psi = \mathrm{EDH}$,

$\theta = \mathrm{GCB} = \mathrm{PBC} = \mathrm{RDC};$

then $\mathrm{ABC} = 180^\circ - (\theta - \varphi)$ and $\mathrm{EDC} = 180^\circ - (\theta - \psi)$,

therefore $\sin \mathrm{ABC} = \sin (\theta - \varphi)$

$\sin \mathrm{EDC} = \sin (\theta - \psi),$

and we have

$$\mathrm{P} : \mathrm{Q} :: \sin (\theta - \varphi) : 2 \sin \varphi \cos \theta,$$

$$\mathrm{R} : \mathrm{Q} :: \sin (\theta - \psi) : 2 \sin \psi \cos \theta,$$

or
$$\frac{\sin (\theta - \varphi)}{\sin \varphi \cos \theta} = \frac{2\mathrm{P}}{\mathrm{Q}}.$$

$$\frac{\sin (\theta - \psi)}{\sin \psi \cos \theta} = \frac{2\mathrm{R}}{\mathrm{Q}}.$$

But
$$\frac{\sin (\theta - \varphi)}{\sin \varphi \cos \theta} = \frac{\sin \theta \cos \varphi}{\sin \varphi \cos \theta} - 1 = \frac{\tan \theta}{\tan \varphi} - 1,$$

$$\frac{\sin (\theta - \psi)}{\sin \psi \cos \theta} = \frac{\sin \theta \cos \psi}{\sin \psi \cos \theta} - 1 = \frac{\tan \theta}{\tan \psi} - 1,$$

therefore

$$\frac{\tan \theta}{\tan \varphi} = \frac{2\mathrm{P}}{\mathrm{Q}} + 1 \ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{\tan \varphi}{\tan \psi} = \frac{2\mathrm{R}}{\mathrm{Q}} + 1. \ldots\ldots\ldots\ldots\ldots (2)$$

To obtain another equation, put $h = \mathrm{AE}$, $\alpha = \mathrm{AB}$, $\beta = \mathrm{DC}$, and $\nu = \mathrm{BC} + \mathrm{CD}$ the remaining part of the string; then

$$\mathrm{AF} = \alpha \sin \varphi, \quad \mathrm{EH} = \beta \sin \psi,$$

$$\mathrm{FH} = h - \alpha \sin \varphi - \beta \sin \psi.$$

But $\mathrm{FH} = (\mathrm{BC} + \mathrm{CD}) \sin \theta = \nu \sin \theta;$

therefore $\nu \sin \theta = h - \alpha \sin \varphi - \beta \sin \psi. \ldots\ldots (3)$

From the equations (1) (2) (3) we can now find the angles θ, φ and ψ.

By putting $m = \frac{2P}{Q} + 1$ and $n = \frac{2R}{Q} + 1$,

we have $\tan\varphi = \frac{\tan\theta}{m}$, $\tan\psi = \frac{\tan\theta}{n}$;

therefore
$$\sin\varphi = \frac{\tan\theta}{\sqrt{(m^2 + \tan^2\theta)}}$$
$$\sin\psi = \frac{\tan\theta}{\sqrt{(n^2 + \tan^2\theta)}},$$

and substituting these values in equation (3) it becomes
$$\nu\sin\theta = h - \frac{\alpha\tan\theta}{\sqrt{(m^2 + \tan^2\theta)}} - \frac{\beta\tan\theta}{\sqrt{(n^2 + \tan^2\theta)}}.$$

From this equation we may find θ, and then φ and ψ become known, and consequently also the position of the points B, C and D.

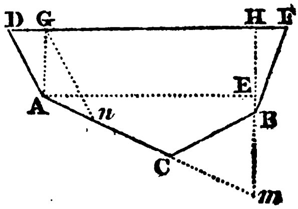
For the tensions at A and E, we have
$$a = P \times \frac{\sin DBC}{\sin ABC} = P \times \frac{\sin\theta}{\sin(\theta - \varphi)}$$
$$d = R \times \frac{\sin RDC}{\sin EDC} = R \times \frac{\sin\theta}{\sin(\theta - \psi)}.$$

When AE is not horizontal, if we put c for the angle it makes with the vertical BF, equation (3) becomes
$$\frac{\nu\sin\theta}{\sin c} = h - \frac{\alpha\sin\varphi}{\sin c} - \frac{\beta\sin\psi}{\sin c};$$
the other two equations remain the same as before.

SECOND SOLUTION, *by* NOV-ANGLUS.

Let DF be the given horizontal line, and A, B, C the position of the three given weights, when in equilibrio.

If the body C were removed and a fixed pully substituted in its place, round which the string ACB, might have a free motion, the position of A and B would evidently not be altered. Hence the tensions of the parts AC, BC must be equal, Since these two tensions are equal and are kept in equilibrio by a fore C acting vertically, their resultant must be in a vertical line; and therefore, as they are equal they must make equal



angles with that vertical: that is, the angles GAC, HBC (drawing AG, BH perpendicular to DF) must be equal. Draw Gn parallel to DA and produce BH, AC to meet in m.

Because the body A is kept at rest by its own weight, acting in the direction AG, the tension of DA, acting in the direction Gn, and that of CA, acting in the direction Am, the three forces must be as these three lines, respectively, or as the sines of their opposite angles; that is,

$$\text{A} : \text{tens}^{n}\ \text{AD} : \text{tens}^{n}\ \text{AC} :: \sin(\text{DAG} + \text{GAC}) : \sin \text{GAC} : \sin \text{DAG}.$$

In like manner, (putting GAC for its equal HBC and tens^{n} AC for tens^{n} BC,)

$$\text{B} : \text{tens}^{n}\ \text{BF} : \text{tens}^{n}\ \text{AC} :: \sin(\text{FBH} + \text{GAC}) : \sin \text{GAC} : \sin \text{FBH}.$$

But the tensions of DA and FB are the same, in a state of equilibrium, whether C be supposed moveable, or fastened at the point C; hence (Bridge's Mechanics, p. 324.) we have the following properties:

$$\text{tens}^{n}\ \text{AD} : \sin \text{FBH} :: \text{tens}^{n}\ \text{BF} : \sin \text{DAG} :: \text{A} + \text{B} + \text{C} : \sin(\text{FBH} + \text{GAD}).$$

Put sin DAG $= x$, sin FBH $= y$, and sin GAC $= z$, we have, after making the proper substitutions and reductions (cos GAC being negative) the two following equations:

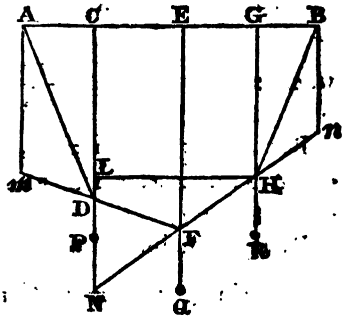
$$\frac{y}{\text{A}}\left\{z\sqrt{(1-x^2)} - x\sqrt{(1-z^2)}\right\} = \frac{x}{\text{B}}\left\{z\sqrt{(1-y^2)} - y\sqrt{(1-z^2)}\right\}$$
$$= \frac{z}{\text{A}+\text{B}+\text{C}}\left\{y\sqrt{(1-x^2)} + x\sqrt{(1-y^2)}\right\}.$$

These two equations are wholly independent of the length of the lines DF, DA, &c. To obtain a third equation involving these lines, put DA $= a$, BF $= b$, AC + CB $= c$ and DF $= d$. Then since Am is = AC + CB, and AmE = supplement of GAC, AE = (AC + CB) sin GAC $= cz =$ GH. Likewise DG = DA × sin DAG $= ax$, and HF = BF sin HBF $= by$. Hence our third equation is

$$ax + by + cz = d.$$

If a numerical solution were required, the usual methods of approximating to the values of two unknown quantities from two equations might be employed with the two first. In making suppositions for the values of x and y and substituting in those equations, wherever z occurs, its value $\frac{d - by - ax}{c}$ must be employed.

THIRD SOLUTION, by Mr. CUNLIFFE.



Let ADFHB represent the position of the string, when the weights are in equilibrio, in the circumstances mentioned in the question; that is, having its ends fastened to the given points A and B in the horizontal line AB; and the given weight P, being fastened at a given distance AD, from A; and the given weight R, fastened at a given distance BH, from B; and the ring to which the given weight Q is attached, at liberty to slide freely upon the string.

Draw DC, FE and HG perpendicular to the horizontal line AB. The ring F being at liberty to slide freely upon the string, when the equilibrium takes place, it is obvious that the angle EFH is equal to EFD, that is, the line FE bisects the angle DFH. Produce HF to meet CD in N. Then, because of the parallels DN, HR,

angle $DNF = FHR$; also angle $NDF = EFD = EFH$;

therefore $DNF = NDF$, and $FN = FD$; and, consequently,

$HN = AF + FN = HF + FD =$ a given length.

When the weights are in equilibrio, they will be to each other respectively, as follows: that is,

the weight P, will be as $\dfrac{\sin ADF}{\sin ADC \sin CDF} = \dfrac{\sin ADF}{\sin ADC \sin EFH}$,

and the weight Q, as $\dfrac{\sin DFH}{\sin^2 EFH} = \dfrac{\sin 2EFH}{\sin^2 EFH}$,

and the weight R, will be as

$$\frac{\sin FHB}{\sin BHG \sin GHF} = \frac{\sin FHB}{\sin BHG \sin EFH}.$$

The demonstration of the foregoing properties may be seen at prop. 66, and corol. 5, of Emerson's Mechanics.

Therefore $\quad P : Q :: \dfrac{\sin ADF}{\sin ADC} : \dfrac{\sin 2EFH}{\sin EFH}$, whence $$P \times \frac{\sin 2EFH}{\sin EFH} = Q \times \frac{\sin ADF}{\sin ADC}.$$

Also $$R : Q : \frac{\sin FHB}{\sin BHG} : \frac{\sin 2EFH}{\sin EFH},$$

whence $$R \times \frac{\sin 2EFH}{\sin EFH} = Q \times \frac{\sin FHB}{\sin BHG}.$$

Put $AB = a$, $AD = b$, $BH = c$, $HN = HF + FD = d$, $\sin \angle ADC = x$, $\sin \angle BHG = y$, and $\sin \angle EFH = v$; the radius being 1.

By the principles of trigonometry

$$\sin \angle ADF = v \sqrt{(1 - x^2)} - x \sqrt{(1 - v^2)},$$

because the angle CDF being obtuse its cosine is negative.

And, for the same reason,

$$\sin \angle FHB = v \sqrt{(1 - y^2)} - y \sqrt{(1 - v^2)}.$$

Also, $$\sin 2\angle EFH = 2v \sqrt{(1 - v^2)}.$$

From whence there will be had

$$\frac{\sin ADF}{\sin ADC} = \frac{v\sqrt{(1-x^2)}-x\sqrt{(1-v^2)}}{x} = \frac{v}{x}\sqrt{(1-x^2)}-\sqrt{(1-v^2)};$$

$$\frac{\sin FHB}{\sin BHG} = \frac{v\sqrt{(1-y^2)}-y\sqrt{(1-v^2)}}{y} = \frac{v}{y}\sqrt{(1-y^2)}-\sqrt{(1-v^2)};$$

and, by means of these values, the expressions

$$P \times \frac{\sin 2EFH}{\sin EFH} = Q \times \frac{\sin ADF}{\sin ADC},$$

and $$R \times \frac{\sin 2EFH}{\sin EFH} = Q \times \frac{\sin BHF}{\sin BHG}$$

before deduced become

$$2P \sqrt{(1 - v^2)} = Q \frac{v}{x} \sqrt{(1 - x^2)} - Q \sqrt{(1 - v^2)},$$

and $$2R \sqrt{(1 - v^2)} = Q \frac{v}{y} \sqrt{(1 - y^2)} - Q \sqrt{(1 - v^2)};$$

whence $$\frac{2P + Q}{Q} \cdot \frac{\sqrt{(1 - v^2)}}{v} = \frac{\sqrt{(1 - x^2)}}{x}.$$

and $$\frac{2R+Q}{Q}\cdot\frac{\sqrt{(1-v^2)}}{v}=\frac{\sqrt{(1-y^2)}}{y}.$$

Putting $m=\frac{2P+Q}{Q}$ and $n=\frac{2R+Q}{Q}$ the two last equations become

$$\frac{m}{v}\sqrt{(1-v^2)}=\frac{\sqrt{(1-x^2)}}{x} \text{ and } \frac{n}{v}\sqrt{(1-v^2)}=\frac{\sqrt{(1-y^2)}}{y},$$

from the first of which $x=\frac{v}{\sqrt{\{m^2-v^2(m^2-1)\}}}$.

and from the other $y=\frac{v}{\sqrt{\{n^2-v^2(n^2-1)\}}}$.

Now draw HL parallel to AB meeting CD in L; then CG = HL: and by the principles of trigonometry and what has been deduced,

$$AC = AD \times x = bx = \frac{bv}{\sqrt{\{m^2-v^2(m^2-1)\}}}.$$

$$BG = BH \times y = cy = \frac{cv}{\sqrt{\{n^2-v^2(n^2-1)\}}};$$

and $$HL = CG = HN \times v = dv,$$

and therefore AC + CG + GB = AB, that is

$$\frac{bv}{\sqrt{\{m^2-v^2(m^2-1)\}}}+dv+\frac{cv}{\sqrt{\{n^2-v^2(n^2-1)\}}}=a,$$

and from this equation we may find v, and from thence x and y may be found, which will give the position of F, D and H.

Or without referring to Mr. Emerson's book of mechanics, the properties upon which the solution of the question depends, may be investigated in the following manner. It was remarked that when the equilibrium takes place, the angle EFH = EFD.

Draw Am and Bn perpendicular to AB, and meeting FD and FH produced in m and n respectively.

By the principles of the resolution of forces, the weight of the body P, and its tension of the string, in the direction FD, are respectively as the lines Am and Dm. And, by trigonometry,

Am : Dm :: sin ADm : sin DAm = sin ADC;

wherefore, $\sin \mathrm{AD}m : \sin \mathrm{ADC} :: \mathrm{P} : \frac{\mathrm{P} \times \sin \mathrm{ADC}}{\sin \mathrm{AD}m} =$ the tension of the string caused by P, in the direction FD: and in the same way we shall find the tension caused by Q, in the direction $\mathrm{DF} = \frac{\mathrm{Q} \times \sin \mathrm{EFH}}{\sin \mathrm{DFN}} = \frac{\mathrm{Q} \times \sin \mathrm{EFH}}{\sin 2\,\mathrm{EFH}}$; and when the weights are in equilibrio, the tensions of the string in the opposite directions FD and DF, must manifestly be equal to each other, that is,

$$\frac{\mathrm{P} \times \sin \mathrm{ADC}}{\sin \mathrm{AD}m} = \frac{\mathrm{Q} \times \sin \mathrm{EFH}}{\sin 2\mathrm{EFH}},$$

and for the same reasons, and in the very same manner we shall obtain

$$\frac{\mathrm{R} \times \sin \mathrm{BHG}}{\sin \mathrm{BH}n} = \frac{\mathrm{Q} \times \sin \mathrm{EFH}}{\sin 2\mathrm{EFH}}.$$

EF, CD and GH being parallel, the angle $\mathrm{CD}m = \mathrm{EFD} = \mathrm{EFH} = \mathrm{GHN}$: wherefore, by the principles of trigonometry and the notation in the foregoing solution,

$$\sin \mathrm{AD}m = v\sqrt{(1 - x^2)} - x\sqrt{(1 - v^2)};$$

$$\sin 2\mathrm{EFH} = 2v\sqrt{(1 - v^2)} \text{ and}$$

$$\sin \mathrm{BH}n = v\sqrt{(1 - y^2)} - y\sqrt{(1 - v^2)};$$

hence the expression $\frac{\mathrm{P} \times \sin \mathrm{ADC}}{\sin \mathrm{AD}m} = \frac{\mathrm{Q} \times \sin \mathrm{EFH}}{\sin 2\,\mathrm{EFH}}$

becomes $$\frac{\mathrm{P} \times x}{v\sqrt{(1 - x^2)} - x\sqrt{(1 - v^2)}} = \frac{\mathrm{Q}}{2\sqrt{(1 - v^2)}},$$

whence $$\frac{2\mathrm{P} + \mathrm{Q}}{\mathrm{Q}} \cdot \frac{\sqrt{(1 - v^2)}}{v} = \frac{\sqrt{(1 - x^2)}}{x}.$$

Also the expression $\frac{\mathrm{R} \times \sin \mathrm{BHG}}{\sin \mathrm{BH}n} = \frac{\mathrm{Q} \times \sin \mathrm{EFH}}{\sin 2\mathrm{EFH}}$

becomes $$\frac{\mathrm{R} \times y}{v\sqrt{(1 - y^2)} - y\sqrt{(1 - v^2)}} = \frac{\mathrm{Q}}{2\sqrt{(1 - v^2)}},$$

whence $$\frac{2\mathrm{R} + \mathrm{Q}}{\mathrm{Q}} \cdot \frac{\sqrt{(1 - v^2)}}{v} = \frac{\sqrt{(1 - y^2)}}{y};$$

which equations are exactly the same as those obtained by the properties quoted from Mr. Emerson's book of mechanics.

XVII. QUESTION 427, *by* C. B.

Prove that $\frac{1}{2} = \frac{1}{\cos\theta} - \frac{1}{\cos 2\theta} + \frac{1}{\cos 3\theta} - \&c.$

SOLUTION, *by* C. B. *the Proposer.*

Put $$\frac{v^{2z}}{1+v^{2z}} = v^{2z} - v^{4z} + v^{6z} - v^{8z} + \&c.$$

Multiply by $(-1)^{-z}$ and integrate, observing that

$$\Sigma(-1)^{-1} v^{2iz} = -\frac{(-1)^{-z}\, v^{2iz}}{v^{2i}+1}, \text{ we have } \Sigma\frac{(-1)^{-z}\, v^{2z}}{1+v^{2z}} =$$

$$-\frac{(-1)^{-z}\, v^{2z}}{v^{2}+1} + \frac{(-1)^{-z}\, v^{4z}}{v^{4}+1} - \frac{(-1)^{-z}\, v^{6z}}{v^{6}+1} + \&c. \ldots\ldots(1)$$

Again $\frac{v^{-2z}}{1+v} = v^{-2z} - v^{-4z} + v^{-6z} - \&c.$ Multiply by $(-1)^{-z}$ and integrate; then $\Sigma\frac{(-1)^{-z}\, v^{-2z}}{1+v^{-2z}} =$

$$-\frac{(-1)^{-z}\, v^{-2z}}{v^{-2}+1} + \frac{(-1)^{-z}\, v^{-4z}}{v^{-4}+1} - \frac{(-1)^{-z}\, v^{-6z}}{v^{-6}+1} + \&c. \ldots\ldots(2)$$

Add (1) and (2) after dividing the numerator and denominator of each term by v^{i} in the first and by v^{-i} in the last

$$\Sigma\left\{\frac{v^{2z}}{1+v^{2z}} + \frac{v^{-2z}}{1+v^{-2z}}\right\}(-1)^{-z} = -(-1)^{-z} \times$$

$$\left\{\frac{v^{2z-1}+v^{-2z+1}}{v+v^{-1}} - \frac{v^{4z-2}+v^{-4z+2}}{v+v^{-2}} + \frac{v^{6z-3}+v^{-6z+3}}{v+v^{-3}} - \&c.\right\}$$

but the integral is $+\,\Sigma(-1)^{-z} = -\frac{(-1)^{-z}}{2} + c$: put $z + \frac{1}{2}$ for z, and let $v = \cos\theta \pm \sqrt{-1}\sin\theta$, then it is

$$-\frac{(-1)^{-z-\frac{1}{2}}}{2} + c =$$

$$-(-1)^{-z-\frac{1}{2}}\left\{\frac{v^{2z}+v^{-2z}}{2\cos\theta} - \frac{v^{4z}+v^{-4z}}{2\cos 2\theta} + \frac{v^{6z}+v^{-6z}}{2\cos 3\theta} - \&c.\right\}$$

$$\text{If } z = \tfrac{1}{2},\ \tfrac{1}{2} + c = \left\{\frac{2\cos\theta}{2\cos\theta} - \frac{2\cos\theta}{2\cos\theta} + \&c.\right\} = \tfrac{1}{2},$$

hence $c = 0$; and multiplying by $-\frac{(-1)^{-z-\frac{1}{2}}}{2}$ we have

$$1 = \frac{v^{2z}+v^{-2z}}{\cos\theta} - \frac{v^{4z}+v^{-4z}}{\cos 2\theta} + \frac{v^{6z}+v^{-6z}}{\cos 3\theta} - \&c.$$

Now put $z = 0$, and dividing by 2, we have

$$\frac{1}{2} = \frac{1}{\cos\theta} - \frac{1}{\cos 2\theta} + \frac{1}{\cos 3\theta} - \frac{1}{\cos 4\theta} + \&c.$$

In the same way it may be proved that

$$1 = \frac{x+x^{-1}}{(\cos\theta)^{n}} - \frac{x^{2}+x^{-2}}{(\cos 2\theta)^{n}} + \frac{x^{3}+x^{-3}}{(\cos 3\theta)^{n}} - \&c.$$

SECOND SOLUTION, *by* A. D.

Let

$$P = \cos\theta - \cos(\theta+\phi) + \cos(\theta+2\phi) - \cos(\theta+3\phi) + \&c.$$

then $2P\cos\frac{\phi}{2} =$

$$2\cos\theta\cos\frac{\phi}{2} - 2\cos\left(\theta+\frac{2\phi}{2}\right)\cos\frac{\phi}{2} + 2\cos\left(\theta+\frac{4\phi}{2}\right)\cos\frac{\phi}{2} - \&c.$$

$$= \cos\left(\theta-\frac{\phi}{2}\right) + \cos\left(\theta+\frac{\phi}{2}\right)$$

$$-\cos\left(\theta+\frac{\phi}{2}\right) - \cos\left(\theta+\frac{3\phi}{2}\right)$$

$$+\cos\left(\theta+\frac{3\varphi}{2}\right)+\cos\left(\theta+\frac{5\varphi}{2}\right)$$

$$-\cos\left(\theta+\frac{5\varphi}{2}\right)-\&c.$$

$$=\cos\left(\theta-\frac{\varphi}{2}\right),$$

therefore

$$\cos\theta-\cos(\theta+\varphi)+\cos(\theta+2\varphi)-\&c.=\tfrac{1}{2}\,\frac{\cos\left(\theta-\frac{\varphi}{2}\right)}{\cos\frac{\varphi}{2}};$$

hence

$$\cos\theta-\cos 2\theta+\cos 3\theta-\&c.=\tfrac{1}{2}\,\frac{\cos\left(\theta-\frac{\theta}{2}\right)}{\cos\frac{\theta}{2}}=\tfrac{1}{2},$$

and

$$\cos\theta-\cos 3\theta+\cos 5\theta-\&c.=\tfrac{1}{2}\,\frac{\cos(\theta-\theta)}{\cos\theta}=\frac{1}{2\cos\theta}.$$

For θ put successively 2θ, 3θ, &c. and change the signs of the alternate equations. We thus get

$$\frac{1}{2\cos\theta}=\cos\theta-\cos 3\theta+\cos 5\theta-\&c.$$

$$-\frac{1}{2\cos 2\theta}=-\cos 2\theta+\cos 6\theta-\cos 10\theta+\&c.$$

$$+\frac{1}{2\cos 3\theta}=\cos 3\theta-\cos 9\theta+\cos 15\theta-\&c.$$

$$\&c. \qquad\qquad \&c.$$

if we add all these equations, the sums of the vertical columns will be $\frac{1}{2}$, $-\frac{1}{2}$, alternately.

Therefore,

$$\frac{1}{\cos\theta}-\frac{1}{\cos 2\theta}+\frac{1}{\cos 3\theta}-\&c.=1-1+1-1+\ldots\ldots$$

$$=\frac{1}{2}. \qquad\qquad Q.E.D.$$

XVIII. QUESTION 428, *by* C. B.

Prove that the sum of the series

$$\frac{1}{1\cdot3\ldots\ldots2n+1}-\frac{1}{3\cdot5\ldots\ldots2n+3}+\text{\&c. is equal to}$$

$$\frac{2^{-1}}{1\cdot2\cdot3\ldots n}\left\{\frac{\pi}{2}-\frac{1}{1}-\frac{1}{1\cdot3}-\frac{1\cdot2}{1\cdot3\cdot5}-\text{\&c. to } n \text{ terms}\right\}.$$

FIRST SOLUTION, *by* C. B. *the Proposer.*

Assume

$$u_n=\frac{1}{1\cdot3\ldots\ldots2n+1}-\frac{1}{3\cdot5\ldots\ldots2n+3}+\frac{1}{5\cdot7\ldots\ldots2n+5}-\text{\&c.}$$

then,

$$u_n-\frac{1}{1\cdot3\ldots2n+1}=-\frac{1}{3\cdot5\ldots2n+3}+\frac{1}{5\cdot7\ldots2n+5}-\text{\&c.}$$

adding these together, we have

$$2u_n-\frac{1}{1\cdot3\ldots2n+1}=$$

$$\frac{2n+3-1}{1\cdot3\ldots\ldots2n+3}-\frac{2n+5-3}{3\cdot5\ldots\ldots2n+5}+\frac{2n+7-5}{5\cdot7\ldots2n+7}-\text{\&c.}$$

but the series on the right hand is equal to $(2n+2)\,u_{n+1}$ hence the equation of first differences

$$2u_n-2(n+1)u_{n+1}=\frac{1}{1\cdot3\ldots\ldots2n+1}.$$

Put $u_n=\frac{w_n}{1\cdot2\cdot3\ldots n}$, then $u_{n+1}=\frac{w_{n+1}}{1\cdot2\cdot3\ldots\ldots n+1}$

and our equation becomes

$$2\left\{w_n\frac{1}{1\cdot2\ldots n}-\frac{n+1}{1\cdot2\ldots\ldots n+1}w_{n+1}\right\}=\frac{1}{1\cdot3\ldots\ldots2n+1}$$

or $w_{n+1} - w_n = -\frac{1}{2}\,\frac{1.2\ldots\ldots n}{1.3.5\ldots\ldots 2n+1}$;

hence $w_n = -\frac{1}{2}\left\{\frac{1}{1} + \frac{1}{1.3} + \frac{1.2}{1.3.5} + \text{\&c. to } n \text{ terms}\right\} + c.$

If $n = 0$, $w_0 = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \text{\&c.} = \frac{\pi}{4} = c$,

hence, the series in question or w_n is equal to

$$\frac{2^{-1}}{1.2\ldots n}\left\{\frac{\pi}{2} - \frac{1}{1} - \frac{1}{1.3} - \frac{1.2}{1.3.5} - \text{\&c. to } n \text{ terms}\right\}.$$

SECOND SOLUTION, *by* A. D.

Let $y_n = \frac{1}{1.3\ldots\ldots 2n+1} - \frac{1}{3.5\ldots\ldots 2n+3} + \text{\&c.}$

then

$$\frac{1}{1.3\ldots 2n+1} - y_n = \frac{1}{3.5\ldots 2n+3} - \frac{1}{5.7\ldots 2n+5} + \text{\&c.}$$

and, by subtraction,

$$2y_n - \frac{1}{1.3\ldots 2n+1} = \frac{(2n+3)-1}{1.3.5\ldots 2n+3} - \frac{(2n+5)-3}{3.5.7\ldots 2n+5} + \text{\&c.}$$

$$= (2n+2)\,y_{n+1}:$$

whence, changing $(n+1)$ into n,

$$y_n = -\frac{1}{1.3\ldots\ldots 2n-1}\,\frac{1}{2n} + \frac{1}{n}\,y_{n-1},$$

$$= a_n + \frac{1}{n}y_{n-1},$$

$$= a_n + \frac{1}{n}\left(a_{n-1} + \frac{1}{n-1}\,y_{n-2}\right),$$

$$= a_n + \frac{1}{n}\,a_{n-1} + \frac{1}{n.n-1}\,a_{n-2} + \frac{1}{n.n-1.n-2}\,a_{n-3}\ldots\ldots$$

$$+\frac{1}{n.\overline{n-1}.\overline{n-2}\ldots 3.2}a_1+\frac{1}{\overline{n-1}.\overline{n-2}\ldots 3.1}u_1.$$

Now

$$u_1 = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots = \tan^{-1}(1) = \frac{\pi}{4};$$

$$a_1 = -\frac{1}{1}\frac{1}{2},\quad a_2 = -\frac{1}{1.3}\frac{1}{4},\quad a_3 = -\frac{1}{1.3.5}\frac{1}{6},\ \&c.$$

and by substitution of these values, the series written backwards is

$$y_n\left(\text{or } \frac{1}{1.3\ldots 2n+1}-\frac{1}{3.5\ldots\ldots 2n+3}+\&c.\right)$$

$$=\frac{1}{1.2.3\ldots \overline{n-1}.n}\,\frac{\pi}{4}-\frac{1}{2.3\ldots\ldots n}\,\frac{1}{1.2}-\frac{1}{3.4\ldots n}\,\frac{1}{4}\,\frac{1}{1.3}$$

$$-\frac{1}{4.5\ldots n}\,\frac{1}{6}\,\frac{1}{1.3.5}-\&c.$$

$$=\frac{2^{-1}}{1.2\ldots n}\left\{\frac{\pi}{2}-\frac{1}{1}-\frac{1}{1.3}-\frac{1.2}{1.3.5}-\&c.\ (\text{to } n \text{ terms})\right\}.\ Q.E.D.$$

Third Solution, *by Mr.* Cunliffe.

Let q denote the quadrantal arc of a circle to radius 1: then it is well-known, that

$$\frac{q}{2} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}\ \&c.$$

and hence, by transposition,

$$1-\frac{q}{2} = \frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+\frac{1}{15}-\frac{1}{17}\ \&c.$$

half the difference of the corresponding terms gives

$$\frac{q-1}{2} = 2^{-1}\times\frac{q-1}{1} = \frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}+\frac{1}{9.11}\ \&c.$$

and, by transposition,

$$\frac{1}{1\cdot3}-\frac{q-1}{2}=\frac{1}{3\cdot5}-\frac{1}{5\cdot7}+\frac{1}{7\cdot9}-\frac{1}{9\cdot11}+\frac{1}{11\cdot13}-\frac{1}{13\cdot15}\ \&c.$$

and one-fourth of the difference of the corresponding terms of these is,

$$\frac{q-1-\frac{1}{1\cdot3}}{4}=\frac{2^{-1}}{1\cdot2}\times\left(q-1-\frac{1}{1\cdot3}\right)=$$

$$\frac{1}{1\cdot3\cdot5}-\frac{1}{3\cdot5\cdot7}+\frac{1}{5\cdot7\cdot9}-\frac{1}{7\cdot9\cdot11}+\frac{1}{9\cdot11\cdot13}-\frac{1}{11\cdot13\cdot15}\ \&c.$$

whence,

$$\frac{1}{1\cdot3\cdot5}-\frac{2^{-1}}{1\cdot2}\times\left(q-1-\frac{1}{1\cdot3}\right)=$$

$$\frac{1}{3\cdot5\cdot7}-\frac{1}{5\cdot7\cdot9}+\frac{1}{7\cdot9\cdot11}-\frac{1}{9\cdot11\cdot13}+\frac{1}{11\cdot13\cdot15}-\frac{1}{13\cdot15\cdot17}\ \&c.$$

One-sixth of the difference of these is,

$$\frac{q-1-\frac{1}{1\cdot3}-\frac{1\cdot2}{1\cdot3\cdot5}}{2\cdot6}=\frac{2^{-1}}{1\cdot2\cdot3}\times\left(q-1-\frac{1}{1\cdot3}-\frac{1\cdot2}{1\cdot3\cdot5}\right)=$$

$$\frac{1}{1\cdot3\cdot5\cdot7}-\frac{1}{3\cdot5\cdot7\cdot9}+\frac{1}{5\cdot7\cdot9\cdot11}-\frac{1}{7\cdot9\cdot11\cdot13}\ \&c.$$

and hence, again by transposition,

$$\frac{1}{1\cdot3\cdot5\cdot7}-\frac{2^{-1}}{1\cdot2\cdot3}\times\left(q-1-\frac{1}{1\cdot3}-\frac{1\cdot2}{1\cdot3\cdot5}\right)=$$

$$\frac{1}{3\cdot5\cdot7\cdot9}-\frac{1}{5\cdot7\cdot9\cdot11}+\frac{1}{7\cdot9\cdot11\cdot13}-\frac{1}{9\cdot11\cdot13\cdot15}\ \&c.$$

One-eighth of the difference of these is,

$$\frac{q-1-\frac{1}{1\cdot3}-\frac{1\cdot2}{1\cdot3\cdot5}-\frac{1\cdot2\cdot3}{1\cdot3\cdot5\cdot7}}{1\cdot2\cdot3\cdot8}=$$

$$\frac{\overset{-1}{2}}{1.2.3.4}\times\left(q-1-\frac{1}{1.3}-\frac{1.2}{1.3.5}-\frac{1.2.3}{1.3.5.7}\right)=$$

$$\frac{1}{1.3.5.7.9}-\frac{1}{3.5.7.9.11}+\frac{1}{5.7.9.11.13}-\frac{1}{7.9.11.13.15}\ \&c.$$

By attentively considering what has been done, the method of continuing the operations without much additional labour, will be sufficiently manifest; for, from the foregoing steps and results, it is evident, that if the expression for the sum of any of the foregoing series is multiplied by 2, and the first term thereof be subtracted from that product, and the remainder divided by the difference of the numerators of the first and second terms of the said series, when reduced to the least common denominator, the quotient will be equal to the sum of the next succeeding series, &c.

The following particulars are also obvious from what has been done:

1st. That the number of factors in the denominator of the expression for the sum of any of the foregoing series, is always *one* less than the number of factors in the denominator of each of the terms of the series.

2nd. The number of terms of the numerator of the expression for the sum, not affected with q, is the same as the number of factors in the denominator of the expression for the sum of the series.

3rd. There are (n + 1) factors in the denominator of every term of the given series: if this is not sufficiently manifest by inspection, it may clearly be made out from the following considerations.

The factors in the denominator of every term of the series, are obviously constructed from the terms of an arithmetical progression, where the common difference is 2; and in the denominator of every term of the series, there are given the two extreme factors, which are the extremes of an arithmetical progression whose common difference is 2; from whence by well-known methods or rules, the number of terms or factors may be easily found. For having given the extremes of an arithmetical progression and the common difference, the number of terms may be found by the following rule, viz. Divide the difference of the extremes, by the common difference, and add one to the quotient, which will give the number of terms of the progression.

If this rule is applied to the denominator of any term of the given series, we shall find that it consists of (n + 1) factors.

We have been thus particular, that no part of the solution should be short of explanation.

By means of the preceding conclusions and observations we shall have

$$\frac{2^{n-1}}{1.2.3.4\ldots n} \times \left\{q-1-\frac{1}{1.3}-\frac{1.2}{1.3.5}-\frac{1.2.3}{1.3.5.7}\text{ \&c. to } n \text{ terms}\right\}$$

$$= \frac{1}{1.3.\ldots.(2n+1)} - \frac{1}{3.5\ldots.(2n+3)} + \frac{1}{5.7\ldots.(2n+5)} -$$

$\frac{1}{7.9\ldots.(2n+7)}$ &c. ad infinitum; and writing $\frac{\pi}{2}$ in the place of q, we shall have the very expression in the question.

This question was also answered by NOV-ANGLUS.

XIX. QUESTION 429, *by* C. B.

Prove that the series

$$\frac{1}{1.2.3\ldots.n} - \frac{1}{2.3\ldots.n+1} + \frac{1}{3.4\ldots n+2} - \text{\&c. is equal to}$$

$$\frac{1}{1.2\ldots.n+1}\left\{2^{n-1}\log.2 - \frac{2^{n-2}}{1} - \frac{2^{n-3}}{2} - \text{\&c.}\ldots - \frac{2^0}{n-1}\right\}.$$

FIRST SOLUTION, *by* C. B. *the Proposer.*

Put $u_n = \frac{1}{1.2\ldots n} - \frac{1}{2.3.\ldots.n+1} + \text{\&c.}$ Then

$$u_n - \frac{1}{1.2\ldots.n} = -\frac{1}{2.3\ldots.n+1} + \frac{1}{3.4\ldots.n+2} - \text{\&c.}$$

$$2u_n - \frac{1}{1.2\ldots.n} = \frac{n+1-1}{1.2\ldots.n+1} - \frac{n+2-2}{2.3\ldots n+2} + \text{\&c.} = nu_{n+1};$$

hence the equation of first differences

$2u_n - nu_{n+1} = \frac{1}{1.2 \ldots n}$. Assume $u_n = \frac{2^n w_n}{1.2 \ldots n-1}$,

then it becomes

$$\frac{2^{n+1} w_n}{1.2 \ldots n-1} - \frac{n 2^{n+1} w_{n+1}}{1.2 \ldots n} = \frac{1}{1.2 \ldots n}, \text{ or } w_n - w_{n+1} = \frac{2^{-n-1}}{n}.$$

Therefore

$$w_n = -\Sigma \frac{2^{-n-1}}{n} = -\frac{2^{-2}}{1} - \frac{2^{-3}}{2} - \frac{2^{-4}}{3} - \&c. - \frac{2^{-n}}{n-1} + c,$$

and

$$u_n = \frac{1}{1.2 \ldots n-1}\left\{2^n c - \frac{2^{n-2}}{1} - \frac{2^{n-3}}{1} - \frac{2^{n-4}}{3} - \&c. - \frac{2^0}{n-1}\right\}.$$

If $n=1$, then $2c = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \&c.$ or $c = \frac{1}{2} \log. 2$.

Therefore the sum of the series is,

$$\frac{1}{1.2 \ldots n-1}\left\{2^{n-1} \log. 2 - \frac{2^{n-2}}{1} - \frac{2^{n-3}}{2} - \frac{2^{n-4}}{3} - \&c. - \frac{2^0}{n-1}\right\}.$$

SECOND SOLUTION, *by* A. D.

Assume

$$u_{n-1} = \frac{1}{1.2 \ldots n-1} - \frac{1}{2.3 \ldots n} + \frac{1}{3.4 \ldots n+1} - \&c. \qquad (1)$$

In order to obtain an equation of finite differences, we will put equation (1) under this form:

$$2u_{n-1} = \frac{1}{1.2 \ldots n-1} + \left\{\frac{1}{1.2 \ldots n-1} - \frac{1}{2.3 \ldots n}\right\}$$
$$- \left\{\frac{1}{2.3 \ldots n} - \frac{1}{3.4 \ldots n+1}\right\} + \&c.$$
$$= \frac{1}{1.2 \ldots n-1} + \frac{n-1}{1.2 \ldots n} - \frac{n-1}{2.3 \ldots n+1}$$

$$+\frac{n-1}{3\cdot 4\,\ldots\ldots\, n+2}-\text{\&c.}$$

$$\therefore\ 2u_{n-1}=\frac{1}{1.2\ldots\ldots n-1}+(n-1)\,u_n \qquad (2)$$

Let $a_{n-1}=-\frac{1}{1.2.3\ldots n-1}\,\frac{1}{n-1}$, then

$$u_n=a_{n-1}+\frac{2}{n-1}\,u_{n-1}$$

$$=a_{n-1}+\frac{2}{n-1}\left\{a_{n-2}+\frac{2}{n-2}\,u_{n-2}\right\}$$

$$=a_{n-1}+\frac{2}{n-1}a_{n-2}+\frac{2^2}{n-1.n-2}a_{n-3}+\frac{2^3}{n-1.n-2.n-3}a_{n-4}+\ldots$$

$$+\frac{2^{n-2}}{n-1\,.\,n-2\ldots 3.2}a_1+\frac{2^{n-1}}{n-1\,.\,n-2\ldots 3.2.1}\,u_1. \qquad (3)$$

But

$$u_1=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots=\log.\ 2,$$

$$a_1=\frac{1}{1}\,\frac{1}{1}.\quad a_2=-\frac{1}{1.2}\,\frac{1}{2},\quad a_3=-\frac{1}{1.2.3}\,\frac{1}{3}\ \text{\&c.}$$

Substituting these values and writing the series backwards, we shall have

$$u_n=\frac{2^{n-1}}{1.2\ldots\ldots n-1}\log.\,2-\frac{2^{n-2}}{2.3\ldots\ldots n-1}\,\frac{1}{1}$$

$$-\frac{2^{n-3}}{3.4\ldots\ldots n-1}\,\frac{1}{1.2}\,\frac{1}{2}-\frac{2^{n-4}}{4.5\ldots\ldots n-1}\,\frac{1}{1.2.3}\,\frac{1}{3}-\text{\&c.}$$

$$= \frac{1}{1.2.3\ldots n-1}\left\{2^{n-1}\log.\,2 - \frac{2^{n-2}}{1} - \frac{2^{n-3}}{2} - \frac{2^{n-4}}{3} - \ldots - \frac{2^{0}}{n-1}\right\},$$

as was to be shewn.

THIRD SOLUTION, *by Mr.* CUNLIFFE.

Put a = hyp. log. 2; then it is well-known that

$$a = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \text{ \&c.}$$

whence

$$1 - a = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} - \frac{1}{9} \text{ \&c.}$$

taking the difference

$$\frac{2a-1}{1} = \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + \frac{1}{5.6} - \frac{1}{6.7} \text{ \&c.} = \frac{2^1 a - 2^0}{1}$$

and transposing

$$\frac{1}{1.2} - \frac{2a-1}{1} = \frac{1}{2.3} - \frac{1}{3.4} + \frac{1}{4.5} - \frac{1}{5.6} + \frac{1}{6.7} - \frac{1}{7.8} \text{ \&c.}$$

taking the difference and dividing by 2

$$\frac{4a - 2 - \frac{1}{2}}{1.2} = \frac{2^2 a - 2^1 - \frac{2^0}{2}}{1.2} =$$

$$\frac{1}{1.2.3} - \frac{1}{2.3.4} + \frac{1}{3.4.5} - \frac{1}{4.5.6} + \frac{1}{5.6.7} - \frac{1}{6.7.8} \text{ \&c.}$$

and hence

$$\frac{1}{1.2.3}-\frac{2^2a-2^1-\frac{2^0}{2}}{1.2}=$$

$$\frac{1}{2.3.4}-\frac{1}{3.4.5}+\frac{1}{4.5.6}-\frac{1}{5.6.7}+\frac{1}{6.7.8}-\frac{1}{7.8.9}\ \&c.$$

taking the difference and dividing by 3,

$$\frac{2^3a-2^2-\frac{2^1}{2}-\frac{2^0}{3}}{1.2.3}=\frac{1}{1.2.3.4}-\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}-\frac{1}{4.5.6.7}\ \&c.$$

and transposing

$$\frac{1}{1.2.3.4}-\frac{2^3a-2^2-\frac{2^1}{2}-\frac{2^0}{3}}{1.2.3}=\frac{1}{2.3.4.5}-\frac{1}{3.4.5.6}+\frac{1}{4.5.6.7}\ \&c.$$

taking the difference and dividing by 4,

$$\frac{2^4a-2^3-\frac{2^2}{2}-\frac{2^1}{3}-\frac{2^0}{4}}{1.2.3.4}=\frac{1}{1.2.3.4.5}-\frac{1}{2.3.4.5.6}+\frac{1}{3.4.5.6.7}\ \&c.$$

From what has been done it is manifest, that the sums of these series may be obtained in succession, in the same manner as those in the solution of question 428, viz. by multiplying the sum of any of the above series by 2, then subtracting the first term from that product, and dividing the remainder by the difference of the numerators of the 1st and 2nd terms, when reduced to the least common denominator, the quotient will be the sum of the next succeeding series, &c. &c.

We shall also perceive, as, in the solution to question 428, that the number of factors in the denominator of the expression for the sum of any of the above series, is always *one* less, than the number of factors in the denominators of the several terms of the series.

But the way of forming the expressions for the sums of the foregoing series is so plainly indicated by the expressions themselves, as to leave no difficulty, in forming by induction, the expression for the sum of any series of the kind, whatever number of factors the denominators of the terms of the series may consist of.

In the series, in the question, the number of factors in the denominator of each term is n, and therefore we gather from the preceding results, that the expression for the sum will be

$$\frac{1}{1.2.3.4...(n-1)} \times \left\{ 2^{n-1} a - 2^{n-2} - \frac{2^{n-3}}{2} - \frac{2^{n-4}}{3} - \frac{2^{n-5}}{4} \,\&c. - \frac{2^{0}}{n-1} \right\}$$

that is,

$$\frac{1}{1.2.3...n} - \frac{1}{2.3.4..(n+1)} + \frac{1}{3.4.5...(n+2)} - \frac{1}{4.5.6....(n+3)} \,\&c.$$

$$= \frac{1}{1.2.3...(n-1)} \times$$

$$\left\{ 2^{n-1} \times h.l.2 - 2^{n-2} - \frac{2^{n-3}}{2} - \frac{2^{n-4}}{3} - \frac{2^{n-5}}{4} \,\&c - \frac{2^{0}}{n-1} \right\},$$

NOV-ANGLUS *likewise sent a solution to this question.*

XX. PRIZE QUESTION 430, *by* MECHANICUS.

If two bodies, connected by an inflexible and inextensible line, be placed on an horizontal plane, and one of them be constrained to move in a straight line given by position, with a uniform velocity; it is required to determine the circumstances of the motion of the other body, and the nature of the curve it describes?

FIRST SOLUTION, *by* M. FRANCAIS,

(From the Annales de Mathématiques).

Upon a horizontal plane, a rectilineal path is traced in which a body P is constrained to move uniformly. This body is connected, by an inflexible and inextensible rod, with another body M which is posited on this plane, and which is supposed to have received some primitive impulse in the direction of this plane. It is required to find the nature of the curve described by the body M and the other circumstances of the motion, abstracting from friction.

Let the straight line in which the body P moves, be taken for the axis of x, and any straight line at right angles to this, for the axis of y.

Let, at the epoch t, x and y be the co-ordinates of the point, M and x' the abscissa of the point P; the rectilineal motion of this last point must be the effect of an accelerative force, directed according to the axis of x and disturbed by the reaction of M upon P. Let p be this accelerative force.

The general equation of motion is then, by supposing t the independent variable,

$$\mathrm{M}\frac{d^2x}{dt^2}\,\delta x + \mathrm{M}\,\frac{d^2y}{dt^2}\,\delta y + \mathrm{P}\left\{\frac{d^2x}{dt^2} - p\right\}\delta x' = 0; \qquad (*)$$

or simply, because $\frac{dx'}{dt}$ is constant

$$\mathrm{M}\,\frac{d^2x}{dt^2}\,\delta x + \mathrm{M}\,\frac{d^2y}{dt^2}\,\delta y - \mathrm{P}p\delta x' = 0. \qquad (1)$$

Designating by a the length of the rod, the connection of the parts of the system is expressed by the single equation

$$(x - x')^2 + y^2 = a^2, \qquad (2)$$

which gives

$$(x - x')(\delta x - \delta x') + y\delta y = 0,$$

whence

$$\delta x' = \delta x - \frac{y}{(x - x')}\,\delta y; \qquad (3)$$

substituting therefore this value in equation (1) it becomes

$$\left\{\mathrm{M}\,\frac{d^2x}{dt^2} - \mathrm{P}p\right\}\delta x + \left\{\mathrm{M}\,\frac{d^2y}{dt^2} - \frac{\mathrm{P}py}{x - x'}\right\}\delta y = 0; \qquad (4)$$

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[*] *Mécanique céleste tom.* 1, *page* 51.

δx and δy being then independent, we have simply

$$\text{M}\,\frac{d^2x}{dt^2} = \text{P}p, \qquad \text{M}\,\frac{d^2y}{dt^2} = \frac{\text{P}py}{x - x'}, \tag{5}$$

whence, by elimination, we conclude

$$y\,d^2x = (x - x')\,d^2y. \tag{6}$$

Since dx' is constant, we must try to find an equation between x' and y. For this purpose differentiate twice consecutively equation (2) and it becomes

$$dx = dx' - \frac{y\,dy}{\sqrt{(a^2 - y^2)}},$$

$$d^2x = -\frac{y\,d^2y}{\sqrt{(a^2 - y^2)}} - \frac{a^2dy^2}{(a^2 - y^2)^{\frac{3}{2}}};$$

but equation (6) gives

$$d^2x = \frac{d^2y\,\sqrt{(a^2 - y^2)}}{y};$$

equating these two values, and reducing

$$\frac{d^2y}{\sqrt{(a^2 - y^2)}} + \frac{y\,dy^2}{(a^2 - y^2)^{\frac{3}{2}}} = 0, \tag{7}$$

an equation which has for its integral

$$\frac{dy}{\sqrt{(a^2 - y^2)}} + c\,dx' = 0. \tag{8}$$

This last equation, integrated again, gives

$$\text{arc}\left(\cos = \frac{y}{a}\right) = cx' + c';$$

or, putting for x its value found from equation (a)

$$\text{arc}\left(\cos = \frac{y}{a}\right) = \mathrm{c}\left\{x - \sqrt{(a^2 - y^2)}\right\} - \mathrm{c}'. \qquad (9)$$

For determining the constants c and c', suppose first that the constant velocity of P is b, or that $\frac{dx'}{dt} = b$. By putting this value in equation (8) it becomes

$$\frac{dy}{dt} + b\mathrm{c}.\sqrt{(a^2 - y^2)} = 0. \qquad (10)$$

Supposing again that at the origin of the time, the point P is the origin of the co-ordinates, and that the rod a forms then an angle α with the axis of x. Suppose also that the initial velocity of M parallel to the axis of y is c, so that for $t = 0$, and $y = a \sin \alpha$ we have $\frac{dy}{dt} = c$: equation (10) becomes

$$c + ab\mathrm{c} \cos \alpha = 0, \qquad \text{whence } \mathrm{c} = -\frac{c}{ab \cos \alpha}.$$

The second integral (9) relative to the same initial state, becomes

$$\text{arc}\,(\cos = \sin \alpha) = \mathrm{c}', \qquad \text{whence } \mathrm{c}' = \tfrac{1}{2}\pi - \alpha.$$

Whence, we have

$$x = \sqrt{(a^2 - y^2)} + \frac{ab \cos \alpha}{c}\left\{\text{arc}\left(\sin = \frac{y}{a}\right) - \alpha\right\}. \qquad (11)$$

This is the equation of the curve described by the body M. We see that this curve is a cycloid, relative to the straight line passed over by the centre of the generating circle: this circle has for its radius the length a of the rod; its centre is the extremity P of this rod, and the ratio of the velocities of translation of the centre and of rotation of the parts of the circumference about this centre is that of $b \cos \alpha$ to c; the cycloid is therefore *prolate*, common, or *curtate*, according as we have

$$b \cos \alpha > c, \quad b \cos \alpha = c, \quad \text{or } b \cos \alpha < c.$$

Equation (11) contains, as one of the given quantities the initial velocity of M in the direction of y, we may easily introduce its velocity in the direction of x. If, in effect, we put in the first integral (8) for $\frac{dy}{\sqrt{(a^2 - y^2)}}$, its value $\frac{dx' - dx}{y}$, we have

$$\frac{dx'}{dt} + cy\,\frac{dx'}{dt} - \frac{dx}{dt} = 0.$$

Let now c' be the initial velocity of M in the direction of x, so that we shall have $\frac{dx}{dt} = c'$, this equation becomes $b + abc \sin a - c' = 0$, whence

$$c = \frac{c' - b}{ab \sin a};$$

introducing this value into the equation of the curve it becomes

$$x = \sqrt{(a^2 - y^2)} + \frac{ab \sin a}{c' - b}\left\{\text{arc}\left(\sin = \frac{y}{a}\right) - a\right\}; \qquad (12)$$

the relation between the initial velocities c and c' being expressed by

$$c' \cos a + c \sin a = b \cos a.$$

We must notice here, that these initial velocities are not those which are imprinted on M by an impulsive force only; they are the resultants of the primitive impulse on M and of the action of P upon M; so that, at the point of impulsion, they are due only to the action of P. The velocity b is not wholly due to the action of the accelerative force p, but to this action modified by the effect of the impulse given to M.

Equation (11) is in fault when we have $a = \frac{1}{2}\pi$; but then we employ equation (12) which becomes

$$x = \sqrt{(a^2 - y^2)} - \frac{ab}{c' - b}\,\text{arc}\left(\cos = \frac{y}{a}\right).$$

In like manner if $a = 0$, equation (12) is in fault, but equation (11) becomes

$$x = \sqrt{(a^2 - y^2)} + \frac{ab}{c}\,\text{arc}\left(\sin = \frac{y}{a}\right).$$

For determining the velocity of M, for any point of the curve, we have the equations

$$\frac{dx}{dt} = \frac{ab \cos \alpha - cy}{a \cos \alpha}, \quad \frac{dy}{dt} = \frac{c\sqrt{(a^2 - y^2)}}{a \cos \alpha};$$

hence

$$v = \frac{ac^2 - 2bcy \cos \alpha + ab^2 \cos^2 \alpha}{a \cos^2 \alpha}.$$

Also; according as we have $y = a$, or $y = -a$, we have

$$v = b - \frac{c}{\cos \alpha}, \text{ or } v = b + \frac{c}{\cos \alpha}.$$

It is very easy to see that these are the least and the greatest velocities of the point M; the first has place at the highest and the second at the lowest point of each cycloid. Hence in the common cycloid for which we have $c = b \cos \alpha$, the velocity of the point M is nothing each time that it comes to the greatest elevation; and that it is double of that of the point P, each time that it comes to the lowest.

The time is found from the formula $\frac{dy}{\sqrt{(a^2 - y^2)}} = \frac{c\,dt}{a \cos \alpha}$, which gives

$$\text{arc}\left(\sin = \frac{y}{a}\right) = \frac{ct}{a \cos \alpha} + C'';$$

and as we have at the same time $y = a \sin \alpha$ and $t = 0$, it follows that $C'' = \alpha$, which gives

$$t = \frac{a \cos \alpha}{c}\left\{\text{arc}\left(\sin = \frac{y}{a}\right) - \alpha\right\}. \qquad (13)$$

Also, when $y = a$, we have

$$t = \frac{a \cos \alpha}{c}\left\{\frac{2n + 1}{2}\pi - \alpha\right\},$$

n being any whole positive number; whence it follows that the time employed in passing over the whole cycloid is $= \frac{\pi a \cos \alpha}{c}$.

The accelerative force $p = \frac{M}{P}\,\frac{d^2x}{dt^2}$; but

$$\frac{dx}{dt} = b - \frac{cy}{a\cos\alpha}, \text{ whence } \frac{d^2x}{dt^2} = -\frac{c\,dy}{a\,dt\,\cos\alpha},$$

and as we have otherwise

$$\frac{dy}{dt} = \frac{c\sqrt{(a^2 - y^2)}}{a\cos\alpha},$$

it follows that we shall have

$$p = -\frac{M}{P}\cdot\frac{c^2\sqrt{(a^2 - y^2)}}{a^2\cos^2\alpha},$$

which gives for the initial value of p, $p = -\frac{M}{P}\cdot\frac{c^2}{a\cos\alpha}$.

Reflections and further researches on the same Problem, by M. GERGONNE.

This problem and several analogous problems were treated of by Clairault in the memoirs *de l' académie des sciences* of Paris for 1736. This geometer proved very easily by considerations purely synthetical, that the elementary space described by the point P in any instant, divided by the angle formed by the two directions of the rod a, at the beginning and at the end of this instant, is a constant quantity; whence it follows that the point M describes a circumference about the point P with an uniform motion during the time that this point is moved uniformly in a straight line, and also that the *Tractoire* is a cycloid.

Clairault's being occupied with this problem occasioned a discussion respecting it with Fontaine, who pretended that, in the whole course of the motion the direction of the rod was always a tangent to the curve; whence he concluded that the *Tractoire* was no other than the curve of equal tangents, and that the problem was thus reduced to a simple problem in geometry.

Notwithstanding the solution of Clairault, many geometers still continued to think, with Fontaine, that there was nothing to distinguish this problem from that of the curve of equal tangents. They concluded that neither the constant nor variable velocity of the point P, the friction or resistance of the medium which acted always in the direction of the motion, which is here the same as that of the rod a, could in any way modify the nature of the curve. As for Clairault, he agreed so far with Fontaine as to admit that when the body M experienced resistance from the horizontal plane and there was not any velocity impressed, the *Tractoire* might be a curve of equal tangents; and that his design was only to demonstrate that the curve must be a cycloid in the case when the friction and other obstacles were noticed.

In a memoir, which I presented several months ago, to *L' académie du Gard*, I have sketched a solution of the general problem of *Tractoires* in a resisting medium, supposing the point P to describe in space any given curve of double curvature with a motion any how varied: we see then that it will, by supposing the rod heavy and flexible, and having regard to its weight and curvature, be sufficient for determining the complete theory of the motion of a Kite.

By particularising my results so as to adapt them to the present case, I find that in general I have obtained similar results to those of M. Francais. But as my way differs a little from his, I think it will not be disagreeable to find here a comparison of the two methods.

For disembarrasing the problem of all foreign considerations, I shall propose the particular case in the question as follows:

Problem. A point P describes the axis of x by moving uniformly with a known velocity equal to b and exercises an attractive or repulsive force, constant or variable, on another point M, posited on the plane of the rectangular co-ordinates, but otherwise perfectly at liberty. The action of P upon M is such that these two points always keep at the same distance from one another. It is required from this to determine the nature of the curve described by the point M, and also the other circumstances of the motion.

Solution. Let, for any epoch, x and y be the ordinates of M, x' the abscissa of P and p the action of P upon M; this action is exercised according to the straight line which joins these two points, and its components parallel to the axis of x and y are $\frac{x-x'}{a}p$, $\frac{y}{a}p$; whence it follows that the equations of motion of the point M, t being the independent variable, are

$$\frac{d^2x}{dt^2} = \frac{x-x'}{a}p \qquad (1) \qquad \frac{d^2y}{dt^2} = \frac{y}{a}p$$

to which we must join

$$(x-x')^2+y^2=a, \quad (3) \qquad \frac{dx'}{dt}=b$$

This supposed, if we differentiate twice the equation (3) and have regard to equation (4) we find

$$(x-x')\left\{\frac{dx}{dt}-b\right\}+y\frac{dy}{dt}=0. \qquad (5)$$

$$(x-x')\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+\left\{\frac{dx}{dt}-b\right\}^2+\left(\frac{dy}{dt}\right)^2=0. \qquad (6)$$

But on the other hand, by eliminating p from the equations (1) and (2), we obtain

$$(x-x')\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=0. \qquad (8)$$

By eliminating x, $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$ from the equations (3), (5), (6), (7), we have

$$(a^2-y^2)\frac{d^2y}{dt^2}+y\left(\frac{dy}{dt}\right)^2=0; \qquad (8)$$

an equation of which the integral is,

$$\frac{dy}{dt}=c\sqrt{(a^2-y^2)}=c\,(x-x'); \qquad (9)$$

the value of which being substituted in equation (5) gives

$$\frac{dx}{dt}=b-cy=-c\sqrt{\left\{a^2-(x-x')^2\right\}}. \qquad (10)$$

Eliminating, therefore, t from the equations (9) and (10) we have for the differential equation of the trajectory

$$\frac{dy}{dx} = \frac{\mathrm{c}\sqrt{(a^2 - y^2)}}{b - \mathrm{c}y} \qquad (11)$$

an equation which we know to belong to the cycloid, which is prolate, common or curtate, according to the different values which may be attributed to the constant c.

For determining the constant c, I suppose that at the epoch when $t = 0$, the point M is found to have the velocity c, either imprinted or otherwise acquired, in a direction forming an angle with the axis of x, and of which the components respectively parallel to the axis of x and y, are consequently $c \cos \beta$ and $c \sin \beta$; I suppose also, with M. Francais, that at the same epoch the point P is at the origin and that the rod a makes an angle α with the axis of x; that is to say, I suppose at the same time

$$t = 0, \quad \frac{dx}{dt} = c \cos \beta, \quad \frac{dy}{dt} = c \sin \beta, \quad x = a \cos \alpha, \quad y = a \sin \alpha.$$

From these suppositions, equations (9) and (10) become

$$c \sin \beta = \mathrm{c}a \cos \alpha,$$

$$c \cos \beta = b - \mathrm{c}a \sin \alpha;$$

whence

$$\mathrm{c} = \frac{c \sin \beta}{a \cos \alpha} = \frac{b - c \cos \beta}{a \sin \alpha}, \qquad (12)$$

which gives the equation of relation

$$c \cos (\beta - \alpha) = b \cos \alpha, \qquad (13)$$

If now, we integrate equation (11), we find

$$x = \sqrt{(a^2 - y^2)} + \frac{ab}{\mathrm{c}} \operatorname{arc} \left(\sin = \frac{y}{a}\right) + \mathrm{c}';$$

the initial circumstances of the motion give, after reduction

$$0 = \frac{ab}{c}\alpha + c',$$

so that we have definitively

$$x = \sqrt{(a^2 - y^2)} + \frac{ab}{c}\left\{\text{arc}\left(\sin = \frac{y}{a}\right) - \alpha\right\}. \qquad (14)$$

an equation in which by virtue of the relation (13) we may substitute for c either of the two values given by equation (12). We have again

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = c^2a^2 + 2cby - b^2.$$

But equation (9) gives by integration and having regard to the initial circumstances of the motion

$$ct - \alpha = \text{arc}\left(\sin = \frac{y}{a}\right);$$

whence

$$y = a \sin(ct - \alpha);$$

therefore

$$v^2 = c^2a^2 + 2abc \sin(ct - \alpha) - b^2. \qquad (15)$$

It appears well established, from what has preceded, that abstractng friction and the resistance of the medium, and supposing the motion of P to be uniform and rectilineal, that the plane *Tractoire* is a cycloid.

To leave nothing to be desired on the subject, I shall finally seek the value of the accelerative force which must act on the point M so as to make it describe the curve of equal tangents, that is to say, I shall resolve the following

Problem. While a point P passes over the axis of x with the uniform velocity b, another point M moves with a varied and curvilineal motion on the plane of xy. The motion of this last point is such that it is always to be found at the same constant distance a from the point P and besides that, the moveable straight line which joins the two points is perpetually a tangent to the curve described by the point M. It is required to find the nature of this curve and the accelerative force which acts upon M.

Solution. Retaining the notation and conventions of the preceding problem, the invariability of the distance between the points M and P, is expressed by the equation

$$(x - x')^2 + y^2 = a^2; \qquad (1)$$

and the property that the straight line which joins them is always a tangent to the curve described by M, is expressed by this other equation

$$\frac{y}{x - x'} = \frac{dy}{dx}, \quad \text{whence } y = (x - x')\frac{dy}{dx}. \qquad (2)$$

Eliminating $x - x'$ from these, we have

$$\frac{dy}{dx} = \frac{y}{\sqrt{(a^2 - y^2)}};$$

an equation of which the integral is,

$$x = \sqrt{(a^2 - y^2)} - a \log \frac{a + \sqrt{(a^2 - y^2)}}{y} + c.$$

If for determining the constant, we suppose, as above, that we have at the same time

$$x = a \cos \alpha, \qquad y = a \sin \alpha$$

it becomes

$$c = a \log \frac{1 + \cos \alpha}{\sin \alpha} = a \log \cot \tfrac{1}{2}\alpha = - a \log \tan \tfrac{1}{2}\alpha;$$

and which gives for the equation of the curve of equal tangents

$$x = \sqrt{(a^2 - y^2)} - a \log . \frac{a + \sqrt{(a^2 - y^2)}}{y \tan \frac{1}{2}\alpha} . \qquad (3)$$

Now, by considering t as the independent variable, we may put equation (2) under this form

$$y \frac{dx}{dt} - \sqrt{(a^2 - y^2)} . \frac{dy}{dt} = 0, \qquad (4)$$

on the other hand, by differentiating equation (1) it becomes, since $\frac{dx'}{dt} = b$,

$$\sqrt{(a^2 - y^2)} . \frac{dx}{dt} + y \frac{dy}{dt} = b \cdot \sqrt{(a^2 - y^2)}. \qquad (5)$$

From these two equations we draw

$$\frac{dx}{dt} = \frac{b\,(a^2 - y^2)}{a^2}, \qquad (6)$$

$$\frac{dy}{dt} = \frac{by\sqrt{(a^2 - y^2)}}{a^2}. \qquad (7)$$

We find again, by a new differentiation,

$$\frac{d^2x}{dt^2} = -\frac{2b^2y^2\sqrt{(a^2 - y^2)}}{a^4}, \qquad (8)$$

$$\frac{d^2y}{dt^2} = +\frac{b^2y\,(a^2 - 2y^2)}{a^4}. \qquad (9)$$

By designating therefore by X and Y, respectively, the components of the accelerative force, parallel to the axis, we have

$$X = -\frac{2b^2y^2\sqrt{(a^2 - y^2)}}{a^4}, \qquad Y = +\frac{b^2y\,(a^2 - 2y^2)}{a^4};$$

and, consequently if we designate this force by ϕ, we have

$$\varphi = \sqrt{(X^2 + Y^2)} = \frac{b^2 y}{a^2},$$

and this makes, with the axis of x, an angle of which the tabular tangent is

$$\frac{Y}{X} = -\frac{a^2 - 2y^2}{2y\sqrt{(a^2 - y^2)}};$$

whence it is easy to conclude that the components, according to the tangent and normal are respectively

$$\frac{b^2 y^2}{a^3}, \qquad \frac{b^2 y \sqrt{(a^2 - y^2)}}{a^3}.$$

We see therefore that the power φ is not in the direction MP.

Another Solution, with some explanations of M. Francais' equation of conditions.

By M. DUBUAT.

1. The equation $c' \cos a + c \sin a = b \cos a$, is no other than the general equation of condition $(x - x')(dx - dx') + ydy = 0$, in which we put for the variables dx', dx, dy, $x - x'$, y the values bdt, $c'dt$, cdt; $a \cos a$, $a \sin a$, which they have at the origin of the motion.

2. Or, the general equation $(x - x')(dx - dx') + ydy = 0$, signifies that the variable velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$ of the point M, in the direction of the axes of the co-ordinates are such that, if from the velocity $\frac{dx}{dt}$ according to the axis of x, we subtract the velocity $\frac{dx'}{dt}$ of the point P, the remaining velocity $\frac{dx - dy}{dt}$ forms with the velocity according to the axis of y, a resultant perpendicular to the radius vector PM: whence it follows that the velocity of the point M, considered either at the commencement or in the continuation of the motion, may always be decomposed into

two velocities, the one parallel to the axis of x constant and equal to b, the other perpendicular to the radius vector, the value of which may be any thing.

3. Hence if the velocity imprinted on the point M, at the origin of the motion is not decomposable into two velocities according to the same law, this velocity is not the initial velocity from which we must determine the constants in the integration.

4. Let, at the origin of motion, V be the velocity imprinted on the point N, and β the angle that the rod makes with the axis of x, the components are V cos β in the direction of x and V sin β in the direction of y.

The first component V cos β is equivalent to the two velocities b and V cos $\beta - b$, of which the first b only subsists by virtue of the equation of condition; but the velocity V cos $\beta - b$ is not totally destroyed by its decomposition into two velocities, the one according to the radius vector, and the other perpendicular to this radius; this last, the expression for which is $(V \cos \beta - b) \sin \alpha$, subsists, whilst the other is destroyed.

The velocity V sin β, imprinted in the way of y, being also decomposed into two velocities, the one according to the radius vector and the other perpendicular to this radius; the second subsists only, and is expressed by V sin β cos α,

5. The initial velocity resulting from the imprinted velocity V, is therefore composed of a velocity b, parallel to the axis of x, and a velocity $(V \cos \beta - b) \sin \alpha + V \sin \beta \cos \alpha$, perpendicular to the radius vector: this gives for the component c' of the initial velocity according to the axis of x,

$$c' = b \pm \left\{ V \sin (\alpha + \beta) - b \sin \alpha \right\} \sin \alpha.$$

and for the component c of the initial velocity according to the axis of y

$$c = \pm \left\{ V \sin (\alpha + \beta) - b \sin \alpha \right\} \cos \alpha.$$

6. But there is another difficulty presented in equations (11) and (12).

If we make in the first $c = 0$, or $c' - b = 0$ in the second, we have $x = \sqrt{(a^2 - y^2)} + \infty$, which has no signification. For removing this difficulty I remark that by virtue of the equation of condition $(c' - b) \cos \alpha + c \sin \alpha = 0$, the hypothesis $c = 0$ gives $(c' - b) \cos \alpha = 0$, and consequently $c' = b$, or $\cos \alpha = 0$.

Let us put $c = 0$, $c' = b$. These two equations signify that the initial velocity of the point M, parallel to the axis of x is nothing, and that its initial velocity parallel to the axis of y is b, and equal of course to the velocity of the point P in the same direction: the two points M and P are therefore animated, at the origin of the motion with velocities equal and parallel: The equation of condition subsists for these two velocities, in the first instant and in the whole course of the motion. The point M describes therefore a straight line parallel to the axis of x, with a velocity constant and equal to b; this gives $y = constant$ and $x = bt + constant$.

Let, in the second place, $c = 0$ and $\cos a = 0$. These two equations signify that the initial velocity of the point M parallel to y is nothing, and that the ordinate at the same point is also nothing at the origin of the motion without determining the initial velocity parallel to x. The two points M and P at the origin of the motion are therefore on the axis of x, and the point P of which the velocity is b, by virtue of the equation of condition, can neither be augmented nor diminished. It is easy to conclude that the system of the two points moves in the first instant, and during the whole course of the motion on the axis of x, with a common velocity b, that is to say we have $y = 0$, $x = bt + constant$.

The problem may also be resolved in the following manner:

7. The equations of condition are, making the radius vector $= 1$,

$$(x - x')^2 + y^2 = 1, \qquad x' = bt;$$

those of motion are

$$\frac{d^2x}{dt^2} = \mu\,(x - x'), \qquad \frac{d^2y}{dt^2} = \mu y;$$

μ being an indeterminate. Let $y = \sin\varphi$ and $x - x' = \cos\varphi$, by substituting these values in the equations of motion, we have

$$-\frac{d^2\varphi}{dt^2}\sin\varphi - \frac{d^2\varphi}{dt^2}\cos\varphi = \mu\cos\varphi,$$

$$\frac{d^2\varphi}{dt^2}\cos\varphi - \frac{d^2\varphi}{dt^2}\sin\varphi = \mu\sin\varphi;$$

and by eliminating μ,

$$d^2\varphi = 0, \quad \text{therefore } \varphi = \text{A}t + \text{A}', \quad \text{and}$$

$$x = bt + \cos(\text{A}t + \text{A}'), \qquad y = \sin(\text{A}t + \text{A}').$$

By determining the constants from the initial velocity V, making with the axis of x the angle β, we have

$$x = bt + \cos\left\{[\text{V}\sin(\alpha + \beta) - b\sin\alpha]\,t + \alpha\right\},$$

$$y = \sin\left\{[\text{V}\sin(\alpha + \beta) - b\sin\alpha]\,t + \alpha\right\},$$

These formulae shew that the point M is moved about the point P, with an uniform and continued motion, and with a velocity $\text{V}\sin(\alpha + \beta) - b\sin\alpha$.

8. If we suppose, as above, that the initial velocity of the point M, parallel to the axis of y, is nothing, and that parallel to the axis of x is b; we find, by making $\text{V} = b$ and $\beta = 0$, $y = \sin\alpha$, $x = bt + \cos\alpha$; a result conformable to that of No. 6. If we suppose again that the initial velocity of the point M, parallel to y is nothing, and that the ordinate of the same point is nothing at the origin of motion, we find, by making $\alpha = 0$, $\beta = 0$, conformable to this hypothesis, $y = 0$, $x = bt + 1$, as above.

SECOND SOLUTION, *by Mr.* LOWRY.

Let P and Q be two bodies connected by an inflexible and inextensible line or rod of a given length, and placed on an horizontal plane; let an impulse be given to one of them as Q, so that this impulse combined with the mutual action of the bodies may cause the other body P to move in a straight line given by position, with an uniform velocity a; it is required to determine the circumstances of the motion of Q, and the curve it describes.

Put

r = the length of the rod,

x and y = the rectangular co-ordinates of Q at the end of any definite time t, x being taken in the direction of the given line and having its origin at the point where the motion of P commences, and y the perpendicular from Q upon x.

x = the distance of P from the foot of y at the end of the time t.

φ = the accelerative force arising from the combined effects of the impulse and the action of the bodies upon each other.

Then $\frac{z}{r} \times \varphi$ will be the effect of that force parallel to x, and $\frac{y}{r} \times \varphi$ the effect in the direction of y; therefore, by the principles of dynamics,

$$\frac{\ddot{x}}{\dot{t}^2} = \frac{z}{r} \times \varphi$$

$$\frac{\ddot{y}}{\dot{t}^2} = \frac{y}{r} \times \varphi$$

and, by eliminating φ,

$$\frac{z\ddot{y}}{\dot{t}^2} - \frac{y\ddot{x}}{\dot{t}^2} = 0, \qquad (1)$$

Now to exterminate z and $\ddot{x}$ from this equation, so as to get the value of $\frac{\dot{y}}{\dot{t}}$, we have, by the question,

$$z^2 + y^2 = r^2,$$

and taking first and second fluxions

$$z\dot{z} + y\dot{y} = 0$$

$$z\ddot{z} + \dot{z}^2 + y\ddot{y} + \dot{y}^2 = 0$$

therefore

$$\ddot{z} = -\frac{y\ddot{y}}{z} - \frac{\dot{z}^2 + \dot{y}^2}{z} = -\frac{y\ddot{y}}{z} - \frac{r^2\dot{y}^2}{z^3},$$

But $x - z$ is the space described by P in the time t, with the uniform velocity a; therefore the fluxion of this space, or of $x - z$, must be constant; consequently $\ddot{x} = \ddot{z}$, therefore

$$\ddot{x} = -\frac{y\ddot{y}}{z} - \frac{r^2\dot{y}^2}{z};$$

and substituting this value for $\ddot{x}$ in equation (1), and remarking that $z^2 = r^2 - y^2$, we have

$$(r^2 - y^2)\frac{\ddot{y}}{\dot{t}^2} + y\left(\frac{\dot{y}}{\dot{t}}\right)^2 = 0.$$

This equation gives by integration

$$\frac{\dot{y}}{\dot{t}} = c\sqrt{(r^2 - y^2)} = cz \qquad (2)$$

and to find $\frac{\dot{x}}{\dot{t}}$, we have $\dot{z} = -\frac{y\dot{y}}{z} = cy\dot{t}$, but $\frac{\dot{x} - \dot{z}}{\dot{t}} = a$; therefore

$$\frac{\dot{x}}{\dot{t}} = a + \frac{\dot{z}}{\dot{t}} = a - cy. \qquad (3)$$

If therefore we put u = the velocity of Q in the direction of y, and v its velocity parallel to x, we have

$$u = c\sqrt{(r^2 - y^2)} \qquad (4)$$

$$v = a - cy \qquad (5)$$

and the velocity of Q in the curve $= \sqrt{(u^2 + v^2)}$

$$= \sqrt{(a^2 - 2acy + c^2r^2)}.$$

To determine the value of the constant quantity c, we must consider the manner in which the bodies are first put in motion. Suppose at that instant, in consequence of the impulse and action of P, that Q has a tendency to move in a direction which makes with the axis of x a given angle e, and with a velocity α, the incipient velocities in the directions of y and x would then be α sin e and α cos e; or $u = \alpha \sin e$ and $v = \alpha \cos e$.

Let β and δ be the values of x and y at this instant or when $t = 0$, then from equations (4) and (5)

$$\alpha \sin e = c\beta$$

$$\alpha \cos e = a - c\delta$$

and by elimination,

$$\alpha = \frac{a\beta}{\delta \sin e + \beta \cos e}$$

$$c = \frac{a \sin e}{\delta \sin e + \beta \cos e}$$

therefore

$$u = \frac{a \sin e}{\delta \sin e + \beta \cos e} \times \sqrt{(r^2 - y^2)}$$

$$v = \frac{a\,(\delta - 1) \sin e + \beta \cos e}{\delta \sin e + \beta \cos e} \times y.$$

Again, to find the equation of the curve, we have, by exterminating t from equations (2) and (3)

$$\dot{x} = \frac{a\dot{y} - cy\dot{y}}{c\sqrt{(r^2 - y^2)}} = \frac{a}{c} \times \frac{\dot{y}}{\sqrt{(r^2 - y^2)}} - \frac{y\dot{y}}{\sqrt{(r^2 - y^2)}}$$

and taking the fluents,

$$x = \frac{a}{c} \times \text{arc}\left(\sin \frac{y}{r}\right) + \sqrt{(r^2 - y^2)} + c';$$

but at the commencement of motion $x = \beta$, and $y = \delta$, therefore

$$\beta = \frac{a}{c} \times \text{arc}\left(\sin \frac{\delta}{r}\right) + \beta + c'$$

or $$c' = -\frac{a}{c} \times \text{arc}\left(\sin \frac{\delta}{r}\right)$$

and $$x = \frac{a}{c} \times \left\{\text{arc}\left(\sin \frac{y}{r}\right) - \text{arc}\left(\sin \frac{\delta}{r}\right)\right\} + \sqrt{(r^2 - y^2)};$$

which is an equation to the cycloid.

To find the time, we have, from equation (2)

$$t = \frac{1}{c} \times \text{arc}\left(\sin \frac{y}{r}\right) + c'';$$

but when $$t = 0,\ \frac{1}{c} \times \text{arc}\left(\sin \frac{\delta}{r}\right) + c'' = 0,$$

therefore $$c'' = -\frac{1}{c} \times \text{arc}\left(\sin \frac{\delta}{r}\right),$$

and $$t = \frac{1}{c} \times \left\{\text{arc}\left(\sin \frac{y}{r}\right) - \text{arc}\left(\sin \frac{\delta}{r}\right)\right\}.$$

The value of t might also have been found by considering that $t = \frac{x - z}{a}$, and if for x we put its value found above, we shall have the very same expression for t as before.

The results thus obtained agree with those in the solutions to the 14th question in the Ladies Diary for the year 1778, but the methods of solution are very different. See vol. III. pa. 41 of the new Republication of the Mathematical Questions in the Ladies Diary.

Solutions, upon different principles, were received from Mr. ANTHONY COOK *and* NOV-ANGLUS.

ARTICLE II.

Solutions to Questions proposed in No. XVII.

I. QUESTION 431, *by* PALABA.

A and B travelled on the same road and at the same rate from H to L. At the 50th mile stone from L, A overtook a drove of geese, which were proceeding at the rate of three miles in two hours; and two hours afterwards met a stage waggon which was moving at the rate of nine miles in four hours. B overtook the same drove of geese at the 45th mile stone and met the same stage waggon exactly forty minutes before he came to the 31st mile stone. Where was B when A reached L?

FIRST SOLUTION, *by* PALABA, *the Proposer.*

Let x = rate per hour at which A or B travel. The geese and waggon travel at the respective rates of $\frac{3}{2}$ and $\frac{9}{4}$ miles per hour. B approaches the waggon at the rate of $x + \frac{9}{4}$ miles per hour. And $\frac{3}{2} : 5 :: 1 : \frac{10}{3}$ = No. of hours B overtook the geese after A ∴ $\frac{10x}{3}$ = space he had passed over in that time ∴ $\frac{10x}{3} - 5$ = B's distance from A. Again A met the waggon $50 - 2x$ miles from L. B met it $31 + \frac{2x}{3}$ miles from L. ∴ it had travelled in that time $\frac{8x}{3} - 19$ miles, and $\frac{9}{4} : \frac{8x}{3} - 19 :: 1 : \left(\frac{8x}{3} - 19\right)\frac{4}{9}$ = time elapsed between A and B meeting the waggon ∴ A's distance from B = $\left(\frac{8x}{3} - 19\right)\frac{4}{9} \times \left(x + \frac{9}{4}\right)$.

$$\therefore \left(\frac{8x}{3} - 19\right) \cdot \left(\frac{4x}{9} + 1\right) = \frac{10x}{3} - 5:$$

Hence $x = 9$ ∴ $\frac{10x}{3} - 5 = 25$ the distance required.

B at the 45th was $\frac{10}{3}$ hours; and B was $\frac{5}{x}$ hours in travelling the last five miles, consequently $\frac{10}{3} - \frac{5}{x}$ is the time A was at the 50th mile stone before B, so that when B arrived at the 50th mile stone A was $\frac{10}{3}x - 5$ miles from it. Again, A met the waggon $50 - 2x$ miles from L, and B met it $31 + \frac{2}{3}x$ miles from L; therefore the waggon moved $31 + \frac{2}{3}x - 50 + 2x = \frac{8}{3}x - 19$ miles in the same time that B travelled $\frac{10}{3}x - 5 - \frac{8}{3}x + 19 = \frac{2x + 42}{3}$ miles, consequently $\frac{2x + 42}{3} : \frac{8x - 57}{3} :: x : \frac{9}{4}$; multiplying means and extremes we shall have $32x^2 - 228x = 18x + 378$, or $x^2 - 7\frac{11}{16}x = 11\frac{13}{16}$, and $x = 9$, therefore $\frac{10}{3} \times 9 - 5 = 25$ miles the distance that B was behind A when A reached L.

FIFTH SOLUTION, *by Mr.* JOHN BAINES, *Doncaster*.

Let x represent the rate, per hour, at which A and B travelled, y the distance gone by A before B started from H, and z the distance the geese were from H when A started; then, by the property of uniform motions, $2x - 3 : 2 :: z : \frac{2z}{2x-3}$ hours before A overtook the geese, $x : 1 :: y : \frac{y}{x}$ hours A started from H before B, and $2 : 3 :: \frac{y}{x} : \frac{3y}{2x}$ miles gone by the geese in the same time; hence the geese were $z + \frac{3y}{2x}$ miles from B when he started from H; therefore, $2x - 3 : 2 :: z + \frac{3y}{2x} : \frac{2z}{2x - 3} + \frac{3y}{2x^2 - 3x}$ hours before B overtook the geese, reckoning from the time of his leaving H, and consequently $\frac{2z}{2x - 3} + \frac{3y}{2x^2 - 3x} + \frac{y}{x}$ hours from the time of A's leaving it; but between the time of the travellers overtaking the geese, the latter had gone 5 miles; which at their rate of moving answers to $3\frac{1}{3}$ hours, hence

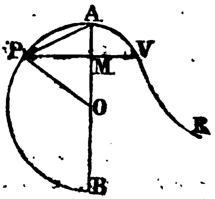
$$\frac{2z}{2x-3}+\frac{3y}{2x^2-3x}+\frac{y}{x}-\frac{2z}{2x-3}=3\tfrac{1}{3}\text{, or } 9y=30x-45.$$

Again, when A met the stage waggon he had gone $2x$ miles from the 50th mile stone, and B, at this time, was $19+y-2x$ miles from the 31st, and y miles from the waggon; hence, $4x+9 : 4 :: y : \frac{4y}{4x+9}$ hours before B met the waggon, reckoning from the time of A's meeting it; also $x : 1 :: 19+y-2x : \frac{19+y}{x}-2$ hours before B arrived at the 31st milestone; therefore $\frac{19+y}{x}-2\frac{2}{3}=\frac{4y}{4x+9}$, or $9y=10\frac{2}{3}x^2-52x-171$; consequently $10\frac{2}{3}x^2-52x-171=30x-45$, or $x=9$ miles the rate per hour and $y=25$ miles, the distance B was from L when A arrived at it.

II. QUESTION 432, *by* PALABA.

The curve AVR and semicircle APB have the ſame abſciſſa, the ordinate MV = tan ½ arc AP; prove that the area AMV = twice the segment AP.

FIRST SOLUTION, *by Mr.* ANDREW JOHNSTON, *Rector of the Academy, Montrose.*



Let arc AP $= z$, AM $= x$, and PO $= 1 =$ R, Then differential of area AMV $=\tan\frac{1}{2}z,\times dx=\tan\frac{1}{2}z\sin zdz$. Let $\frac{1}{2}z=\varphi$, then this becomes $\tan\varphi\sin 2\varphi d(2\varphi)=2\tan\varphi\sin 2\varphi d\varphi=4\tan\varphi\sin\varphi\cos\varphi d\varphi=4\sin^2\varphi d\varphi$. Wherefore area AMV $=4\int\sin^2\varphi d\varphi=4\left(\frac{\varphi}{2}-\frac{\sin\varphi\cos\varphi}{2}\right)=2\varphi-2\sin\varphi\cos\varphi=z-\sin z=2\left(\frac{1}{2}\text{R}z-\frac{1}{2}\text{R}\sin z\right)$ = 2 (sector AOP — triangle AOP) = 2 segment AP.

SECOND SOLUTION, *by Mr.* JOHN WALLACE.

Put AM $= x$, MP $= y$, the arc AP $= z$ and AO, the radius, $= r$. By the nature of the circle we have

$$y = \sqrt{(2rx - x^2)};$$

whence by differentiating we obtain

$$dy = \frac{(r - x)\,dx}{\sqrt{2rx - x^2}}.$$

Multiplying both sides of this equation by x, and adding to the right hand side of the result, and subtracting from it the same quantity;—namely $\frac{rxdx}{\sqrt{2rx - x^2}}$, we have

$$xdy = \frac{(2rx - x^2)\,dx}{\sqrt{2rx - x^2}} - \frac{rxdx}{\sqrt{2rx - x^2}},$$

or by substituting y for its value we have

$$xdy = ydx - \frac{rx}{y}\,dx.$$

If to both sides of this equation we add ydx, the left hand side of the result will evidently be the differential of xy; so that by integrating we shall find

$$xy = \int 2ydx - \int \frac{rx}{y}\,dx;$$

and by transposition

$$\int \frac{rx}{y}\,dx = \int 2ydx - xy.$$

But by the arithmetic of sines $\frac{rx}{y}$, that is, $\frac{r\,(1 - \cos z)}{\sin z}$ $= r \tan \frac{1}{2}z =$ MV, by hypothesis. Hence we have

$$\int \text{MV}dx = 2\int ydx - xy.$$

Now $\int$MVdx is the expression for the area AMV, and $2\int ydx - xy$ is the expression for twice the area of the segment AP. Wherefore

area AMV $=$ 2 segment AP.

THIRD SOLUTION, *by* PALABA, *the Proposer.*

Join PO. Let AM $= x$, MV $= y$, AO $= a$, arc AP $=$ A.

$$\text{Then } y^2 = \tan^2 \frac{\text{A}}{2} = a^2 \cdot \frac{a - \cos \text{A}}{a + \cos \text{A}} = \frac{a^2 x}{2a - x} = \frac{a^2 x^2}{2ax - x^2},$$

$$\therefore y = \frac{ax}{\sqrt{(2ax - x^2)}} \text{ and } \int y\dot{x} = a \int \frac{x\dot{x}}{\sqrt{(2ax - x^2)}} = -$$

$$a \int \frac{a\dot{x} - x\dot{x}}{\sqrt{(2ax - x^2)}} + a \int \frac{a\dot{x}}{\sqrt{(2ax - x^2)}} = -a\sqrt{(2ax - x^2)}$$

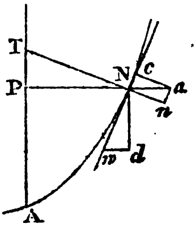
$+ a \times$ A $=$ 2 sector AOP $-$ 2 △ AOP $=$ 2 segment AP.

Ingenious Solutions were also received from Messrs. BUTTERWORTH, FUNIDUS, *and* S. H.

III. QUESTION 433, *by* PALABA.

A parabola revolves round its axis, which is vertical, in a given time, and the angular motion will just prevent a body at any point of the curve from descending: required the parameter of the parabola.

FIRST SOLUTION, *by* PALABA, *the Proposer.*



Let the body be at any point N, draw the ordinate NP and normal NT. Then the body is kept at rest by the action of three forces, gravity in the direction TP, the centrifugal force in the direction PN and the reaction of the curve in the direction NT; therefore the sides of the triangle TPN will represent the forces.

Call t'' the time, and (v) the velocity of revolution, and let $g = 32\frac{1}{6}$ feet and $p =$ parameter of the parabola.

$$\text{Then } t'' : 1'' :: 2\pi \times \text{PN} : \text{arc in } 1'' = \frac{2\pi}{t} \times \text{PN} = v,$$

∴ the centrifugal force at N, which $= \frac{v^2}{PN}, = \frac{4\pi^2}{t^2} \times PN$

$$\therefore \frac{4\pi^2}{t^2} \times PN : g :: PN : \frac{gt^2}{4\pi^2} = PT = \frac{p}{2}$$

$$\therefore p = \frac{gt^2}{2\pi^2}.$$

SECOND SOLUTION, *by Mr.* E. C. TYSON.

Let AT be the axis of the parabola AN: and let N be the situation of a body revolving round at the distance PN. Take N*a* to represent the centrifugal force and N*d*, perpendicular to the horizon, to represent the force of gravity; resolve them into N*c*, *ca*, N*w*, *wd*, of which N*c* and N*w* are the effective parts. Then when N*c* = N*w* the body is at rest.

Now let π = circumference of a circle to radius 1, m = $16\frac{1}{12}$ feet and y = PN.

Then (prin. sect. 2. p. 4 cor. 9) the arc which a body describes in a circle in any time (t) is a mean proportional between the diameter of the circle and the space fallen thro' by a body in the same time and acted upon by the same force.

∴ supposing (t) to be the periodic time in a circle.

∴ $2y : \pi y :: \pi y : \frac{\pi^2 y}{2}$ = space described by the action of the central force in (t). Now $F \times t^2 \propto S$, and here t is given ∴ $F \propto S$, and central force : gravity :: $\frac{\pi^2 y}{2} : mt^2$;

$$\therefore \text{centrifugal} = \frac{\pi^2 y}{2}.$$

Now by similar triangles

$$Na : Nc :: TN : TP, \quad TN \text{ being a normal,}$$

and $\quad Nc \text{ or } (Nw) : Nd :: NP : TN$

$\therefore \quad Na : Nd :: PN : TP :: y : TP$

or $\quad \frac{\pi^2 y}{t^2} : 2m :: y : TP = 2m \times \frac{t^2}{\pi^2}$

$\therefore \quad 2PT = 4m \times \frac{t^2}{\pi^2}$, = latus rectum of the parabola.

Other Solutions were given to this Question by Messrs. BUTTERWORTH *and* S. H.

IV. QUESTION 434, *by* PALABA.

To divide a given arc (A), less than a quadrant, into two such parts (p) and (q) that $\tan^m p \times \tan^n q$ may = max.

FIRST SOLUTION, *by Mr.* JOHN WALLACE, *R. M. College.*

Since $\tan^m p \times \tan^n q =$ max. we obtain, by taking the differential and putting it equal to 0,

$$m \tan q \frac{dp}{\cos^2 p} + n \tan p \frac{dq}{\cos^2 q} = 0.$$

But since $q = A - p$, we have $dq = -dp$: therefore, transposing, we find

$$\frac{m \tan q}{\cos^2 p} = \frac{n \tan p}{\cos^2 q};$$

hence

$$m \tan q \cos^2 q = n \tan p \cos^2 p.$$

By the Arithmetic of sines, a being any arc, $\tan a = \frac{1 - \cos 2a}{\sin 2a}$, and $\cos^2 a = \frac{1}{2}(1 + \cos 2a)$; so that we find $\tan a \cos^2 a = \frac{1}{2} \sin 2a$.

Hence, by substitution, the above equation becomes

$$m \sin 2q = n \sin 2p.$$

Now, put $p - q = B$, then shall we have $2p = A + B$ and $2q = A - B$.

Substituting therefore these values for $2p$ and $2q$, we obtain

$$m \sin (A - B) = n \sin (A + B).$$

Hence we derive

$$\frac{\tan A + \tan B}{\tan A - \tan B} \left(= \frac{\sin (A + B)}{\sin (A - B)}\right) = \frac{m}{n}.$$

From this equation we obtain

$$\tan B = \frac{m - n}{m + n} \tan A.$$

The difference of the arcs p and q being thus determined and their sum being given, the arcs themselves become known.

SECOND SOLUTION, *by Mr.* ANDREW JOHNSON.

Since $\tan^m p \times \tan^n q$ = maximum, we have

$$m \tan^{m-1} p \times \tan^n q \times \frac{dp}{\cos^2 p} + n \tan^{n-1} q \times \tan^m p \times \frac{dq}{\cos^2 q} = 0,$$

But evidently $dq = -dp$, wherefore, reducing,

$$m \tan q \cos^2 q = n \tan p \cos^2 p, \text{ or}$$

$$m \sin q \cos q = n \sin p \cos p, \text{ whence}$$

$$2 \sin p \cos p : 2 \sin q \cos q :: m : n, \text{ or}$$

$$\sin 2p : \sin 2q :: m : n,$$

$$\therefore \sin 2p + \sin 2q : \sin 2p - \sin 2q :: m + n : m - n.$$

But $\sin 2p + \sin 2q : \sin 2p - \sin 2q$

$:: \tan (p + q) : \tan (p - q) :: \tan A : \tan (p - q):$ whence $m + n : m - n :: \tan A : \tan (p - q)$, and the two parts are thus found easily.

THIRD SOLUTION, *by* S. H.

Since $\tan^m p \times \tan^n q$ is a maximum,

$$\therefore m \tan^{m-1} p \times \sec^2 p \times \tan^n q + n \tan^{n-1} q \times \sec^2 q \times \tan^m p \, \frac{dq}{dp} = 0.$$

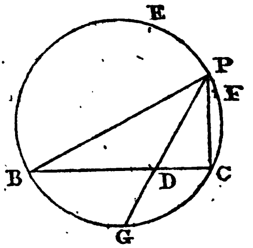
Also, since $p + q = A \therefore 1 + \frac{dq}{dp} = 0, \therefore \frac{dq}{dp} = -1,$

$$\therefore m \times \sec^2 p \times \tan q = n \times \sec^2 q \times \tan p,$$

$$\therefore m \times \sin q \times \cos q = n \times \sin p \times \cos p$$

$$\therefore m \times \sin 2q = n \times \sin 2p:$$

Whence the following construction.



In a circle whose radius is equal to that of the given arc A, take an arc BPC = 4A; draw BC and divide it in D making BD : DC :: m : n. From the middle point G of the arc opposite to BPC draw GD, meeting BPC in P. Take PE a fourth part of arc PB, and PF a fourth part of arc PC, ∴ arc EF = one fourth of arc BC = A, and PE = p, and PF = q.

For, since BG = GC, ∴ GP bisects ∠ BPC, ∴ BP : PC :: BD

$: DC :: m : n$. But $\sin 2PE : \sin 2PF :: \text{chord } 4PE : \text{chord } 4PF :: BP : PC :: m : n$, $\therefore m \times \sin 2PF = n \times \sin 2PE$ and $PF + PE = A$. And A is so divided into two parts p and q that $PE = p$ and $PF = q$, and therefore

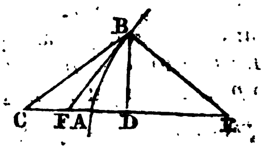
$$m \times \sin 2q = n \times \sin 2p.$$

Thus very nearly, was the question answered by Messrs. **Baines, Butterworth, A. B. L., Palaba,** *and* **Tyson.**

V. QUESTION 435, *by* G. V.

A normal at any point of an equilateral hyperbola is equal to the distance of that point from the centre. Required the demonstration.

First Solution, *by Mr.* John Butterworth.

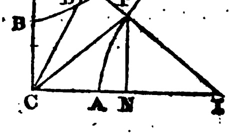


Let AB be an equilateral hyperbola, whose centre is C and vertex A; draw the normal BE, and ordinate BD; join CB, and draw the tangent BF meeting the axis in F. Then, by the property of the hyperbola, $BD^2 = CD^2 - CA^2$ and $CA^2 = CF \times CD$, therefore, $BD^2 = CD^2 - CF \times CD = CD \times DF$: Again, $BD^2 = DE \times DF$, $\therefore CD \times DF = DE \times DF$, or $CD = DE$; consequently $CD^2 + DB^2 = DE^2 + DB^2$, or $CB^2 = BE^2$, and $CB = BE$.

Second Solution, *by Mr.* Cunliffe.

By a well-known property of the hyperbola,
$\text{Trans}^2 : \text{conj}^2 :: CD : DE$; but in the equilateral hyperbola the transverse and conjugate axes are equal to one another, therefore $CD = DE$; consequently the two right-angled triangle CBD, EBD are equal in every respect; therefore $CB = BE$.

Third Solution, *by* A. B. L.



Let PC, DC be semi-conjugate diameters of the equilateral hyperbola whose semi-axes are AC, BC; and IP be drawn at right angles to a tangent at P and produced to meet the semi-diameter DC in F. Then, by the writers on conic sections, $IP \times PF = BC^2$, $CD \times PF = AC \times BC$ and $CP^2 - CD^2 = AC^2 - BC^2$; but in the equilateral hyperbola $AC = BC$, therefore $IP \times PF = AC^2$, $DC \times PF = AC^2$, and $PC^2 - CD^2 = 0$, or $CP = CD$; whence $IP \times PF = CD \times PF = CP \times PF$ or $IP = CP$.

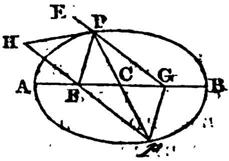
Solutions were also received from Messrs. **Baines, Johnson,** *and* **S. H.**

VI. QUESTION 436, *by* G. V.

C is the centre of an ellipse, and F either focus, PH a tangent at P. Draw the diameter PCp and pH meeting PH in H. Prove that pH is equal to the transverse diameter.

Solution, *by* Funidus.

Let G be the other focus, draw GPE, and join Gp. Then since FG is bisected in C, and Pp also bisected in C, the figure $PFpG$ is a parallelogram. Now $PF + PG = AB$, or $FP + Fp = AB$. But since the tangent by the nature of the ellipse bisects the angle EPF, we have $\angle EPH = \angle HPF$, and since GE and pH are parallel, the angle $EPH = pHF = HPF$, therefore $PF = FH$, and therefore $FH + Fp = AB$ or $pH = AB$.



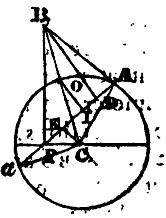
This question was likewise answered nearly in the same manner by Messrs. **Butterworth, Johnson, A. B. L., S. H.** *and* **J. Wallace.**

VII. QUESTION 437, *by* G. V.

P is any point in the diameter of a circle, and PB perpendicular to that diameter; draw any line PA, and a tangent AB meeting PB in B, and draw BD and CE (C the centre) perpendicular to PA; then PE = DA. Required a demonstration.

First Solution, *by Mr.* Butterworth.

Join CA and CB, and from O, the middle of CB, demit upon PA the perpendicular OI. Then the angles CPB, CAB being right angles, the points P, B, A, C are in the circumference of a circle, of which the diameter is CB. Again, O being the centre of the circle and PA a chord, I will be the middle of PA, and consequently $EI = ID$, therefore $PE = DA$.



Thus the question was answered by Mr. **John Wallace.**

SECOND SOLUTION, *by Mr.* E. C. TYSON.

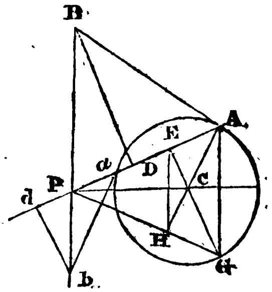
Produce AP to *a* and join *a*C, AC (*preceding fig.*). The angle BAD is equal to the angle in the alternate segment, or equal to half the angle *a*CA or equal to ACE, and the angles at E and D are right angles; therefore the triangles BDA, AEC are similar. Again, the angle BPD + BPC = EPC + ECP, ∴ BPD = ECP and the triangles BDP, CEP are similar:

Hence PE : EC :: DB : DP, and CE : EA :: AD : DB; ∴ PE : EA :: AD : DP, and PE + EA : EA :: AD + DP : DP, or PA : EA :: PA : DP, ∴ EA = DP, and PE = DA.

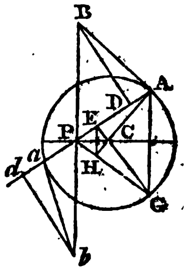
And thus nearly the question was answered by S. H. *Other demonstrations were received from Messrs.* BAINES, FUNIDUS, *and* JOHNSON.

THIRD SOLUTION, *by* Mr. LOWRY, *R. M. College.*

Draw HP perpendicular to the radius AC produced, and let it meet EC produced in G: Join AG and EH. Because BA is a tangent at A, it is at right angles to AH, and PG is also at right angles to AH by construction; therefore AB is parallel to PG.



Again, because of the right angles at H and E, the points P, E, C, and H are in the circumference of a circle, and so are also the points H, E, A, and G, wherefore the angles EPC, EHC, and EGA are equal to one another; but the sum of the two angles BPA and EPC, and also the sum of the two, EAG and EGA, is a right angle; and it has been shewn that EPC is equal to EGA, therefore BPA is equal to EAG, and AG parallel to BP. Therefore the figure PBAG is a parallelogram, and consequently PG = AB. The two right-angled triangles GPE and ADB are therefore equal in all respects; hence PE = AD.



In the same way it may be shewn that, if the tangent *ab* be drawn at the other end of the chord AP*a* to meet BP in b, and bd be drawn perpendicular to Aa produced, ad shall be = PE; and consequently ad = DA.

Again, because ad = PE, if aP be added to each, we have dP = aE = AE = PD; therefore the right angled triangles Pbd, PBD are equal in every respect: Wherefore Pb = PB, and bd = BD. Also the tangent ab = the tangent AB.

From these properties we may infer that,

If from any point P in the diameter of a circle, or in the diameter produced, a straight line BPb be drawn perpendicular to the diameter, and through the point P any chord aPA be drawn, and the tangents at its extremities produced to meet the perpendicular in B and b: the segments BP, bP, adjacent to the given point P, shall be equal to one another; and the tangent AB equal to the tangent ab.

Also, if the perpendiculars BD and bd be drawn from B and b, upon the chord Aa, these perpendiculars shall be equal to one another; the segment AD equal to the segment ad; and PD equal to Pd.

It is evident that any of these propositions may be enunciated in the porismatic form.

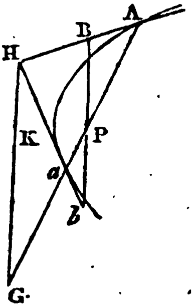
The first proposition, for instance, may be enunciated thus:

A straight line and a circle being given by position, a point in that line may be found, such, that if any straight line whatever be drawn through it to meet the circle in two points, and tangents be drawn to the circle at these points and produced to meet the straight line given by position, they shall cut off equal segments from that line, adjacent to the point found; or (taking no notice of these segments) the intercepted parts of the tangents shall be equal to one another.

From what has been done we are led, by analogy, to the discovery of some pretty enough properties of the conic sections, and of similar properties when the figure is described on the surface of a sphere. I shall here only notice the following properties of the conic sections.

If a straight line BPb be drawn at any point P in the diameter of a conic section, or the diameter produced, parallel to a tangent at the extremity of that diameter: and through that point any straight line be drawn to meet the section in two points, the tangents, drawn to the curve at these points shall cut off equal segments PB, Pb from the straight line BPb.

For, let *a*PA be any straight line drawn through P, and let the tangents AB and *ab* be produced to meet in H; and draw HK parallel to BP*b* meeting *a*A in G. By similar triangles



PA : AG :: PB : HG

P*a* : *a*G :: P*b* : HG.

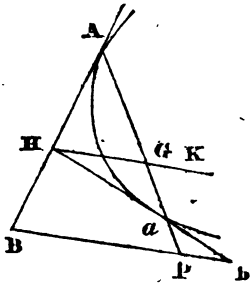
But it is a well-known property of the conic sections that the line PA is harmonically divided in the points P, *a*, G, and A; therefore

PA : AG :: P*a* : *a*G,

And by equality of ratios

PB : HG :: P*b* : HG,

therefore PB = P*b*.

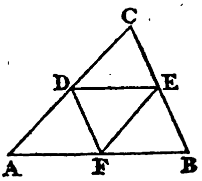


VIII. QUESTION 438, *by* PROTEUS.

Within a given triangle, suppose another triangle to be inscribed, by joining the middle points of its sides; and again within this triangle, suppose another triangle to be inscribed by joining the middle points of its sides, and so on *ad infinitum;* What will be the limit of the aggregate of the sum of the squares of the sides of all the triangles so formed?

SOLUTION, *by Mr.* BUTTERWORTH.

Let ACB be a given triangle, and DEF a triangle formed by joining the middle points of its sides. Now it is pretty evident that each of the sides of the inner triangle will be equal to one half of the side to which it is parallel, consequently the sum of the squares of its sides will be equal to one fourth of the sum of the squares of the sides of the original triangle. If another triangle be formed by joining the middle points of the sides of the triangle DEF, the



sum of the squares of its sides will be equal to one fourth of the sum of the squares of the sides of the triangle DEF; therefore the sum of the squares of the sides of any of the triangles is equal to one fourth of the sum of the squares of the sides of the preceding triangle. Whence the sums are in a geometrical progression, whose first term is one fourth of the sum of the squares of the sides of the triangle ACB, and ratio 4; consequently the aggregate of the sum of the squares of the sides of all the triangles, that can be formed in the way specified in the question, is one third of the sum of the squares of the sides of the original triangle ACB.

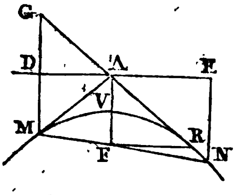
Similar answers were received from Messrs. BAINES, CUNLIFFE, *and* S. H.

IX. QUESTION 439, *by Mr.* CUNLIFFE.

Let any right line be drawn through the focus of a given conic section, terminating in the curve; then a fourth proportional to the whole line and the two segments thereof, made by the focus, will always be of the same constant length; required a demonstration.

FIRST SOLUTION, *by Mr.* CUNLIFFE.

Let the curve MVN be a conic section, FA its axis passing through the vertex V, and meeting the directrix DE in A. Also let MN be any right line drawn through the focus F terminating in the curve, in M and N; draw MD, NE perpendicular to DE; also draw the lines MA, NA.



MD, FA, and NE are parallels, therefore MF : FN :: DA : AE; and by the property of the curve MF : FN :: MD : NE; therefore DA : AE :: MD : NE, whence it follows that the right-angled triangles ADM and AEN are similar, and therefore the angle EAN = DAM.

Produce NA to meet MD in G; then since ∠ EAN = ∠ DAG = ∠ DAM, the right-angled triangles DAG and DAM are equal and similar, and therefore DG = MD, or MG = 2MD.

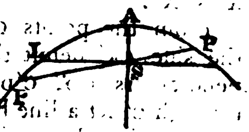
Again, by reason of the parallels FA, MG; NM : NF :: MG = 2MD : FA :: MD : $\frac{1}{2}$FA, and hence NM × $\frac{1}{2}$FA = MD × FN.

Draw FR perpendicular to FA, and meeting the curve in R;

then, by the known property of the directrix, MD : FM :: FA : FR :: $\frac{1}{2}$FA : $\frac{1}{2}$FR, and hence MD × FN : FM × FN :: NM × $\frac{1}{2}$FA : NM × $\frac{1}{2}$FR; now we have proved that MD × FN = NM × $\frac{1}{2}$FA, consequently FM × FN = NM × $\frac{1}{2}$FR, and hence NM : FN :: FM : $\frac{1}{2}$FR: but FR in a given conic section is given, therefore the property stated in the question has been demonstrated.

SECOND SOLUTION, *by Mr.* JOHN BAINES.

Let PLA*p* be any given conic section, S the focus, PS*p* a right line drawn through it, terminating in the curve at P and *p*, and SL half the *Latus Rectum*.



It is proved by writers on Conics, (vid. Peacock's Conic Sections, Prop. v. of the Parabola, &c.) that 2SP × S*p* = SL × P*p*, which turned into an analogy is P*p* : S*p* :: 2SP : SL, or P*p* : S*p* :: SP : $\frac{1}{2}$SL; but $\frac{1}{2}$SL is = $\frac{1}{4}$ of the *Latus Rectum*, and is therefore a constant quantity.

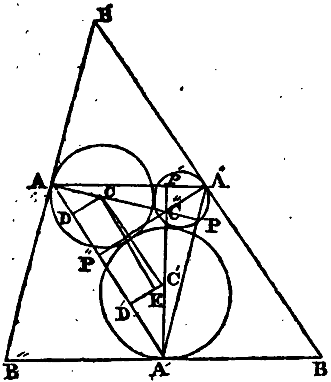
Messrs. JOHNSON *and* S. H. *answered this question.*

X. QUESTION 440, *by* ———

A triangle being given, it is required to describe three circles, so that each circle shall touch the other two and a side of the triangle at the point of bisection.

SOLUTION, *by Mr.* J. B. DURRANDE, *taken from the Annales de Mathématiques.*

Let B, B′, B″ be any triangle, and A, A′, A″ the points where the sides are bisected, we propose to describe three circles which shall touch each other two and two, and also the sides of the triangle in the points A, A′, A″. Join these points, and draw AP, A′P′, A″P″ perpendicular to the sides of the triangle B, B′, B″; they will also be perpendicular to the sides of the triangle A A′ A″, because the sides of this triangle are respectively parallel to the sides of the other. The centres of the required circles will also be in these perpendiculars; let C, C′, C″ be the centres, r, r', r'' their unknown radii, and a, a', a'' the three sides of the triangle A A′ A″.



From the points C, C′ on the side AA′ demit the perpendiculars CD, C′D′; and through C′ let a line be drawn parallel to AA′ meeting CD in E; we shall then have $\mathrm{C'E} = \mathrm{DD'} = \mathrm{AA'} - (\mathrm{AD} + \mathrm{A'D'})$:

$$\text{but } \mathrm{C'E} = \sqrt{(\mathrm{CC'}^2 - \mathrm{CE}^2)} = \sqrt{(\mathrm{CC'}^2 - (\mathrm{CD} - \mathrm{C'D'})^2)};$$

therefore

$$\sqrt{(\mathrm{CC'}^2 - (\mathrm{CD} - \mathrm{C'D'})^2)} = \mathrm{AA'} - (\mathrm{AD} + \mathrm{A'D'});$$

whence, by squaring and remarking that

$$\mathrm{AD}^2 + \mathrm{CD}^2 = \mathrm{AC}^2, \qquad \mathrm{A'D'}^2 + \mathrm{C'D'}^2 = \mathrm{A'C'}^2,$$

and transposing

$$\left.\begin{array}{l} \mathrm{CC'}^2 - \mathrm{AC}^2 - \mathrm{A'C'}^2 + 2(\mathrm{CD}\cdot\mathrm{C'D'} - \mathrm{AD}\cdot\mathrm{A'D'}) \\ \qquad + 2\mathrm{AA'}(\mathrm{AD} + \mathrm{A'D'}) - \mathrm{AA'}^2 \end{array}\right\} = 0.$$

But we have

$$\mathrm{CC'} = r + r', \quad \mathrm{AC} = r, \quad \mathrm{A'C'} = r', \quad \mathrm{AA'} = a'',$$

$$\mathrm{CD} = \mathrm{AC} \sin \mathrm{CAD} = r \cos \mathrm{A'}; \quad \mathrm{C'D'} = \mathrm{A'C'} \sin \mathrm{C'A'D'} = r' \cos \mathrm{A},$$

$$\mathrm{AD} = \mathrm{AC} \cos \mathrm{CAD} = r \sin \mathrm{A'}; \quad \mathrm{A'D'} = \mathrm{A'C'} \cos \mathrm{C'A'D'} = r \sin \mathrm{A},$$

whence

$$\begin{aligned} \mathrm{CD}\cdot\mathrm{C'D'} - \mathrm{AD}\cdot\mathrm{A'D'} &= rr'(\cos \mathrm{A} \cos \mathrm{A'} - \sin \mathrm{A} \sin \mathrm{A'}) \\ &= rr' \cos(\mathrm{A} + \mathrm{A'}) = -rr' \cos \mathrm{A''}; \end{aligned}$$

and by substituting and remarking that $1 - \cos \mathrm{A''} = 2 \sin^2 \tfrac{1}{2}\mathrm{A''}$, we have

$$4rr' \sin^2 \tfrac{1}{2}\mathrm{A''} + 2a''(r \sin \mathrm{A'} + r' \sin \mathrm{A}) - a''^2 = 0;$$

But, putting T for the area of the triangle, we have

$$\sin A = \frac{2T}{a'a''}; \quad \sin A' = \frac{2T}{aa''};$$

$$\sin^2 \tfrac{1}{2}A'' = \frac{4T^2}{aa(a+a'+a'')(a+a'-a'')}$$

consequently the above equation becomes

$$\frac{4Tr}{a} , \frac{4Tr'}{a'} + (a+a'+a'')(a+a'-a'')\left\{\frac{4Tr}{a} + \frac{4Tr'}{a'}\right\}$$
$$-a''^2(a+a'+a'')(a+a'-a'') = 0;$$

and as the three points C, C′, C″, compared two and two, must give analogous results, it follows by supposing

$$\frac{4Tr}{a} = x, \quad \frac{4Tr'}{a'} = x', \quad \frac{4Tr''}{a''} = x''$$

that x, x', x'' will be given by the equations

$$\left.\begin{array}{l} xx' + (a+a'+a'')(a+a'-a'')(x+x') \\ \quad -a''^2(a+a'+a'')(a+a'-a'') \end{array}\right\} = 0$$

$$\left.\begin{array}{l} x'x'' + (a+a'+a'')(a'+a''-a)(x'+x'') \\ \quad -a^2(a+a'+a'')(a'+a''-a) \end{array}\right\} = 0$$

$$\left.\begin{array}{l} x''x + (a+a'+a'')(a''+a-a')(x''+x) \\ \quad -a'^2(a+a'+a'')(a''+a-a') \end{array}\right\} = 0.$$

Putting therefore, for the sake of abridging, $2s$ for the sum of the three sides of the triangle AA′A″, these equations become

$$xx' + 4s(s-a'')(x+x'-a''^2) = 0,$$
$$x'x'' + 4s(s-a)(x'+x''-a^2) = 0,$$
$$x''x + 4s(s-a')(x''+x-a'^2) = 0.$$

If the values of x' and x'', drawn from the extreme equations, be substituted in the intermediate equation, it is clear that it will then include only one unknown quantity x, raised to the second power, from whence the value of x may be easily deduced, and consequently those of x' and x'', and also those of r, r', r'' by means of the above relations.

XI. QUESTION 441, by ———.

Required the curve that has at each point the radius of curvature a fourth proportional to the abscissa, the ordinate and a given straight line.

SOLUTION, *by* α. β.

Let x be the abscissa, y the ordinate, a the given line, and R the radius of curvature:
then, by the question,

$$x : y :: a : R,$$

$$\text{or } R = \frac{ay}{x}.$$

But R is $= \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{-dx\, d^2y}$, where dx is constant;

$$\text{therefore } \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{-dx\, d^2y} = \frac{ay}{x},$$

the fluxionary equation of the curve; but this equation, I rather think, has never been integrated. The question was proposed in *Tome* I. of the Annales de Mathematiques, but no solution to it has appeared in any of the succeeding *Tomes* of that valuable work, owing probably to the difficulty of the integration.

Euler and some other authors, have resolved several problems of a similar kind, a few of which I shall subjoin. They may perhaps be acceptable to those readers of the Repository who have not the opportunity of consulting the original works. The perusal of them may also perhaps suggest some method of integrating the above equation.

I. To find the nature of the curve in which the radius of curvature at any point is equal to a given function of the corresponding abscissa.

That is, let $\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{-dx\, d^2y} = f(x)$, some function of x.

Put $dy = p\,dx$, then $\frac{-(1 + p^2)^{\frac{3}{2}} dx}{dp} = f(x)$ or $\frac{dp}{-(1 + p^2)^{\frac{3}{2}}}$

$= \frac{dx}{f(x)}$, and $\frac{-p}{\sqrt{(1 + p^2)}} = \int \frac{dx}{f(x)}$.

Let $\int \frac{dx}{f(x)} = v$, then $p = v\sqrt{(1 + p^2)}$ and $p = \frac{v}{\sqrt{(1 - v)}}$ a function of x; whence

$$dy = p\,dx = \frac{v\,dx}{\sqrt{(1 - v^2)}}$$

and $y = \int \frac{vdx}{\sqrt{(1 - v^2)}}$.

Ex. If the radius be reciprocally proportional to the abscissa.

Assume $f(x) = \frac{a^2}{2x}$, then $v = \int \frac{2xdx}{a^2} = \frac{x^2 + ab}{a^2}$:

hence $y = \int \frac{(x^2 + ab)\,dx}{\sqrt{\{a^4 - (x^2 + ab)^2\}}}$, which is the equation to the elastic curve.

II. When the radius of curvature at any point is equal to a given function of the corresponding ordinate:

That is $\frac{dx^2 + dy^2}{-dxdy^2} = f(y)$, some function of y.

Suppose $dx = udy$, which gives $\frac{(1 + u^2)^{\frac{3}{2}} dy}{du} = (fy)$, or

$\frac{du}{(1 + u^2)^{\frac{3}{2}}} = \frac{dy}{f(y)}$, and $\frac{u}{\sqrt{(1 + u^2)}} = \int \frac{dy}{f(y)}$.

Let $\int \frac{dy}{f(y)} = w$, then $\frac{u}{\sqrt{(1 + u^2)}} = w$, and consequently $u = \frac{w}{\sqrt{(1 - w^2)}}$, a function of y: whence $dx = udy = \frac{wdy}{\sqrt{(1 - w^2)}}$

and $x = \int \frac{wdy}{\sqrt{(1 - w^2)}}$.

III. To find the equation of the curve in which the radius of curvature at any point is proportional to the angle which the tangent to the curve at that point makes with the abscissa.

That is when $\frac{dy\sqrt{(dx^2 + dy^2)}}{d^2x} = a \cdot \text{arc}\left(\tan \frac{dy}{dx}\right)$, where $\sqrt{(dx^2 + dy^2)}$ is constant.

Put $dy = pdx$, then $\sqrt{(dx^2 + dy^2)} = dx\sqrt{(1 + p^2)}$, and by differentiating, supposing p and dx both to vary, there results

$$d^2x\sqrt{(1 + p^2)} + \frac{pdxdp}{\sqrt{(1 + p^2)}} = 0$$

which gives $d^2x = \frac{-pdx\,dy}{1 + p^2}$, and the proposed equation then

becomes $\frac{-dx(1 + p^2)^{\frac{3}{2}}}{dp} = a \cdot \text{arc}\,(\text{tang} = p)$.

Substituting the value of dx found from this equation in $dy = pdx$, we have

$$dx = \frac{-adp}{(1+p^2)^{\frac{3}{2}}} \text{ arc (tang} = p)$$

$$dy = \frac{-apdp}{(1+p^2)^{\frac{3}{2}}} \text{ arc (tang} = p)$$

and by integration

$$x = \frac{-a}{\sqrt{(1+p^2)}} - \frac{ap}{\sqrt{(1+p^2)}} \text{ arc (tang} = p) + c$$

$$y = \frac{-ap}{\sqrt{(1+p^2)}} + \frac{a}{\sqrt{(1+p^2)}} \text{ arc (tang} = p) + c'.$$

IV. To find the curve in which the radius of curvature is a fourth proportional to the fluxions of the ordinate and abscissa, and a given line.

That is, when $\frac{dy\sqrt{(dx^2+dy^2)}}{d^2x} = a \times \frac{\dot{x}}{\dot{y}}$,

By making the same substitution, and proceeding as in the last example, the given equation reduces to

$$\frac{-dx\,(1+p^2)^{\frac{3}{2}}}{dp} = \frac{a}{p},$$

therefore by substituting in the equation

$$dy = pdx,$$

we get

$$dx = \frac{-adp}{p\,(1+p^2)^{\frac{3}{2}}}:$$

$$dy = \frac{-adp}{(1+p^2)^{\frac{3}{2}}}.$$

Hence, by integrating,

$$x = c - \frac{a}{\sqrt{(1+p^2)}} + a \log. \left(\frac{1+\sqrt{(1+p^2)}}{p}\right)$$

$$y = c' - \frac{ap}{\sqrt{(1+p^2)}}.$$

V. To find the equation of the curve in which the radius of curvature at any point is proportional to the distance of that point from the origin of the co-ordinates.

That is, when $\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dxd^2y} = n\sqrt{(x^2 + y^2)}$.

Assume $dy = pdx$, $dp = qdx$, $y = ux$, $qx = z$; then $dy = xdu + udx$, or $pdx = xdu + udx$, and $\frac{du}{p - u} = \frac{dx}{x}$, also $\frac{dx}{x} = \frac{dp}{z}$, therefore $\frac{dp}{z} = \frac{du}{p - u}$.

By substitution, the given equation becomes

$$(1 + p^2)^{\frac{3}{2}} = nz\sqrt{(1 + u^2)},$$

and putting for z its value found from the equation $\frac{dp}{z} = \frac{du}{p - u}$, we have

$$n(p - u)\,dp\sqrt{(1 + u^2)} = (1 + p^2)^{\frac{3}{2}}\,du$$

or

$$\frac{n(p - u)}{\sqrt{(1 + p^2)}} \cdot \frac{dp}{(1 + p^2)} = \sqrt{(1 + u^2)} \cdot \frac{du}{(1 + u^2)} \quad \ldots.(a)$$

in which equation we remark that $\frac{dp}{1 + p^2}$ and $\frac{du}{1 + u^2}$ are the differentials of two arcs the tangents of which are p and u, we may therefore suppose $p = \tan b$, $u = \tan \beta$; this hypothesis gives

$$\sqrt{(1 + p^2)} = \frac{1}{\cos b}, \quad \sqrt{(1 + u^2)} = \frac{1}{\cos \beta}$$

$$p - u = \frac{\sin b \cos \beta - \cos b \sin \beta}{\cos b \cos \beta} = \frac{\sin (b - \beta)}{\cos b \cos \beta},$$

and the equation (a) becomes

$$ndb \sin (b - \beta) = d\beta,$$

which, under the hypothesis $b - \beta = \varphi$, is transformed into

$$ndb \sin \varphi = db - d\varphi,$$

whence, we draw

$$db = \frac{d\varphi}{1 - n \sin\varphi}, \quad d\beta = \frac{d\varphi}{1 - n \sin \varphi} - d\varphi = \frac{n \sin \varphi\, d\varphi}{1 - n \sin \varphi},$$

therefore

$$\frac{dx}{x} = \frac{du}{p - a} = \frac{\cos b \cos \beta\, d(\tan \beta)}{\sin (b - \beta)} = \frac{d\beta \cos b}{\cos \beta \sin \varphi},$$

or putting again $\cos\varphi\cos\beta - \sin\varphi\sin\beta$ for $\cos b = \cos(\beta+\varphi)$, we find

$$\frac{dx}{x} = \frac{nd\varphi\cos.\varphi}{1-n\sin\varphi} - \frac{d\beta\sin\beta}{\cos\beta}.$$

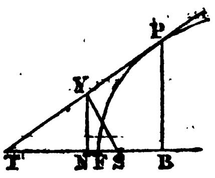
The integral of this equation is $x = \frac{c\cos\beta}{1-n\sin\varphi}$, and since $y = ux = x\,\text{tang}\,\beta = \frac{x\sin\beta}{\cos\beta}$ we have $y = \frac{c\sin\beta}{1-n\sin\varphi}$. These values are complete because the arc β, determined from the equation $\beta = -\varphi + \int\frac{d\varphi}{1-n\sin\varphi}$, includes only one constant arbitrary quantity.

If we put in place of β its value $b-\varphi$ we have $x = \frac{c\cos(b-\varphi)}{1-n\sin\varphi}$, $y = \frac{c\sin(b-\varphi)}{1-n\sin\varphi}$, $b = \int\frac{d\varphi}{1-n\sin\varphi}$: the integral of the last member of this equation being found, we may take away b and φ from the above expressions for x and y, and we shall then have their values expressed by means of a single arc.

XII. QUESTION 442, *by* PALABA.

To determine the nature of the curve, such, that the perpendicular from a given point upon the tangent shall be a mean proportional between a given line and the segment of the axis intercepted between the tangent and this same given point.

FIRST SOLUTION, *by* PALABA, *the Proposer.*



Let TB $= x$, BP $= y$, $z =$ arc of curve.

Then TP $= \frac{y\dot z}{\dot y}$ and from sim. Δs,

$\frac{y\dot z}{\dot y} : y :: \frac{y\dot x}{\dot y} - x :$ SY $= \frac{y\dot x - x\dot y}{\dot z}$,

$\therefore \frac{(y\dot x - x\dot y)^2}{\dot z^2} = a\,.\,\frac{y\dot x - x\dot y}{\dot y}$, by the quest.

or $y\dot x - x\dot y = \frac{a}{\dot y}\,.\,\dot z^2 = a\dot y + \frac{a\dot x^2}{\dot y}$.

Take the fluxions making $\dot{y}$ constant

$$\therefore \dot{y}\dot{x} + y\ddot{x} - \dot{y}\dot{x} = \frac{2a\dot{x}\ddot{x}}{\dot{y}}$$

$$\therefore y\ddot{x} = \frac{2a\dot{x}\ddot{x}}{\dot{y}} \text{ or } \left\{y - \frac{2a\dot{x}}{\dot{y}}\right\}\ddot{x} = 0$$

$$\therefore y - \frac{2a\dot{x}}{\dot{y}} = 0 \text{ and } \ddot{x} = 0.$$

Because $y - \frac{2a\dot{x}}{\dot{y}} = 0$, $\frac{\dot{x}}{\dot{y}} = \frac{y}{2a}$ and substituting this value of $\frac{\dot{x}}{\dot{y}}$ in the equation

$$y = (a + x)\frac{\dot{y}}{\dot{x}} + \frac{a\dot{x}}{\dot{y}}$$

we have

$$y = (a + x) \cdot \frac{2a}{y} + \frac{y}{2} \therefore \frac{y}{2} = \frac{2a \cdot (a + x)}{y}$$

and $y^2 = 4a \cdot (a + x)$: hence the curve is a parabola whose focus is S and parameter $(4a)$.

Again, because $\ddot{x} = 0$, $\dot{x} = c\dot{y}$ $\therefore \frac{\dot{x}}{\dot{y}} = c$

$\therefore y = \frac{a + x}{c} + ac$ the equation to a straight line, as it ought to be.

SECOND SOLUTION, *by Mr.* JOHN BUTTERWORTH.

Let S be the given point in the axis FB, and FP the curve; from any point, as P, of the curve draw the tangent TP meeting the axis produced in T, and draw the ordinate PB and demit SY perpendicular to TP in Y. Put SB = x, BP = y, and a = the given line.

Then we shall have $\dot{y} : \dot{x} :: \text{BP} = y : \text{BT} = \frac{y\dot{x}}{\dot{y}}$,

therefore $\text{ST} = \frac{y\dot{x}}{\dot{y}} - y$. Again, by similar triangles,

$\text{TP}^2 = \frac{y^2\dot{x}^2}{\dot{y}^2} + y^2 : \text{BP}^2 = y^2 :: \text{ST}^2 = \left(\frac{y\dot{x}}{\dot{y}} - x\right)^2 : \text{SY}^2 = \frac{y^2\dot{x}^2 - 2yx\dot{x}\dot{y} + x^2\dot{y}^2}{\dot{x}^2 + \dot{y}^2}$. Whence, by the question,

$$\frac{y^2\dot{x}^2 - 2yx\dot{x}\dot{y} + x^2\dot{y}^2}{\dot{x}^2 + \dot{y}^2} = \frac{ay\dot{x}}{\dot{y}} - ax = \frac{ay\dot{x} - ax\dot{y}}{\dot{y}}$$, which divided by $y\dot{x} - x\dot{y}$ gives $\frac{y\dot{x} - x\dot{y}}{\dot{x}^2 + \dot{y}^2} = \frac{a}{\dot{y}}$,

therefore $y\dot{x}\dot{y} - x\dot{y}^2 = a\dot{x}^2 + a\dot{y}^2$,

or $a + x - y\frac{\dot{x}}{\dot{y}} + a\frac{\dot{x}^2}{\dot{y}^2} = 0$.

By substituting $a + x = \text{P}$, $-y = \text{Q}$, and $a = \text{S}$ in the formula at page 145, No. 12, of the Repository, we shall have

$$4a^2 + 4ax - y^2 = 0,\ \text{or}\ 4a\,(a + x) = y^2,$$

which is an equation to the parabola, whose parameter is $4a$.

THIRD SOLUTION, *by Mr.* LOWRY, *R. M. College.*

Let $x =$ the abscissa of the required curve, reckoned from the given point, $y =$ the corresponding ordinate, $v =$ the distance between the given point and the intersection of the tangent and axis, and $a =$ the given line.

Then $v + x =$ the subtangent,

$(v + x)^2 + y^2 =$ the square of the tangent;

and, by similar triangles,

$(v + x)^2 + y^2 : y^2 :: v^2 :$ the square of the perpendicular from the given point upon the tangent, which, by the question, is $= av$;

Therefore $a(v + x)^2 + ay^2 = vy^2$ ....(1).

Taking the fluxion of this equation, supposing v only to vary, and dividing by $\dot{v}$, we have

$$2av + 2ax = y^2$$

or $v = \frac{y^2 - 2ax}{2a}$ and $v + x = \frac{y^2}{2a}$.

These values being substituted instead of v and $v + x$ in equation (1), it becomes

$$\frac{y^4}{4a} + ay^2 = \frac{y^2 - 2ax}{2a} \times y^2$$

or $4a\,(a + x) = y^2$,

an equation to the common parabola, which is therefore the curve required.

FOURTH SOLUTION, *by* S. H.

Let S (*see the preceding fig.*) be the given point, SY a perpendicular on a line PT touching the curve in P, and meeting the axis in T. Draw PB, YN perpendicular to the axis.

Since TS : SY :: SY : SN ∴ SN is = to the given line.

Let SN $= a$, NB $= x$, PB $= y$.

Now TS : SY :: TP : PB and SY : SN :: TP : BP

∴ TS : SN :: TP² : BP²,

i. e. $y\frac{dx}{dy}+a-x : a :: dx^2+dy^2 : dy^2$;

therefore by reduction

$$\frac{dx^2}{dy^2}-\frac{y}{a}\cdot\frac{dx}{dy}+\frac{y^2}{4a^2}=\frac{y^2-4ax}{4a^2}$$

$$\therefore \frac{dx}{dy}-\frac{y}{2a}=\pm\frac{\sqrt{(y^2-4ax)}}{2a}$$

$$\therefore 2adx-y\,dy=\pm\sqrt{(y^2-4ax)}\,.\,dy$$

$$\therefore dy=\mp\frac{ydy-2adx}{\sqrt{(y^2-4ax)}}=\mp d\sqrt{(y^2-4ax)}$$

$$\therefore c+y=\mp\sqrt{(y^2-4ax)}$$

(c being some quantity which must disappear by differentiation)

$$\therefore c^2+2cy+y^2=y^2-4ax \;\therefore c^2+2cy+4ax=0,$$

the equation to the curve, in which c remains to be determined.

Now the entire differential of the equation is

$$cdc+cdy+ydc-2adx=0.$$

But $2adx=ydy\pm\sqrt{(y^2-4ax)}\,.\,dy=-cdy$

$$\therefore cdc+cdy+ydc-cdy=0$$

$$\therefore (c+y)\frac{dc}{dy}=0$$

$$\therefore c+y=0 \quad \text{or} \quad \frac{dc}{dy}=0.$$

If $c+y=0$ then $\mp\sqrt{(y^2-4ax)}=0 \therefore y^2=4ax$, the equation to a parabola, which is the curve required.

If $\frac{dc}{dy}=0$, then c is an arbitrary constant and the equation is that of a straight line, which will coincide with any tangent PT according to the assumed value of c. This, however, is not the proper solution of the problem, as it introduces a new condition of limitation, not stated in it—viz. the value of the arbitrary constant line c.

XIII. QUESTION 443, *by* PALABA.

To determine the equation to the curve whose tangent is a mean proportional between the segment of the axis intercepted between it and a given point, and that same segment augmented by a given line.

FIRST SOLUTION, *by Mr.* LOWRY.

Let x, y, and v represent the same parts as in the solution to the preceding question, and let a be the given line; then, by the conditions of the question,

$$(v + x)^2 + y^2 = v \times (v + a)$$

$$\text{or } y^2 = (a - 2x)\,v - x^2 \ldots\ldots\ldots\ldots(1).$$

But $v + x =$ the subtangent $= \dfrac{y\dot{x}}{\dot{y}}$.

Therefore $v = \dfrac{y\dot{x}}{\dot{y}} - x$,

and this value being substituted for v in equation (1), it becomes

$$y^2 = (a - 2x)\left(\frac{y\dot{x}}{\dot{y}} - x\right) - x^2$$

$$\text{or } y^2\dot{y} = (a - 2x)\,y\dot{x} - (ax - x^2)\,\dot{y}$$

and dividing by y^2

$$\dot{y} = \frac{(a - 2x)\,y\dot{x} - (ax - x^2)\,\dot{y}}{y^2}.$$

Taking the fluents

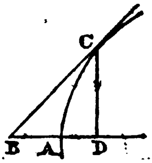
$$y = \frac{ax - x^2}{y}$$

$$\text{or } y^2 = ax - x^2.$$

An equation to the circle whose diameter is a.

SECOND SOLUTION, *by Mr.* BUTTERWORTH.

Let A be the given point, AC the curve whose equation is required; to any point C of the curve draw the tangent BC meeting the axis DA produced in B, and draw the ordinate CD. Put AD $= x$, DC $= y$, and $a =$ the given line;



then $\dot y : \dot x ::$ DC $= y :$ DB $= \frac{y\dot x}{\dot y}$;

therefore AB $= \frac{y\dot x}{\dot y} - x$;

$$\text{also BC}^2 = \text{BD}^2 + \text{DC}^2 = \frac{y^2\dot x^2}{\dot y^2} + y^2;$$

therefore by the question

$\left(\frac{y\dot x}{\dot y} - x + a\right) . \left(\frac{y\dot x}{\dot y} - x\right) = \frac{y^2\dot x^2}{\dot y^2} + y^2$, which being reduced gives

$$ay\dot x - ax\dot y - 2yx\dot x + x^2\dot y = y^2\dot y,$$

or $\frac{ay\dot x - ax\dot y}{y^2} - \frac{2yx\dot x - x^2\dot y}{y^2} = \dot y$; the fluent of which is $\frac{ax}{y} - \frac{x^2}{y} = y$, therefore $ax - x^2 = y^2$, an equation to a circle whose diameter is a.

THIRD SOLUTION, *by* PALABA, *the Proposer.*

Let the given point be the origin of the abscissa; then the subtang. $= \frac{y\dot x}{\dot y}$, the segment of the axis $= \frac{y\dot x}{\dot y} - x$: call (a) the given line.

$$\text{Then } \left\{\frac{y\dot x}{\dot y} - x\right\} . \left\{\frac{y\dot x}{\dot y} - x + a\right\} = \frac{y^2\dot x^2}{\dot y^2} + y^2$$

$$\text{or } \left(\frac{y\dot x}{\dot y} - x\right)^2 + a . \left(\frac{y\dot x}{\dot y} - x\right) = \frac{y^2\dot x^2}{\dot y} + y^2$$

$$\therefore x^2 - \frac{2xy\dot x}{\dot y} + a . \left(\frac{y\dot x}{\dot y} - x\right) = y^2.$$

Take the fluxions, considering $\dot{y}$ as constant,

$$\therefore, -\frac{2y\dot{x}^2}{\dot{y}} - \frac{2yx\ddot{x}}{\dot{y}} + \frac{ay\ddot{x}}{\dot{y}} = 2y\dot{y}$$

$$\text{or } (a-2x)\frac{\ddot{x}}{\dot{y}^2} = 2 \cdot \left(1 + \frac{\dot{x}^2}{\dot{y}^2}\right).$$

$$\text{Assume } \frac{\dot{x}}{\dot{y}} = p, \therefore, \frac{\ddot{x}}{\dot{y}^2} = \frac{\dot{p}}{\dot{y}} = \frac{p\dot{p}}{\dot{x}}$$

$$\therefore (a-2x)\frac{p\dot{p}}{\dot{x}} = 2 \cdot (1+p^2) \therefore \frac{2\dot{x}}{a-2x} = \frac{p\dot{p}}{1+p^2}$$

$$\text{L}\left(\frac{a}{a-2x}\right) = \text{L}\sqrt{1+p^2} \therefore 1+p^2 = \frac{a^2}{(a-2x)^2}$$

$$\therefore p^2 = \frac{4ax-4x^2}{(a-2x)^2}, p = \frac{2\sqrt{(ax-x^2)}}{a-2x}$$

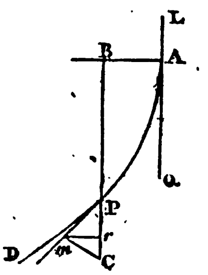
$$\therefore \dot{y} = \tfrac{1}{2} \cdot \frac{a\dot{x}-2x\dot{x}}{\sqrt{ax-x^2}} \therefore y = \sqrt{(ax-x^2)}$$

$\therefore y^2 = ax - x^2$, and the curve is a circle whose diameter is (a).

XIV. QUESTION 444, *by* PALABA.

A body urged by a force perpendicular to the horizon, describes the quadrant of a circle. Required the law of force which will make it recede uniformly from the horizontal radius, and the time elapsed and the velocity acquired at any point of the descent.

FIRST SOLUTION, *by* PALABA, *the Proposer.*



Let APD be any curve whose vertex is A touching the vertical line LAQ, AB, BP the rectangular coordinates to any point P; produce BP to C and take PC to represent the force (φ) required, let fall C*m* a perpendicular on tan. P*m* and *mr* a perpendicular on PC.

Let AB $= x$, BP $= y$, AP $= z$, then on the same scale that P*m* represents $\dot{z}$, P*r* will represent $\dot{x}$ and *rm* $\dot{y}$.

Call (c) the vertical velocity, v the velocity, and F the force in the curve.

Then $\dot{v}v = \text{F}\dot{z}$.

But $\varphi : \text{F} :: \text{PC} : \text{P}m :: \text{P}m : \text{P}r :: \dot{x} : \dot{y}$

∴ $F\dot{z} = \varphi\dot{y}$ ∴ $v\dot{v} = \varphi\dot{y}$. Also $v : c :: Pm : Pr :: \dot{z} : \dot{y}$

∴ $v = \frac{c\dot{z}}{\dot{y}}$ ∴ $v^2 = \frac{c^2\dot{z}^2}{\dot{y}^2}$ and $v\dot{v} = \frac{c^2\dot{z}\ddot{z}}{\dot{y}^2} = \varphi\dot{y}$ ∴ $\varphi = \frac{c^2\dot{z}\ddot{z}}{\dot{y}^3}$

But $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$ ∴ $\dot{z}\ddot{z} = \dot{x}\ddot{x}$ ∴ $\varphi = \frac{c^2\dot{x}\ddot{x}}{\dot{y}^3} \propto \frac{\dot{x}\ddot{x}}{\dot{y}^3}$.

If the curve is a circle,

$x = 1 - \cos. z$, $y = \sin. z$ ∴ $\dot{x} = -\dot{z}\sin. z$, $\dot{y} = \dot{z}\cos. z$

∴ $\frac{\dot{x}}{\dot{y}} = -\frac{\sin. z}{\cos. z} = -\tan. z$ and $\frac{\dot{x}^2}{\dot{y}^2} = \tan.^2 z$ ∴ $\frac{\dot{x}\ddot{x}}{\dot{y}^2} =$ $\dot{z}\tan. z \sec.^2 z$,

and $\frac{\dot{x}\ddot{x}}{\dot{y}^3} = \frac{\dot{z}}{\dot{y}} . \tan. z . \sec.^2 z = \frac{1}{\cos. z} . \tan. z . \sec.^2 z \propto \tan. z . \sec.^3 z$.

Also ∴ $v\dot{y} = c\dot{z}$, $v = c . \frac{\dot{z}}{\dot{y}} = \frac{c}{\cos. z} = c \sec. z$, and $t = \frac{y}{c} = \frac{\sin. z}{c}$.

SECOND SOLUTION, *by Mr.* LOWRY, *R. M. College.*

Let x and y be the rectangular co-ordinates of the body at the end of any indefinite time t, x being taken on the horizontal radius and reckoned from the beginning of the motion; and let z be the arc described by the body. Also let φ be the force required, which acting in the direction of y causes the body to recede uniformly from the axis of x with the constant velocity a; then the force in the direction of the curve will be $= \frac{\dot{y}}{\dot{z}} \times \varphi$; and, by the principles of dynamics,

$$\frac{\ddot{z}}{\dot{t}^2} = \frac{\dot{y}}{\dot{z}} \times \phi$$

$$\frac{\dot{y}}{\dot{t}} = a.$$

By eliminating $\dot{t}$ from these two equations we get

$$\varphi = \frac{a^2\dot{z}}{\dot{y}^2} \times \frac{\ddot{z}}{\dot{y}} \ldots\ldots\ldots\ldots (1)$$

which is a general expression for the force required whatever may be the nature of the curve.

In most cases, however, it will be more commodious to have an expression for the force in terms of the co-ordinates, and this may be easily obtained from the above, by considering that $\dot z^2 = \dot y^2 + \dot x^2$ and $\dot y$ constant, whence by taking second fluxions $\dot z\ddot z = \dot x\ddot x$, and the above expression becomes

$$\varphi = \frac{a^2\dot x}{\dot y^2} \times \frac{\ddot x}{\dot y} \ldots\ldots\ldots\ldots (2).$$

Ex. 1. When the curve is a circle, as in the question, the radius being 1, $\dot y$ is $= \dot z \cos. z$, or $\frac{\dot z}{\dot y} = \frac{1}{\cos. z}$,

therefore $\frac{a^2\dot z}{\dot y^2} = \frac{a^2}{\dot z \cos.^2 z}$ and $\frac{\ddot z}{\dot y} = \frac{\dot z \sin. z}{\cos.^2 z}$,

and by formula (1)

$$\varphi = \frac{a^2}{\dot z \cos.^2 z} \times \frac{\dot z \sin. z}{\cos.^2 z} = \frac{a^2 \sin. z}{\cos.^4 z}.$$

Again, for the time,

$$\dot t = \frac{\dot y}{a} = \frac{\dot z \cos. z}{a},$$

and

$$t = \int \frac{\dot z \cos. z}{a} = \frac{\sin. z}{a}.$$

Also, the velocity in the curve is

$$\frac{\dot z}{\dot t} = \frac{a\dot z}{\dot y} = \frac{a}{\cos. z}.$$

Ex. 2. When the curve is the common parabola, having its axis in a horizontal position, and its parameter $= p$.

Here, $px = y^2$, and $p\dot x = 2y\dot y$; therefore $\frac{\dot x}{\dot y} = \frac{2y}{p}$,

$$\frac{a^2\dot x}{\dot y^2} = \frac{2a^2 y}{p\dot y},\ \frac{\ddot x}{\dot y} = \frac{2\dot y}{p},$$ and by formula (2)

$$\varphi = \frac{2a^2 y}{p\dot y} \times \frac{2\dot y}{p} = \frac{4a^2 y}{p^2}.$$

Again, for the velocity in the curve, $\dot z$ is $= \dot y \sqrt{\left(\frac{4y^2}{p^2} + 1\right)}$,

therefore $\frac{\dot z}{\dot t} = \frac{a\dot z}{\dot y} = a\sqrt{\left(\frac{4y^2}{p^2} + 1\right)} = a\sqrt{\left(\frac{4x+p}{p}\right)}$, the velocity required.

XV. QUESTION 445, *by* ———.

The characteristic property of the circle is that all the chords which pass through a certain determinate point in its plane are equal: but there exists an indefinite number of curves which possess this property. It is required to find the most general equation to curves of this nature.

SOLUTION, *by* ———— ; *taken from the Annales de Mathematiques.*

Construction. Let P be the given point, and $2r$ the constant length of the chords which pass in all directions through this point. Let there be traced at pleasure a curve A, either continued or discontinued; and through the point P let a series of straight lines D be drawn to cut the curve A in a series of points C. Lastly from each of these points C as a centre with the constant radius r let a series of circles be described cutting the line D in which it centre is situated in two points M and N, then all the points M and N are found in the two branches of the curve which satisfies, in all its extent, the conditions of the problem; and this curve is continued or discontinued according to the nature of the arbitrary curve A.

Analysis. Let the arbitrary curve A be referred to two rectangular axes passing through P, and suppose that its equation is

$$y = f(x):$$

the general equation of the straight line D is

$$y = ax$$

a being an indeterminate quantity.

We obtain again the equations of all the points C; by combining the two equations

$$y = f(x), \qquad y = ax$$

whence we conclude first, $f(x) = ax$.

If we suppose this equation when resolved relative to x, to give $x = \varphi(\alpha)$, the points C will have for their equations

$$x = \varphi(\alpha); \quad y = \alpha\phi(\alpha).$$

The circles which have these points for their centres and r for radius, will therefore have for their general equation

$$\{x - \phi(\alpha)\}^2 + \{y - r\phi(\alpha)\}^2 = r^2;$$

by combining this equation, with $y = \alpha x$, the equation of D, we shall find the points M and N corresponding to each particular value of the indeterminate α.

Eliminating this indeterminate from these two equations, the resulting equation in terms of x and y will be that of the geometrical locus of all the points M and N, and will therefore be the equation of the curve sought.

The most general equation of the curves which satisfy the conditions of the proposed Problem is therefore

$$\left\{x - \phi\left(\frac{y}{x}\right)\right\}^2 + \left\{y - \frac{y}{x}\phi\left(\frac{y}{x}\right)\right\}^2 = r^2.$$

in which ϕ is a function absolutely arbitrary.

REMARK. We suppose here that the function ϕ is no other than the function of α which is obtained for the value of x, by resolving the equation

$$f(x) = \alpha x.$$

If, on the contrary, it was the function ϕ which was given, and we wanted to deduce the function f, this last function is no other than the function of x which is obtained by eliminating α from the two equations,

$$x = \phi(\alpha), \qquad y = \alpha\phi(\alpha)$$

and resolving again this equation relative to y.

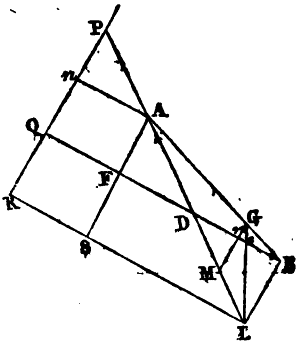
This question was also answered by S. H. *An elegant solution to it, by Mr.* HERSCHEL, *may also be seen in section* X. of a "Collection of Examples of the application of the Calculus of Finite Differences." Lately published.

XVI. QUESTION 446, *by Mr.* CUNLIFFE, *R. M. College.*

A given rod or beam, has one end suspended by a cord of a given length, fixed at a given point above an inclined plane of a given inclination, and the other end of the beam is sustained by the inclined plane; it is required to determine the position of the beam, weight sustained by the cord, and pressure against the inclined plane, when the beam is in equilibrio.

FIRST SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

Let P, be the given point to which the end P of the given cord PA is fixed; AB the rod or beam, resting with its end B upon the given inclined plane QB, and having the end A, sustained by the cord PA.



Draw BL perpendicular to BQ, and produce PA to meet it in L. Let G be the centre of gravity of the beam, and draw GL cutting QB in e; also draw GrM perpendicular to QB in r, and meeting PL in M.

When the beam is in equilibrio as specified in the question, the line GL, joining the centre of gravity and the intersection of PA and BL will be vertical or perpendicular to the horizon, and consequently the angle eLB $=$ eGr is equal to the angle of the plane's inclination to the horizon, and is given by the question. And the weight of the beam, the tension of the cord, PA, and the pressure of the end B, against the inclined plane QB will be respectively, as the lines GL, ML, and GM and in those several directions. Draw PQ at right angles to QB; and LR, An perpendicular to PQ; also draw AFS parallel to PQ.

Put $\mathrm{PA} = r$, $\mathrm{PQ} = p$, $\mathrm{BG} = a$, $\mathrm{AG} = b$, $\mathrm{QF} = n\mathrm{A} = x$, $\mathrm{FA} = \mathrm{Q}n = y$, and $\mathrm{BL} = \mathrm{QR} = v$; then $\mathrm{P}n = p - y$, $\mathrm{PR} = p + v$ and $\mathrm{BF} = \sqrt{(\mathrm{AB}^2 - \mathrm{AF}^2)} = \sqrt{\left\{(a+b)^2 - y^2\right\}}$.

By reason of the parallels nA, RL; $\mathrm{P}n = p - y : n\mathrm{A} = x ::$ $\mathrm{PR} = p + v : \mathrm{QB} = \mathrm{QF} + \mathrm{FB} = x + \sqrt{\left\{(a+b)^2 - y^2\right\}}$, whence

$$v = \frac{p-y}{x}\sqrt{\left\{(a+b)^2 - y^2\right\}} - y.$$

The line GrM is perpendicular to the given inclined plane QB, and GeL is a vertical line, therefore the angle rGe $=$ eLB, is equal to the plane's inclination to the horizon, and is given by the question, therefore let the tangent of this angle to the radius 1 be denoted by t.

Because of the parallels AF, GM; $\mathrm{AB} : \mathrm{AF} :: \mathrm{BG} : \mathrm{G}r = \dfrac{\mathrm{BG} \times \mathrm{AF}}{\mathrm{AB}} = \dfrac{ay}{a+b}$; $1 : t :: \mathrm{G}r : re = \dfrac{aty}{a+b}$; $1 : t ::$ $\mathrm{LB} : \mathrm{B}e = tv$; $\mathrm{AB} : \mathrm{BG} :: \mathrm{BF} : \mathrm{B}r = \dfrac{\mathrm{BG} \times \mathrm{BF}}{\mathrm{AB}} = \dfrac{a}{a+b} \times$

whatever manner the beam may be sustained; whether as in the question, or by two strings of given lengths, attached to its ends and fastened to given points; two strings drawn by weights; two props; two inclined planes; two curve surfaces; or by any two of these.

Therefore to determine the real position of the beam, in any of these cases, we must find another equation between θ and ϕ, or two equations between θ, ϕ, and c, deduced from the particular data of the problem under consideration. In the present case we must take into account the length of the string, and the relative positions of the given point and the inclined plane.

Now $\quad \text{PM} = s \sin\theta,$

and $\quad \text{AH} = \text{KM} = (a+b)\sin\phi;$

but $\quad \text{PM} + \text{KM} = \text{PK};$

therefore $\quad s\sin\theta + (a+b)\sin\phi = p \ldots\ldots\ldots\ldots (2),$

From equations (1) and (2) we can now find the values of θ and ϕ, and consequently the position of the beam.

It is evident that the beam may rest in another position, as $\text{A}'\text{B}'$ in the figure, but the equations will remain the same in both cases with the exception of the change of signs.

To compare the tension of the string and the pressure against the plane, with the weight of the beam, draw GV parallel to PD; then the three forces are respectively as GV, VD and GD.

Put $\text{T} =$ the tension, $\text{P} =$ the pressure, and $\text{W} =$ the weight of the beam.

Then $\sin \text{GVD} : \sin \text{VDG} = \sin c :: \text{DG} : \text{VG} :: \text{W} : \text{T}$

$\sin \text{GVD} : \sin \text{VGD} :: \text{DG} : \text{VD} :: \text{W} : \text{P}$

but $\quad \sin \text{GVD} = \sin \text{GDV} = \sin \text{APM} = \cos\theta$

therefore

$$\text{T} = \frac{\sin c}{\cos\theta} \times \text{W},$$

$$\text{P} = \frac{\cos(\theta + c)}{\cos\theta} \times \text{W}.$$

When the given plane is vertical the beam can only rest in one position. In this case $c = 90^\circ$, $\sin c = 1$; and the two equations become

$$(a+b)\sin(\phi - \theta) = a\sin\phi\cos\theta$$

$$(a+b)\sin\phi - s\sin\theta = p.$$

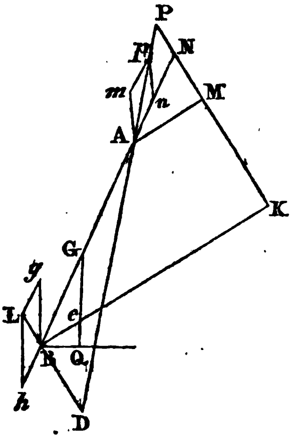
Also

$$\text{T} = \frac{\text{W}}{\cos\theta}$$

$$\text{P} = \frac{\sin\theta}{\cos\theta}\,\text{W}.$$

The same answered otherwise.

Let Ap represent the tension of the string, and BL the pressure against the plane. Ap may be resolved into the two forces Am or pn, and mp or An, the former in the direction of gravity, the latter in the direction of the beam.



In like manner BL may be resolved into the forces Bg or hL, and Lg or Bh, the former in the direction of gravity, the latter in the direction of the beam.

Now in the case of an equilibrium, the forces Am and Bg must, by the property of the lever, be reciprocally as their distances from the centre of gravity G, therefore

$$b \times Am = a \times Bg. \ldots\ldots\ldots\ldots\ldots (\alpha)$$

The forces in the direction of the beam must also be equal to one another,

$$\text{or} \quad An = Bh. \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\beta)$$

From the triangles Apm, Apn,

$$\sin Amp : \sin mpA :: Ap = T : Am,$$
$$\sin Amp : \sin mAp :: Ap : An;$$

but
$$\sin Amp = \sin AGC = \sin BGC = \cos (c + \varphi),$$
$$\sin mpA = \sin PAN = \sin (\theta - \varphi),$$
$$\sin mAp = \sin (90^\circ - (\theta + c)) = \cos (\theta + c);$$

therefore
$$Am = T \times \frac{\sin (\theta - \varphi)}{\cos (c + \varphi)},$$
$$An = T \times \frac{\cos (\theta + c)}{\cos (c + \varphi)}.$$

Again, from the triangles BLg, BLh,

$$\sin LgB : \sin BLg :: BL = P : Bg,$$
$$\sin LgB : \sin LBg :: BL : Bh;$$

but
$$\sin LgB = \sin BGC = \cos (c + \varphi)$$
$$\sin BLg = \sin LBG = \cos \varphi, \text{ and } \sin LBg = \sin c;$$

therefore
$$Bg = P \times \frac{\cos \varphi}{\cos (c + \varphi)}$$
$$Bh = P \times \frac{\sin c}{\cos (c + \varphi)}.$$

Therefore by equation (a), leaving out the common denominator,

$$b \times \text{T} \times \sin(\theta - \varphi) = a \times \text{P} \times \cos\varphi$$

and from equation (β)

$$\text{T} \times \cos(\theta + c) = \text{P} \times \sin c.$$

From the first of these

$$\frac{\text{P}}{\text{T}} = \frac{b \sin(\theta - \varphi)}{a \cos\varphi}.$$

From the second

$$\frac{\text{P}}{\text{T}} = \frac{\cos(\theta + c)}{\sin c}.$$

Therefore by equality

$$\frac{b \sin(\theta - \varphi)}{a \cos\varphi} = \frac{\cos(\theta + c)}{\sin c}$$

or $$b \sin c \sin(\theta - \varphi) = a \cos\varphi \cos(\theta + c)$$

and, by adding $a \sin c \sin(\theta - \varphi)$ to each side of this equation,

$$(a + b) \sin c \sin(\theta - \varphi) = a \cos\theta \cos(c + \varphi),$$

which is the same as equation (1) in the preceding solution.

Again, since the sum of the two vertical forces $\text{A}m$ and $\text{B}g$ must be equal to the weight of the beam, we have

$$\text{T} \times \frac{\sin(\theta - \varphi)}{\cos(c + \varphi)} + \text{P} \times \frac{\cos\varphi}{\cos(c + \varphi)} = \text{W},$$

and $$\text{T} \times \cos(\theta + c) = \text{P} \sin c$$

therefore, by substitution,

$$\text{T} \times \frac{\sin c \sin(\theta - \varphi) + \cos\varphi \cos(\theta + c)}{\sin c \cos(c + \varphi)} = \text{W}$$

$$\text{P} \times \frac{\sin c \sin(\theta - \varphi) + \cos\varphi \cos(\theta + c)}{\cos(\theta + c) \cos(c + \varphi)} = \text{W}$$

or, by reduction,

$$\text{T} \times \frac{\cos\theta}{\sin c} = \text{W}$$

$$\text{P} \times \frac{\cos\theta}{\cos(\theta + c)} = \text{W}.$$

XVII. QUESTION 447, *by* G. V.

Find the rectification and quadrature of the magnetic curve, which is such, that if lines be drawn from the poles to any point in the curve and ϕ and ψ be the angles they make with the line joining them, cos. ϕ + cos. $\psi = c$, a constant quantity.

To this Question no answer has been received.

XVIII. QUESTION 448, *by* ERATOSTHENES.

Required a number consisting of six digits *a b c d e f* such that its multiples by 1, 2, 3, 4, 5, 6 may contain amongst them the following arrangements

$$\begin{matrix} a & b & c & d & e & f \\ b & c & d & e & f & a \\ c & d & e & f & a & b \\ d & e & f & a & b & c \\ e & f & a & b & c & d \\ f & a & b & c & d & e. \end{matrix}$$

FIRST SOLUTION, *by* ERATOSTHENES, *the Proposer.*

Let N be the number, then the sum of these different multiples will be

$$(1+2+3+4+5+6)\,N = (a+b+c+d+e+f)\,(10^5+10^4+10^3+10^2+10+1)$$

or

$$21N = (a+b+c+d+e+f)\,.\,(111111) \ldots\ldots (1)$$

But since 3N is a multiple of 3 the sum of the digits in 3N is a multiple of 3, therefore the sum of the digits in N (which equal that in 3N) is a multiple of 3; hence the sum of the digits in 3N is a multiple 9 and consequently also the sum of the digits in N is a multiple of 9, hence

$$9q = a+b+c+d+e+f \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Divide (1) by (2) then

$$\frac{21N}{9q} = 111111$$

or

$$N = q \times \frac{333333}{7} = q \times 47619$$

but the first figure of N must be 1, for if it were greater the multiple by 6 would produce seven figures, therefore q can only be either 3 or 4. On trial 3 succeeds but 4 does not.

For if $N = 3 \times 47619 = 142857$

it has the required properties.

SECOND SOLUTION, *by* Mr. JOHN BAINES.

Put $S = 100000\,a + 10000\,b + 1000\,c + 100\,d + 10\,e + f$, the number represented by the digits *abcdef*; then a + 1000000b

$+ 10000c + 1000d + 100e + 10f =$ the number represented by the digits $bcdefa$, and this by the question is $= 2s$, $3s$ $4s$, $5s$, or $6s$. From 10 times the first equation subtract the second, and $999999a = 8s, 7s, 6s, 5s$, or $4s$, but a cannot be greater than 1, if it were, $6s$ would consist of 7 digits, which is inadmissible; therefore $a = 1$, and hence $8s, 7s, 6s, 5s$, or $4s = 999999$; but s must be a whole number, which will only be so when $7s = 999999$; that is $s = 142857$, the number required.

For $142857 \times 2 = 285714 = cdefab$; $142857 \times 3 = 428571 = bcdefa$; $142857 \times 4 = 571428 = efabcd$; $142857 \times 5 = 714285 = fabcde$; and $142857 \times 6 = 857142 = defabc$.

THIRD SOLUTION, *by* S. H.

Suppose the least number to be N, and that $a + b + c + d + e + f = \text{S}$

$$
\begin{aligned}
\therefore\ & 1000000a + 10000b + 1000c + 100d + 10e + f \\
+\ & 100000b + 10000c + 1000d + 100e + 10f + a \\
+\ & 100000c + 10000d + 1000e + 100f + 10a + b \\
+\ & 100000d + 10000e + 1000f + 100a + 10b + c \\
+\ & 100000e + 10000f + 1000a + 100b + 10c + d \\
+\ & 100000f + 10000a + 1000b + 100c + 10d + e \\
=\ & \text{N} + 2\text{N} + 3\text{N} + 4\text{N} + 5\text{N} + 6\text{N}, \\
\text{i.e. }\ & 111111\text{S} = 21\text{N} \ \therefore\ \text{N} = 5291\text{S}.
\end{aligned}
$$

But since 6N contains only 6 digits it is less than 1000000

$$\therefore\ \text{N is less than } \frac{1000000}{6}\ \therefore \text{ less than } 166667,$$

$$\therefore\ \text{S is less than } \frac{166667}{5291}\ \therefore \text{ less than } 32.$$

Now as $4\frac{1}{2}$ is the mean value of a digit, and S is the sum of 6 digits it is probable that $\text{S} =$ about $6 \times 4\frac{1}{2}$ or 27.

Let 27 be tried, then $\text{N} = 27 \times 5291 = 142857$ and $1 + 4 + 2 + 8 + 5 + 7 = 27$; so far correct, which encourages farther trial, $2\text{N} = 285714$, $3\text{N} = 428571$, $4\text{N} = 571428$, $5\text{N} = 714285$, $6\text{N} = 857142$, as was required, so that $a = 1$, $b = 4$, $c = 2$, $d = 8$, $e = 5$, $f = 7$.

XIX. QUESTION 449, *by* ERATOSTHENES.

It is well known that the formula $x^2 + x + 41$ contains a great number of primes; prove that when x is of either of the two following forms it cannot be a prime, $x = 53a + 3$ and $x = 97a - 8$.

FIRST SOLUTION, *by* ERATOSTHENES, *the Proposer.*

Let $x = av \pm b$ then

$$\begin{array}{rcl} x^2 & = & a^2v^2 \pm 2abv + b^2 \\ x & = & av \pm b \\ 41 & = & 41 \\ \hline x^2 + x + 41 & = & av\,(av \pm 2b + 1) + b^2 \pm b + 41. \end{array}$$

Now if v is a multiple of $b^2 \pm b + 41$ this expression has a factor of that form: $x^2 + x + 41$ is therefore never a prime when

$$x = (b^2 \pm b + 41)\,a \pm b.$$

If we take the lower sign and put $b + 1$ for b

$$\text{we have } x = (b^2 + b + 41)\,a - b - 1$$

and the upper gives $x = (b^2 + b + 41)\,a + b.$

Let $b = 3$ in the second and $b = 7$ in the first and we have the two cases in the question

$$x = 53\,a + 3 \text{ and } x = 97\,a - 8.$$

If we except from the expression $x^2 + x + 41$ all values of x contained in the two forms $(b^2 + b + 41)\,a + b$ and $(b^2 + b + 41)a - b - 1$ from $x = 0$ to $x = 200$ give nothing but primes.

SECOND SOLUTION, *by Mr.* CUNLIFFE.

To determine cases wherein the formula $x^2 + x + 41$ will not be a prime number.

Put $x = na + b$, then $x^2 + x + 41 = n^2a^2 + na\,(2b + 1) + b^2 + b + 41$. Now take $n = b^2 + b + 41$, and then $x^2 + x + 41 = n \times \left\{na^2 + a\,(2b + 1) + 1\right\}$ which is evidently not a prime: that is, when $n = b^2 + b + 41$, and b is expounded by any integer at pleasure either positive or negative.

Ex. 1. Put $b = 3$, then $n = 53$ and $x = na + b = 53a + 3$.

Ex. 2. Put $b = 8$, then $n = 97$ and $x = 97a - 8$.

Ex 3. Put $b = 2$, then $n = 47$ and $x = 47a + 2$.

Ex 4. Put $b = 1$, then $n = 43$ and $x = 43a + 1$.

It is manifest that an infinite variety of forms for the value of x may be obtained from what has been done, each of which will make the expression, $x^2 + x + 41$, not a prime number.

THIRD SOLUTION, *by* A. B. L.

By making successively $x = 0, 1, 2, 3, 4 \ldots\ldots$&c. this formula will give a series of 41, 47, 53, 71 ....&c. the first forty terms of which are prime numbers.

Let $x = \text{P}a + m$, then the formula becomes

$\text{P}^2a^2 + \text{P}a\,.\,\overline{2m+1} + m^2 + m + 41$. Now if P, and $m^2 + m + 41$ have a common divisor, that common divisor will divide the formula and therefore it cannot be a prime.

In the first case $\text{P} = 53$ and $m = 3$, $m^2 + m + 41 = 53 = \text{P}$ and in this case must oe divisible by 53. Again in the second $\text{P} = 97$ and $m = -8$, $m^2 + m + 41 = 97$ and the formula must be divisible by 97, a being any whole number, which will be satisfactory for an infinite number of forms besides the above two. Hence it appears that in any algebraical formula such a value may be given to the indeterminate quantity as will render it divisible by some other number and therefore no algebraical function of x can in all cases represent a prime. Euler in his Memoires of Berlin, 1772, has mentioned the above formula, and likewise the two following $x^2 + x + 17$ and $2x^2 + 29$, the 1st of these formulæ by making successively $x = 0, 1, 2, 3, 4,$ &c. gives a series of, which the 1st seventeen terms are prime numbers and the second in the same manner gives twenty-nine prime numbers.

It is well known that Fermat asserted that $2^n + 1$ would always be a prime while n was taken any number in the geometrical progression 1, 2, 4, 8, 16 &c. but Euler has proved the case to fail when $n = 5$.

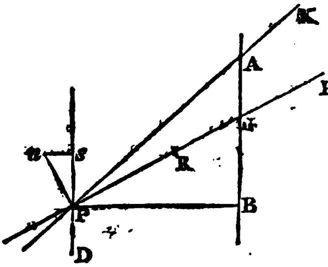
These cases prove the danger of drawing conclusions from induction, and shew the little dependance we ought to place on that method in mathematical investigations, since there are few cases of this kind in which we have more reason to infer a general law from particular results than in the above formulæ.

Messrs. BAINES *and* S. H. *likewise answered this question.*

XX. PRIZE QUESTION 450, *by* MECHANIQUE.

If the beam in Question 401 instead of resting upon the curve, be at liberty to slide freely over the prop and descend by the force of gravity, it is required to determine the motion of its centre of gravity.

SOLUTION, *by Mr.* LOWRY, *R. M. College.*



Let P be the top of the prop DP over which the beam is supposed to slide; PAK the position of the beam at the beginning of the motion, and PGH its position at the end of any time t. Also, let G be the centre of gravity, and R the centre of gyration of the beam; and draw the vertical line AGB to meet the horizon-line PB in B. Because the beam is supposed to slide freely over the prop, the only force that acts upon it at P is the re-action of the prop, which will always be at right angles to the beam; and therefore can have no effect in pushing the centre of gravity G in the direction of the beam.

Put $a = PB$, $b = BA$, $d = GR$, $\angle PGB = \varphi$, radius 1.

F = the force at R, to turn the beam about G,

v = the velocity of G in the direction GB,

u = the angular velocity of R about G,

m = the mass of the beam,

$x = BG$; $g = 32\frac{1}{6}$ feet, the measure of the accelerative force of gravity.

Let Pn at right angles to PG, represent the re-action of the prop against the beam, and draw ns perpendicular to the vertical line DP produced; then Ps will represent the force in the direction of gravity, and by the resolution of forces $Ps = Pn \times \sin Pns = Pn \sin \varphi$. But by the property of the lever

$$Pn : F :: GR : GP :: d : \frac{a}{\sin \varphi}, \text{ or } Pn = F \times \frac{d \sin \varphi}{a};$$

therefore $$Ps = F \times \frac{d \sin^2 \varphi}{a}.$$

By the property of the centre of gyration,

$$F = M \times \frac{\dot{u}}{\dot{t}} = M \times \frac{d\ddot{\varphi}}{\dot{t}^2},$$

Therefore $PS = \frac{M\sin^2\varphi}{a} \times \frac{d^2\ddot{\varphi}}{\dot{t}^2}$ = the motive force in the direction of gravity, arising from the re-action of the prop, and which being taken from Mg, the whole motive force of the beam, gives

$$Mg - \frac{M\sin^2\varphi}{a} \times \frac{d^2\ddot{\varphi}}{\dot{t}^2}$$

for the motive force on G, in the direction GB. Therefore the accelerative force in the same direction is

$$g - \frac{\sin^2\varphi}{a} \times \frac{d^2\ddot{\varphi}}{\dot{t}^2}.$$

But $\frac{\ddot{x}}{\dot{t}^2}$ is the measure of the accelerative force in that direction; therefore

$$g - \frac{\sin^2\varphi}{a} \times \frac{d^2\ddot{\varphi}}{\dot{t}^2} = \frac{\ddot{x}}{\dot{t}^2}.$$

Again, $x = a\cot\varphi$, therefore $-\dot{x} = \frac{a\dot{\varphi}}{\sin^2\varphi}$; and multiplying the first and last terms of the above equation by $-2\dot{x}$, and the second term by $\frac{2a\dot{\varphi}}{\sin^2\varphi}$, which is $= -2\dot{x}$, and taking the correct fluents, we have

$$2g(b-x) = \frac{d^2\dot{\varphi}^2}{\dot{t}^2} + \frac{\dot{x}^2}{\dot{t}^2},$$

or, because $\dot{\varphi} = -\frac{\dot{x}\sin^2\varphi}{a}$,

$$2g(b-x) = \frac{d^2\sin^4\varphi + a^2}{a^2} \times \frac{\dot{x}^2}{\dot{t}^2} = \frac{d^2\sin^4\varphi + a^2}{a^2} \times v^2;$$

therefore $2g(b-x)a^2 = (d^2\sin^4\varphi + a^2) \times v^2$,

or $v = a\sqrt{\left\{\frac{2g(b-x)}{d^2\sin^4\varphi + a^2}\right\}}$.

ARTICLE III.

Solutions to Questions proposed in No. XVIII.

I. QUESTION 451, *by Mr.* JOHN BAINES, *Dewsbury.*

A gentleman has a triangular field, a mile round, and a perpendicular from one angle cuts the opposite side in extreme and mean proportion. Required the sides and area, when the latter is a maximum?

SOLUTION, *by Mr.* BAINES, *the Proposer.*

Let m and n represent the greater and less parts of unity divided in extreme and mean proportion. Put mx and nx = the segments of the base made by the perpendicular, xy = the perpendicular, and a = 80 chains = 1 mile = the perimeter; then $x\sqrt{(m^2+y^2)}$ = the greater side, $x\sqrt{(n^2+y^2)}$ = the less, and $x+x\sqrt{(m^2+y^2)}+x\sqrt{(n^2+y^2)}=a$, the perimeter (by the question); hence $x=\frac{a}{1+\sqrt{(m^2+y^2)}+\sqrt{(n^2+y^2)}}$, and $\frac{x^2y}{2}=\frac{x^2y}{2(1+\sqrt{(m^2+y^2)}+\sqrt{(n^2+y^2)})}$ = the area, a maximum. This in fluxions and reduced gives $\frac{y^2-m^2}{\sqrt{(y^2+m^2)}}+\frac{y^2-n^2}{\sqrt{(y^2+n^2)}}=1$, from which we obtain $y=\cdot 76066$; hence $x=26\cdot 48068$ chains, the base of the triangle; $28\cdot 30628$ = the greater side, $25\cdot 21305$ the less, $23\cdot 09539$ = the perpendicular, and $305\cdot 790816$ sq. chains = 30 A. 2 R. 12·653 P. = the area. The sides of the triangle having the above properties are as the numbers 1, 1·05028, and 1·12264; and the angles are 54° 40′ 39″, 58° 58′ 23″, and 66° 20′ 58″. When $m=n=\frac{1}{2}$, the equation becomes $\frac{2(y^2-\frac{1}{4})}{\sqrt{(y^2+\frac{1}{4})}}=1$; hence $y=\frac{1}{2}\sqrt{3}$, $x=\frac{1}{3}a$, the sides are each equal $\frac{1}{3}a$, and the triangle is equilateral, as it ought to be.

Mr. JERWOOD *answered this question.*

II. QUESTION 452, *by Mr.* NOSYT.

If (R) and (*r*) be the radii of the circles, inscribed and circumscribed about a plane triangle. Prove

$$\text{R} . r = \frac{abc}{2(a+b+c)};\ a,\ b,\ c \text{ being the sides.}$$

FIRST SOLUTION, *by* S. F.

Since the whole triangle is made up of the three triangles whose bases are the three sides and whose perpendicular heights are radii of the inscribed circle, therefore *a*, *b*, *c* being the sides, and R the radius of the inscribed circle, the area of the given triangle $= \frac{\text{R} . (a+b+c)}{2}$. But (by prop. C book VI, Simson's Euclid) if *r* be the radius of the circumscribing circle, *a* and *c* the sides containing the vertical angle, *b* the base and *p* the perpendicular height $2pr = ac$.

$$\text{Hence } 2prb = acb,$$

$$\text{and } \therefore\ pb = \frac{acb}{2r};$$

but $p . b =$ twice the area of the given triangle

$$\therefore \text{ area of the given triangle} = \frac{acb}{4r};$$

$$\text{but the area of the given triangle} = \frac{\text{R} . (a+b+c)}{2}$$

$$\frac{\text{R} . (a+b+c)}{2} = \frac{acb}{4r}$$

$$\therefore \text{R} . r = \frac{abc}{2(a+b+c)}.$$

SECOND SOLUTION, *by Mr.* CHARLES BONNYCASTLE.

In the circumscribing circle, the lines drawn from the centre, to the extremities of the base, include an angle equal to twice the vertical angle; wherefore if this last be put equal φ, we have

$$\text{R} = \frac{a}{2\sin\varphi} = \frac{a}{4\sin\frac{1}{2}\varphi . \cos\frac{1}{2}\varphi}.$$

And in the inscribed circle, if perpendiculars be drawn from the centre to the sides, the equal segments which they cut off on each side of the vertical angle are equal to $\frac{b+c-a}{2}$; wherefore

$$r = \tfrac{1}{2}(b+c-a)\tan\tfrac{1}{2}\varphi,$$

and

$$\text{R} \,.\, r = \tfrac{1}{8}(b+c-a) \,.\, \frac{a}{\cos^2\frac{1}{2}\varphi} \;\ldots\ldots\ldots\ldots(1).$$

But putting φ, θ and ε for the angles of the triangle, we have the well known equations

$$a\cos\theta + b\cos\varphi = c,$$
$$a\cos\varepsilon + c\cos\varphi = b,$$
$$b\cos\varepsilon + c\cos\theta = a.$$

Whence by elimination

$$\cos\varphi = \frac{b^2+c^2-a^2}{2bc},$$

and

$$\cos^2\tfrac{1}{2}\varphi = \frac{b^2+c^2-a^2}{4bc}.$$

Substituting which in the expression (1) it becomes

$$\text{R} \,.\, r = \tfrac{1}{2}(b+c-a) \,.\, \frac{abc}{(a+b+c)\,.\,(b+c-a)}$$
$$= \frac{abc}{2(a+b+c)}.$$

This question was also answered by Messrs. BAINES, FUNIDUS, JERWOOD, *and* NOSYT, *the Proposer.*

III. QUESTION 453, *by Mr.* NOSYT.

Find the sum of the recurring decimal $\cdot qppp$ &c. in infinitum. (q) and (p) containing (m) and (n) digits respectively.

FIRST SOLUTION, *by Mr.* JAMES JERWOOD, *Poughill.*

The numbers composing every circulating or recurring decimal are geometric progressions whose common ratios are $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. according to the number of factors in the recurring decimal, hence in the present case the series of fractions which represent the value of the proposed expression are $\frac{q}{10^m}$ + (geometric series) $\frac{p}{10^{m+n}} + \frac{p}{10^{m+2n}} + \frac{p}{10^{m+3n}}$ &c. the sum of which $= \left(\frac{p}{10^{m+n}}\right) \div \left(1 - \frac{1}{10^{n}}\right) = \frac{p}{10^{m+n} - 10^{m}}$, and therefore $\frac{q}{10^{m}} + \frac{p}{10^{m+n} - 10^{m}} = \frac{q(10^{m+n} - 10^{m}) + 10^{m} p}{10^{2m+n} - 10^{2m}} =$ $\frac{q(10^{n} - 1) + p}{10^{m+n} - 10^{m}} =$ the sum required.

Second Solution, *by* Nosyt.

Let $\cdot qppp$ &c. $=$ s $\therefore q \cdot ppp$ &c. $= 10^{m} \,.\, \text{s}$.

Again, let $\cdot pppp$ &c. $= \Sigma \therefore p \cdot ppp$ &c. $= 10^{n} \,.\, \Sigma$

$\therefore p + \Sigma = 10^{n} \Sigma$ and $p = (10^{n} - 1) \Sigma \therefore \Sigma = \frac{p}{10^{n} - 1}$.

Also $q \cdot ppp$ &c. $= 10^{m}\,\text{s} \therefore q + \frac{p}{10^{n} - 1} = 10^{m}\,\text{s}$

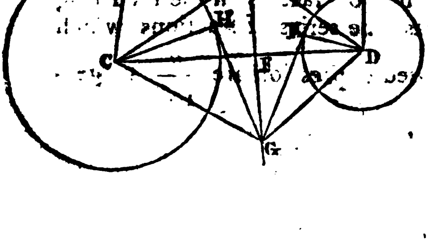
$\therefore \text{s} = \frac{q}{10^{m}} + \frac{p}{10^{m}(10^{n} - 1)} = \frac{q \,.\, (10^{n} - 1) + p}{10^{m}(10^{n} - 1)}$.

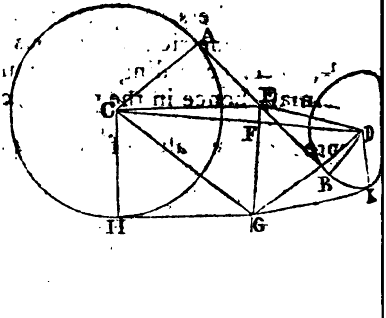
IV. QUESTION 454, *by* A. D.

AB is a common tangent to two circles AH, BK, bisected in E. EF is perpendicular to the line of the centres; then if any point G is taken in EF, the tangents GH, GK are equal. Required the investigation?

First Solution, *by* Funidus.

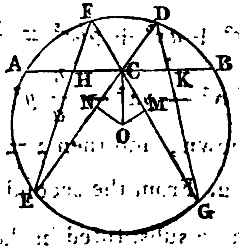
From the centres draw the radii CA, DB to the touching points A and B; draw also CH and DK from the centres to the touching points H and K; join CE, DE, GC, GD. Then it is obvious that





SOLUTION, *by* A. D. *the Proposer.*

Let AC $= a$, CD $= p$, CG $= q$, $\angle$ BCD $= \theta$, $\angle$ BCG $= \phi$, CK $= x$. Then



Δ DCG $= \Delta$ DCK + KCG;

$\therefore pq \sin(\theta + \phi) = px \sin\theta + qx \sin\phi$,

$$\therefore x = \frac{pq \sin(\theta + \phi)}{p \sin\theta + q \sin\phi} \ldots\ldots (1).$$

Similarly, let CE $= p'$, CF $= q'$, CH $= x'$, then

$$x' = \frac{p'q' \sin(\theta + \phi)}{p' \sin\theta + q' \sin\phi}:$$

but $pp' = qq' = a^2$,

$$\therefore x' = \frac{a^2 \sin(\theta + \phi)}{q \sin\theta + p \sin\phi},$$

$$\text{and } x = \frac{a^2 \sin(\theta + \phi)}{q' \sin\theta + p' \sin\phi}.$$

From the centre O drop OM, ON perpendiculars on FG, ED.

$$\text{then OC} = \frac{\text{CN}}{\cos \text{OCN}} = \frac{\text{CM}}{\cos \text{OCM}},$$

$\therefore$ since OCN, OCM, are complements of θ and ϕ,

CN $\sin\phi =$ CM $\sin\theta$, or $(p' - p) \sin\phi = (q - q') \sin\theta$,

or $p \sin\phi + q \sin\theta = p' \sin\phi + q' \sin\theta$;

$\therefore x' = x$. A geometrical demonstration is given in the Repository, Vol. I. Part I. pa. 49, where Mr. W. Wallace has extended the property to the Conic Sections.

VI. QUESTION 456, *by the Rev. Mr.* W. CALLOW.

Solve the following equations.

1. Given to find x and z the two equations

$$z - ax = x^4 + 3b, \text{ and } \frac{az^4}{x} = \left(\frac{b}{x^2}\right)^3.$$

2. Given to find x and z the two equations

$$z + ab \cdot \left(\frac{b}{x}\right)^3 = 5x - 5b, \text{ and } abz = \overline{1 + a} \cdot x^2.$$

3. Given to find x, y and z, the three equations

$$x^{12} + ax^9 y^3 + 3bx^7 y^5 = \frac{b^3}{a} y^{12}$$

$$x^4 + xz = 5cx^2 - 5c^2$$

$$\text{and} \qquad y^2 z = d.$$

SOLUTION, *by the Rev. Mr.* W. CALLOW, *the Proposer.*

1. From the first equation we find $z = x^4 + ax + 3b$ which value substituted in the second we have

$x^9 + ax^6 + 3bx^5 = \frac{b^3}{a}$: the roots of which equation are those of $x^3 - yx^2 - \frac{b}{a}y^2 = 0$, where $y = \sqrt{-a}$; x therefore is known; and then $z = x^4 + ax + 3b$ is also known.

2. From the second equation we get $z = \frac{1+a}{ab}x^2$, which value substituted in the first we have
$(1+a) . x^5 - 5abx^4 + 5ab^2x^3 + a^2b^3 = 0$, the roots of which equation are $x = \frac{b\sqrt[5]{a^2}}{\sqrt[5]{-a} - 1}$, then x being known we have

$z = \frac{1+a}{ab}x^2$.

3. Put $x = ry$, then the first equation when divided by y^{12} becomes $r^{12} + ar^9 + 3br^7 = \frac{b^3}{a}$: the roots of which are those

of $r^4 - ur^3 - \frac{b}{a}u^2 = 0$, where $u = \sqrt[3]{-a}$; r therefore is known, and since $y = \frac{x}{r}$ our third equation becomes $z = \frac{dr^2}{x^2}$; substituting the value of z in our second we have
$x^5 - 5cx^3 + 5c^2x + dr^2 = 0$, whence $x = \frac{1}{s} + cs$ where $s = \sqrt[5]{-t}$, being a root of $t^2 - \frac{dr^2}{c^5}t + \frac{1}{c^5} = 0$.

Then x being known, $y = \frac{x}{r}$, and $z = \frac{dr^2}{x^2}$ are also known.

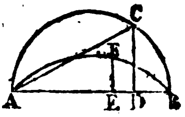
VII. QUESTION 457, *by Mr.* JOHN BUTTERWORTH.

Let ACB be a given semicircle, CD any ordinate, and let AC be joined. It is required to determine the quadrature of the curve which is the *locus* of the centre of the circle inscribed in the triangle ACD.

FIRST SOLUTION, *by* Mr. BUTTERWORTH, *the Proposer.*

From F the centre of the inscribed circle, demit FE perpendicular to AD in E; put AB $= a$ and AD $= x$, and we shall have AC $= \sqrt{(ax)}$, and DC $= \sqrt{(ax - x^2)}$;

$\therefore x + \sqrt{(ax-x^2)} - \sqrt{(ax)} = 2$DE $= 2$EF, consequently



AE $= \frac{x}{2} - \frac{\sqrt{(ax-x^2)}}{2} + \frac{\sqrt{(ax)}}{2}$, the fluxion of which is

$\frac{\dot{x}}{2} - \frac{\frac{1}{2}a\dot{x} - x\dot{x}}{2\sqrt{(ax-x^2)}} + \frac{\sqrt{a}\,\dot{x}}{4\sqrt{x}}$: Hence the fluxion of the area is

$$\left\{\frac{x}{2} + \frac{\sqrt{(ax-x^2)}}{2} - \frac{\sqrt{(ax)}}{2}\right\} \cdot \left\{\frac{\dot{x}}{2} - \frac{\frac{1}{2}a\dot{x} - x\dot{x}}{2\sqrt{(ax-x^2)}} + \frac{\sqrt{a}\,\dot{x}}{4\sqrt{x}}\right\}$$

$$= \frac{x\dot{x}}{4} - \frac{\frac{1}{2}ax\dot{x} - x^2\dot{x}}{4\sqrt{(ax-x^2)}} + \frac{\sqrt{(ax)}\dot{x}}{8} + \frac{\dot{x}\sqrt{(ax-x^2)}}{4} - \frac{a\dot{x}}{8} + \frac{x\dot{x}}{4}$$

$$+ \frac{\sqrt{(a)}\dot{x}\sqrt{(a-x)}}{8} - \frac{\sqrt{(ax)}\dot{x}}{4} + \frac{\frac{1}{2}a^{\frac{3}{2}}\dot{x} - \sqrt{(a)}x\dot{x}}{4\sqrt{(a-x)}} - \frac{a\dot{x}}{8}$$

$$= \frac{x\dot{x}}{2} + \frac{\frac{1}{2}ax\dot{x} - x^2\dot{x}}{4\sqrt{(ax-x^2)}} - \frac{\sqrt{(ax)}\dot{x}}{8} + \frac{\dot{x}\sqrt{(ax-x^2)}}{4} - \frac{a\dot{x}}{4}$$

$$+ \frac{\sqrt{(a)}\,\dot{x}\sqrt{(a-x)}}{2} + \frac{a^{\frac{3}{2}}\dot{x}}{8\sqrt{(a-x)}} - \frac{\sqrt{(a)}\,x\dot{x}}{4\sqrt{(a-x)}};$$

the fluent of which is

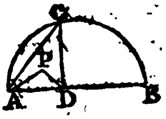
$$\frac{x^2}{4} - \frac{x\sqrt{(ax-x^2)}}{4} + \frac{s}{4} - \frac{\sqrt{(a)}\,x^{\frac{3}{2}}}{12} + \frac{s}{4} - \frac{ax}{4} -$$

$$\frac{\sqrt{(a)}(a-x)^{\frac{3}{2}}}{12} - \frac{a^{\frac{3}{2}}\sqrt{(a-x)}}{4} + \frac{\sqrt{(a)}\,x\sqrt{(a-x)}}{2} + \frac{\sqrt{(a)}(a-x)^{\frac{3}{2}}}{3}$$

$$= \frac{x^2}{4} - \frac{x\sqrt{(ax-x^2)}}{4} + \frac{s}{2} - \frac{\sqrt{(a)}\,(x^{\frac{3}{2}}+(a-x)^{\frac{3}{2}})}{12} - \frac{ax}{4} +$$

$\sqrt{(a-x)}\left(\frac{\sqrt{(a)}x}{2} - \frac{a^{\frac{3}{2}}}{4}\right) + \frac{\sqrt{(a)}\,(a-x)^{\frac{3}{2}}}{3}$ (s being put for the area of ACD) which is correct, for when $x = 0$ it vanishes, as it ought to do; but when $x = a$ it becomes $\frac{a^2}{4} + Q - \frac{a^2}{12} - \frac{a^2}{4} = Q - \frac{a^2}{12}$ the area of the curve AFBA, where Q denotes a quadrant of the semicircle ACB.

SECOND SOLUTION, *by Mr.* JOHN BAINES.



Let AP, DP, bisecting the angles CAD, ADC, meet in P; then P is the centre of the circle inscribed in the triangle ACD. Put AB $= a$, and the angle PAD $= \theta$. By trigonometry AC $= a \cos 2\theta$, and AD $= a \cos^2 2\theta$. Now, the angle ADP $=$ 45°, and consequently ∠ APD $= 135^\circ - \theta$; $\sin(135^\circ - \theta)$: AD :: $\sin 45^\circ$: AP $= \dfrac{a \cos^2 2\theta}{\cos\theta + \sin\theta}$ $= a \cos 2\theta \times (\cos\theta - \sin\theta)$, the polar equation of the curve, which evidently passes through A and B, for when $\theta = 45^\circ$ and 0°, AP $= 0$ and AB.

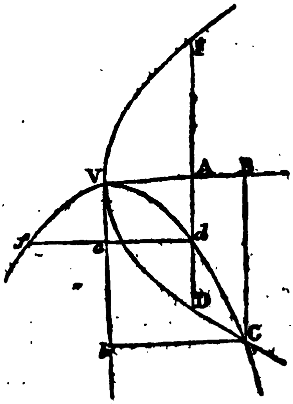
Quadrature. $\int \frac{1}{2}\,\mathrm{AP}^2 \times d\theta = \frac{1}{2}a^2 \int d\theta\,(\cos\theta - \sin\theta)^2 \cos^2 2\theta =$ $\frac{1}{2}a^2 \int d\theta \cos^2 2\theta\,(1 - \sin 2\theta) = \frac{1}{2}a^2 \int d\theta\,(1 - \sin 2\theta - \sin^2 2\theta + \sin^3 2\theta)$ $= \frac{1}{2}a^2 \int d\theta\,(-\frac{1}{4}\sin 6\theta + \frac{1}{2}\cos 4\theta - \frac{1}{4}\sin 2\theta + \frac{1}{2}) = \frac{1}{2}a^2\,(\frac{1}{24}\cos 6\theta$ $+ \frac{1}{8}\sin 4\theta + \frac{1}{8}\cos 2\theta + \frac{1}{2}\theta) = a^2 \left(\frac{\pi}{16} - \frac{1}{12}\right) = \cdot 309366a^2 =$ the area, taken between $\theta = 0^\circ$ and $\theta = 45^\circ$.

VIII. QUESTION 498, *by the Rev. Mr.* L. EVANS.

It is recorded that Menechmus was one of those ancient Mathematicians who solved the Delian Problem by two methods; one of which was, by the construction of two parabolas, having one common summit, with their axes perpendicular to each other, and for their respective parameters the side of the given cube and double that side; then the two ordinates drawn to the point of intersection of the two curves are two mean proportionals between those parameters, the first of which, Menechmus discovered would be the side of the cube sought. Required a demonstration. See Bossut's General History of Mathematics, whence the above originated.

SOLUTION, *by the late Rev. Mr.* L. EVANS, *the Proposer.*

Let *fvdo* and EVBC be two parabolas, whose axes are at right angles and having one common vertex at V, with their parameters fd and FD, the one, namely, fd = side of the given cube, and the other FD = double that side. Draw BC, bC, two ordinates to the point of intersection of the two curves. Now, we are to prove in the first place, that $fd : b\text{C} :: \text{BC} : \text{DF}$.



Since, by the construction, a and A are the two foci of the respective parabolas, $\text{V}a = \frac{1}{4}df$, $df = \text{AD}$, $\text{V}b = \text{BC}$, $\text{VB} = b\text{C}$,

Then, by the property of the parabolas

$$\text{V}a : ad^2 :: \text{V}b : b\text{C}^2$$

$$\text{VA} : \text{AD}^2 :: \text{VB} : \text{BC}^2,$$

and by taking the equals

$$\tfrac{1}{4}df : \tfrac{1}{4}df^2 :: \text{BC} : b\text{C}^2$$

$$\tfrac{1}{4}\text{DF} : \tfrac{1}{4}\text{DF}^2 :: b\text{C} : \text{BC}^2$$

or

$$1 : df :: \text{BC} : b\text{C}^2$$

$$1 : \text{DF} :: b\text{C} : \text{BC}^2$$

$$1 : df . \text{DF} :: \text{BC} . b\text{C} : b\text{C}^2\text{BC}^2$$

$$1 : df . \text{DF} :: 1 : b\text{C} . \text{BC}$$

$$df . \text{DF} = b\text{C} . \text{BC}, \text{ therefore}$$

$$df : b\text{C} :: \text{BC} : \text{DF}.$$

Now, to find whether bC or BC, the less or greater of these two means, is the true side of the cube sought, we proceed thus,

Find two mean proportionals between df and $2df$.

Put x = ratio, then the series will be df, $df . x$, $df . x^2$, $df . x^3$; this fourth term must be equal to $2df$, therefore $df . x^3 = 2df$, hence $x = \sqrt[3]{2}$.

Then the terms will be df, $df\sqrt[3]{2}$, $df\sqrt[3]{2^2}$ and $2df$, the second term being $df\sqrt[3]{2}$.

Again, if the side of a cube be df, what is the side of double this cube, in terms of df.

Put y = side of this double cube,

Then $y^3 = \overline{df}|^3 \times 2$ hence $y = df\sqrt[3]{2}$ which is exactly equal to the second term of the series before found. Therefore, bC the least of the above two means, is the true side of the cube sought.

Mr. BAINES *likewise answered this Question.*

IX. QUESTION 459, *by Mr.* JOHN BAINES.

Admit the matter of a planet, whose axis is to its equatorial diameter as 399 to 400, were in a state of fluidity at the commencement of rotation. Required the time of one revolution on the axis.

SOLUTION, *by Mr.* BAINES, *the Proposer.*

Denoting the axis and equatorial diameter of the planet by $399n$ and $400n$, the density of the earth to the density of the planet as $1 : r$, the earth's equatorial diameter $= 7935$ miles, by e, and the space fallen by a heavy body at the equator in the 1st second $= 16{\cdot}04$ feet, by g. we have by the principles of central forces, $\frac{e^3 \times 1}{e^2} : \frac{400^3n^3 \times r}{400^2n^2}$, or $e : 400nr ::$ the force of gravity at the earth's surface : the force of gravity at the planet's surface; hence, $e : 400nr :: g : \frac{400nrg}{e} =$ the space fallen by a heavy body in the first second at the planet's surface; because the space descended at the surfaces of different bodies is directly as the force of gravity at their surfaces. Again, by Simpson's Flux. Davis's ed. Art. 217, $3{\cdot}141592 \times \sqrt{\left(400n \times 5280 \times \frac{e}{400nrg}\right)} = 3{\cdot}141592 \sqrt{\frac{5280e}{gr}} = 5077{\cdot}4r^{-\frac{1}{2}}$ seconds $=$ the time in which a heavy body would revolve round the planet at its surface, equi-poising the force of gravity there. Put $x =$ the time, in seconds, in which the planet revolves on its axis, and $5077{\cdot}4r^{-\frac{1}{2}} = Ar^{-\frac{1}{2}}$; then, by Marrat's Mech. Art. 438, $\frac{1}{A^2r^{-1}} : \frac{1}{x^2}$, or $\frac{r}{A^2} : \frac{1}{x^2}$ expresses the ratio of the force of gravity at the planet's surface to the centrifugal force at the equator arising from the rotation on its axis. But the ratio of $\frac{r}{A^2} : \frac{1}{x^2}$ is the same as $1 : \frac{A^2}{rx^2}$; and, by Simpson's Flux. Art. 397, $\frac{5}{4}\left(\frac{A^2}{rx^2}\right) + \frac{125}{224}\left(\frac{A^4}{r^2x^4}\right) +$ &c. expresses the excess of the planet's equatorial above its axis, when the latter is unity. Hence, putting $\frac{5}{4}\left(\frac{A^2}{rx^2}\right) + \frac{125}{224}\left(\frac{A^4}{r^2x^4}\right) + \&c. = \frac{1}{399}$, we have $x = A\sqrt{\frac{1995 + 5\sqrt{159771}}{8r}} = \frac{22{\cdot}3427A}{r^{\frac{1}{2}}}$.

Ex. 1. If $r = 1$, or the density of the planet is equal to that of the earth's, $x = 113442\frac{1}{4}'' = 31^h\ 30^m\ 42\frac{1}{4}^s$, the time of rotation.

Ex. 2. If $r = 2$, $x = 80215\frac{3}{4}'' = 22^h\ 16^m\ 55\frac{3}{4}''$.

X. QUESTION 460, *by Mr.* CUNLIFFE, *R. M. College.*

Having a given, even, square integer; it is required to determine a certain number of integers, such, that if they are severally augmented or diminished by the given square, the several results shall all be rational squares.

SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

Let $4m^2$ denote the given square, and x one of the required integers. By the question we shall have

$$x + 4m^2 = a^2$$
$$x - 4m^2 = b^2$$

whence $x = \frac{a^2 + b^2}{2}$; and $a^2 - b^2 = 8m^2$. Put $a + b = \frac{4rm}{s}$, and $a - b = \frac{2sm}{r}$; then $2a = \frac{4rm}{s} + \frac{2sm}{r} = 2m \times \frac{(2r^2 + s^2)}{rs}$, $a = m \times \frac{(2r^2 + s^2)}{rs}$, and $b = m \times \frac{(2r^2 + s^2)}{rs}$.

Whence $x = \frac{a^2 + b^2}{2} = m^2 \times \frac{(4r^4 + s^4)}{r^2s^2}$; where r and s may be taken at pleasure.

Suppose $s = 1$, and let r, be successively expounded by 2, 3, 4, 6, 12, &c. then by a proper application of what has been deduced, we shall obtain $4m^2 = 576$ for the given square, or rather the square to be given, and 2340, 5200, 9225, 20740, and 82945 for four integers of the kind required. That is to say, if each of them be augmented or diminished by 576, the several results will all be rational squares.

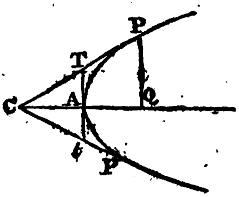
We may also add 1105 and 1360, to the foregoing numbers; the first of which is found by taking $r = 4$ and $s = 3$, and the other by taking $r = 3$ and $s = 2$, and then we shall have 1105, 1360, 2340, 5200, 9225, 20740, and 82945 for seven numbers that will answer.

XL, QUESTION 461, *by Mr.* Noswit.

Suppose T and *t* to be parts of the tangent at the vertex A of a rectangular hyperbola (whose semi-axis = 1) cut off by lines CP, C*p*, drawn from the centre to the curve; shew that if sector CAP = (*n*) times sector CA*p*,

$$\frac{1-T}{1+T} = \left\{\frac{1-t}{1+t}\right\}^n.$$

First Solution, *by the late Mr.* Anthony Cook.



Draw PQ perpendicular to CA produced, and put CQ $= x$, then PQ $= \sqrt{x^2-1}$, and the fluxion of the semiseg. APQ $=$ PQ $.\ \dot{\text{AQ}} =$ PQ $.\ \dot{\text{CQ}} = \dot{x}\sqrt{x^2-1}$; also area $\triangle$ CQP $= \frac{1}{2}$ CQ QP $= \frac{1}{2}x\sqrt{x^2-1}$. Whence the area of the sector CAP $= \triangle$ CQP $-$ semiseg. AQP $= \frac{1}{2}x\sqrt{x^2-1} - \int \dot{x}\sqrt{x^2-1} = \int\left\{\frac{2x^2\dot{x}-\dot{x}}{2\sqrt{x^2-1}} - \dot{x}\sqrt{x^2-1}\right\}$

$$= \int \frac{\dot{x}}{2\sqrt{x^2-1}}.$$

By similar triangles, CA $= 1$: AT $=$ T :: CQ $= x$: PQ $= \sqrt{x^2-1}$.

$\therefore \text{T}x = \sqrt{x^2-1}$, hence $x^2 = \frac{1}{1-\text{T}^2}$, $x^2-1 = \frac{\text{T}^2}{1-\text{T}^2}$,

$x = \frac{1}{\sqrt{1-\text{T}^2}}$, and $\dot{x} = \frac{\text{T}\dot{\text{T}}}{(1-\text{T}^2)^{\frac{3}{2}}}$. Consequently by substitution, we have area of the sector CAP $= \int \frac{\dot{\text{T}}}{2(1-\text{T}^2)} =$ $\frac{1}{4} \times$ h. l. $\frac{1+\text{T}}{1-\text{T}}$.

In like manner it is shewn that the area of the sector CA*p* $= \frac{1}{4} \times$ h. l. $\frac{1+t}{1-t}$. Therefore if the sector CAP $= n \times$ sector CA*p*, we shall have h. l. $\frac{1+T}{1-T} = n \times$ h. l. $\frac{1+t}{1-t}$.

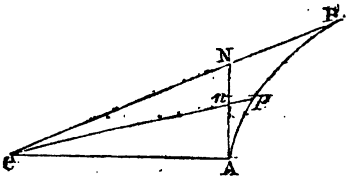
And $\therefore \frac{1+T}{1-T} = \left(\frac{1+t}{1-t}\right)^n$, or $\frac{1-T}{1+T} = \left(\frac{1-t}{1+t}\right)^n$. *Q. E. D.*

And thus the question was answered by FUNIDUS.

SECOND SOLUTION, *by* S. F.

Let ApP be the hyperbola, AN the tangent at A and let CP, Cp cut AN in N and n.

Then if $CP = q$, $Cp = q'$, $ACP = \theta$, $ACp = \theta'$, $AN = T$, $An = t$.



By the property of the rectangular hyperbola (semi-axis $= 1$)

$$q^2 = \frac{1}{\cos^2\theta - \sin^2\theta},$$

and $ACP = \int q^2 d\theta = \int \frac{d\theta}{\cos^2\theta - \sin^2\theta} = \int \frac{d(\tan\theta)}{1 - \tan^2\theta} =$ $\frac{1}{2}$ h. l. $\frac{1-\tan\theta}{1+\tan\theta} = \frac{1}{2}$ h. l. $\frac{1-T}{1+T}$.

No correction is necessary, because when $\theta = 0$, $ACP = 0$, and h. l. $1 = 0$.

In the same way $ACp = \frac{1}{2}$ h. l. $\frac{1-t}{1+t}$.

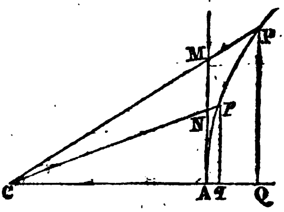
But $ACP = n \times ACp$

$\therefore \frac{1}{2}$ h. l. $\frac{1-T}{1+T} = \frac{n}{2}$ h. l. $\frac{1-t}{1+t}$ $\therefore$ h. l. $\frac{1-T}{1+T} = n \times$ h. l. $\frac{1-t}{1+t}$.

$\therefore$ h. l. $\frac{1-T}{1+T} =$ h. l. $\left(\frac{1-t}{1+t}\right)^n$ $\therefore \frac{1-T}{1+T} = \left(\frac{1-t}{1+t}\right)^n$.

THIRD SOLUTION, *by the Proposer.*

Let AP be the curve, C its centre, $x = CQ$, $y = PQ$. Then $y = \sqrt{x^2 - 1}$ is the equation to the curve.



Now $CPQ = \frac{xy}{2}$ $\therefore \dot{\overline{CPQ}} = \frac{x\dot{y} + y\dot{x}}{2}$;

$$\therefore \dot{\overline{\text{CAP}}} = \frac{x\dot{y} + y\dot{x}}{2} - y\dot{x} = \frac{x\dot{y} - y\dot{x}}{2}.$$

$$\text{Now } \dot{y} = \frac{x\dot{x}}{\sqrt{x^2 - 1}} \;\therefore \dot{\overline{\text{CAP}}} = \frac{\dot{x}}{2\sqrt{x^2 - 1}};$$

$\therefore$ sector CAP $= \frac{1}{2}$ hyp. log. $x + \sqrt{x^2 - 1}$; wants no corr. Similarly, if $w =$ any abscissa Cq,

$$\text{sector CA}p = \tfrac{1}{2} \text{ hyp. log. } w + \sqrt{w^2 - 1},$$

and by supp. CAP $= (n)$ sector CAp;

$\therefore x + \sqrt{x^2 - 1} = (w + \sqrt{w^2 - 1})^n$ by the nature of logarithms, also from the nature of surds

$$x - \sqrt{x^2 - 1} = (w - \sqrt{w - 1})^n$$

Now from similar triangles

$x : \sqrt{x^2 - 1} :: 1 : \text{T} \;\therefore x - \sqrt{x^2 - 1} : x + \sqrt{x^2 - 1} :: 1 - \text{T} : 1 + \text{T}$

and

$w : \sqrt{w^2 - 1} :: 1 : t \;\therefore w - \sqrt{w^2 - 1} : w + \sqrt{w^2 - 1} :: 1 - t : 1 + t$

$$\therefore \frac{x - \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}} = \frac{1 - \text{T}}{1 + \text{T}}, \text{ and } \frac{w - \sqrt{w^2 - 1}}{w + \sqrt{w^2 - 1}} = \frac{1 - t}{1 + t},$$

$$\text{and, since } \frac{x - \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}} = \left(\frac{w - \sqrt{w^2 - 1}}{w + \sqrt{w^2 - 1}}\right)^n \therefore \frac{1 - \text{T}}{1 + \text{T}} = \left(\frac{1 - t}{1 + t}\right)^n.$$

XII. QUESTION 462, *by Mr.* THOS. WHITE, *Dumfries.*

Required the sum of the infinite series

$$1 + \frac{x}{a} + \frac{x^2}{b} + \frac{x^3}{c} + \frac{x^4}{d} + \text{\&c.}$$

converging ever so slowly, by a direct method purely algebraical?

SOLUTION, *by the Proposer.*

(*a*) Put s $=$ the required sum; then s $- 1 = \frac{x}{a} + \frac{x^2}{b} + \frac{x^3}{c} + \frac{x^4}{d} + \ldots =$ (suppose) to the seriesal

$$\frac{\text{A}x + \text{A}_1 x^2 + \text{A}_2 x^3 + \text{A}_3 x^4 + \ldots}{\frac{1}{\text{B}} - \frac{x}{\text{B}_1} + \frac{x^2}{\text{B}_2} - \frac{x^3}{\text{B}_3} + \frac{x^4}{\text{B}_4} - \ldots};$$

then (as is done in Prize Quest. Gent.'s Diary, 1820.) we shall find

$$A = \frac{1}{aB} \qquad A_2 = \frac{1}{cB} - \frac{1}{bB_1} + \frac{1}{aB_2}$$

$$A_1 = \frac{1}{bB} - \frac{1}{aB_1} \quad A_3 = \frac{1}{dB} - \frac{1}{cB_1} + \frac{1}{bB_2} - \frac{1}{aB_3}$$

$$A_4 = \frac{1}{eB} - \frac{1}{dB_1} + \frac{1}{cB_2} - \frac{1}{bB_3} + \frac{1}{aB_4}$$

$$A_5 = \frac{1}{fB} - \frac{1}{eB_1} + \frac{1}{dB_2} - \frac{1}{cB_3} + \frac{1}{bB_4} - \frac{1}{aB_5}$$

$$A_6 = \frac{1}{gB} - \frac{1}{fB_1} + \frac{1}{eB_2} - \frac{1}{dB_3} + \frac{1}{cB_4} - \frac{1}{bB_5} + \frac{1}{aB_6} \text{ \&c.}$$

The law is evident.

Having found A, A_1, A_2, ... in terms of B, B_1, B_2, which are yet to be determined; let us suppose $\alpha A_1 = A$ and find B_1; $\beta A_2 = A_1$, and find B_2; $\gamma A_3 = A_2$ and find B_3; $\delta A_4 = A_3$ and find B_4, &c. &c. and consider B and α, β, γ, δ, ... as whole numbers assumable, at pleasure, we shall find, by these equations

$$\frac{1}{B_1} = \frac{a}{B}\left(\frac{1}{b} - \frac{1}{a\alpha}\right);$$

$$\frac{1}{B_2} = \frac{a}{B_1}\left(\frac{1}{b} - \frac{1}{a\beta}\right) - \frac{a}{B}\left(\frac{1}{c} - \frac{1}{b\beta}\right);$$

$$\frac{1}{B_3} = \frac{a}{B_2}\left(\frac{1}{b} - \frac{1}{a\gamma}\right) - \frac{a}{B_1}\left(\frac{1}{c} - \frac{1}{b\gamma}\right) + \frac{a}{B}\left(\frac{1}{d} - \frac{1}{c\gamma}\right);$$

$$\frac{1}{B_4} = \frac{a}{B_3}\left(\frac{1}{b} - \frac{1}{a\delta}\right) - \frac{a}{B_2}\left(\frac{1}{c} - \frac{1}{b\delta}\right) + \frac{a}{B_1}\left(\frac{1}{d} - \frac{1}{c\delta}\right) - \frac{a}{B}\left(\frac{1}{e} - \frac{1}{d\delta}\right);$$

and so on; which, because a is always less, by hypothesis, than either b, or c, or d, or ...., are all *proper* fractions constituting the coefficients of the alternating series in the denominator; and the law of formation is obvious and singularly beautiful.

Moreover, because

$$A \text{ is} = \frac{1}{aB};\; A_2 = \frac{1}{aB} \cdot \frac{1}{\alpha\beta};\; A_4 = \frac{1}{aB} \cdot \frac{1}{\alpha\beta\gamma\delta};$$

we have

$$A_1 = \frac{1}{aB} \cdot \frac{1}{\alpha};\; A_3 = \frac{1}{aB} \cdot \frac{1}{\alpha\beta\gamma}; \qquad \text{\&c.}$$

Hence, the numerator of the seriesal fraction (when $x = 1$) is $= \frac{1}{aB}\left(1 + \frac{1}{\alpha} + \frac{1}{\alpha\beta} + \frac{1}{\alpha\beta\gamma} + \ldots\right)$ of which we command the convergency; and, by assuming α (any whole number we please) $= \beta = \gamma = \delta = \ldots$, it becomes a decreasing geometrical series.

Hence, the given series $\frac{x}{a}+\frac{x^2}{b}+\frac{x^3}{c}+\ldots\ldots(=s-1)$ is transformed to the seriesal fraction

$$\frac{1}{a^2\mathrm{B}}\cdot\frac{x+\frac{x^2}{a}+\frac{x^3}{\alpha\beta}+\frac{x^4}{\alpha\beta\gamma}+\frac{x^5}{\alpha\beta\gamma\delta}+\ldots\ldots}{\frac{1}{\mathrm{B}}\left(\frac{1}{a}\right)-\frac{1}{\mathrm{B}}\left(\frac{1}{b}-\frac{1}{a\alpha}\right)x+\left[\frac{1}{\mathrm{B}_1}\left(\frac{1}{b}-\frac{1}{a\beta}\right)-\frac{1}{\mathrm{B}}\left(\frac{1}{c}-\frac{1}{b\beta}\right)\right]x^2-\left[\frac{1}{\mathrm{B}_2}\left(\frac{1}{b}-\frac{1}{a\gamma}\right)-\frac{1}{\mathrm{B}_1}\left(\frac{1}{c}-\frac{1}{b\gamma}\right)+\frac{1}{\mathrm{B}}\left(\frac{1}{d}-\frac{1}{c\gamma}\right)\right]x^3+}$$

where B and $\alpha, \beta, \gamma, \ldots$ are whole numbers assumable at pleasure. If B be positive and $\alpha, \beta, \gamma, \ldots$ negative, the numerator will, like the denominator, be an alternating series; and thus an *affirmative* series may be made to depend on two *alternating* series, a thing hitherto unknown. It remains to sum, generally, any *alternating* converging infinite series $a-a_1+a_2-a_3+a_4-\ldots\ldots$

(b) Let s = required sum, then $s=(a-a_1)+(a_2-a_3)+(a_4-a_5)+(a_6-a_7)+\ldots\ldots;$

then $s-a$ is $=(a_1-a_2)-(a_3-a_4)-(a_5-a_6)-(a_7-a_8)-\ldots\ldots$

hence $2s-a$ = alternating series .... $(a-a_1)-(a_1-a_2)+(a_2-a_3)-(a_3-a_4)+\ldots\ldots=s_1$, suppose;

$=(a-2a_1+a_2)+(a_2-2a_3+a_4)+(a_4-2a_5+a_6)+\ldots\ldots$

Also, $s_1-(a-a_1)=-(a_1-2a_2+a_3)-(a_3-2a_4+a_5)-(a_5-2a_6+a_7)-\ldots\ldots$

hence $2s_1-(a-a_1)$ = alternating series ....

$(a-2a_1+a_2)-(a_1-2a_2+a_3)+(a_2-2a_3+a_4)-(a_3-2a_4+a_5)+\ldots=s_2$ (sup.)

$=(a-3a_1+3a_2-a_4)-(a_3-3a_4+3a_5-a_6)+(a_4-3a_5+3a_6-a_7)+\ldots\ldots$

Also $s_2-(a-2a_1+a_2)=-(a_1-3a_2+3a_3-a_4)-(a_3-3a_4+3a_5-a_6)-\ldots$ hence $2s_2-(a-2a_1+a_2)=$ alternating series ...... $=s_3$, supp. &c. &c.

Hence $2s - a = s_1$

$$2s_1 - (a - a_1) = s_2$$

$$2s_2 - (a - 2a_1 + a_2) = s_3$$

$$2s_3 - (a - 3a_1 + 3a_2 - a_3) = s_4$$

$$2s_4 - (a - 4a_1 + 6a_2 - 4a_3 + a_4) = s_5$$

&c. hence

$$s = \frac{a}{2} + \frac{s_1}{2}$$

$$s_1 = \frac{a - a_1}{2} + \frac{s_2}{2}$$

$$s_2 = \frac{a - 2a_1 + a_2}{2} + \frac{s_3}{2}$$

$$s_3 = \frac{a - 3a_1 + 3a_2 - a_3}{2} + \frac{s_4}{2}$$

$$s_4 = \frac{a - 4a_1 + 6a_2 - 4a_3 + a_4}{2} + \frac{s_5}{2}$$

&c. consequently

$$s = \frac{a}{2} + \frac{s_1}{2}; \ldots\ldots \text{ 1st approximation.}$$

$$= \frac{a}{2} + \frac{a - a_1}{4} + \frac{s_2}{2}; \ldots\ldots \text{ 2nd.}$$

$$= \frac{a}{2} + \frac{a - a_1}{4} + \frac{a - 2a_1 + a_2}{8} + \frac{s_3}{x}; \ldots\ldots \text{ 3rd.}$$

$$= \frac{a}{2} + \frac{a - a_1}{4} + \frac{a - 2a_1 + a_2}{8} + \frac{a - 3a_1 + 3a_2 - a_3}{16} + \frac{s_4}{16}; \ldots\ldots \text{ 4th.}$$

$$= \frac{a}{2} + \frac{a - a_1}{4} + \frac{a - 2a_1 + a_2}{8} + \frac{a - 3a_1 + 3a_2 - a_3}{16} + \frac{a - 4a_1 + 6a_2 - 4a_3 + a_4}{32} + \frac{s_5}{32};\ \text{5th.}$$

&c. &c.

If either S_3, S_4, or .... vanish, s is finite. This is the well known case of the 3rd, 4th, ... differences being $=0$; and follows as a cor. to the above. All is thus reduced to the summation of an alternating series. To exemplify;

Let the sum s of the series $\frac{1^a}{b} - \frac{1^{a_1}}{b+c} + \frac{1^{a_2}}{b+2c} - \frac{1^{a_3}}{b+3c} +$ be required, we have ....

I. $s = \frac{1}{2b} + \frac{2c^2}{2}\left[\frac{1\,(A)}{b\cdot b+c\cdot b+2c} + \frac{1\,(B)}{b+c\cdot b+3c\cdot b+4c} + \frac{1\,(C)}{b+4c\cdot b+5c\cdot b+6c} + \ldots\ldots\right]$ first approximation.

II. $s = \frac{1}{2b} + \frac{c}{4b\cdot b+c} + \frac{2\cdot 3c^2}{4}\left[\frac{1\,(A_1)}{b\cdot b+b\cdot b+2c\cdot b+3c} + \frac{1\,(B_1)}{b+2c\cdot b+3c\cdot b+4c\cdot b+5c} + \ldots\ldots\right]$ second approximation.

III. $s = \frac{1}{2b} + \frac{c}{4b\cdot b+c} + \frac{2c^2}{8b\cdot b+c\cdot b+2c} + \frac{2\cdot 3\cdot 4c^4}{8}\left[\frac{1\,(A_2)}{b\cdot b+c\cdot b+2c\cdot b+3c\cdot b+4c} + \frac{1\,(B_2)}{b+2c\cdot b+3c\cdot b+4c\cdot b+5c\cdot b+6c} + \ldots\right]$ third approximation. Put $\frac{1}{2b} + \frac{c}{4b\cdot b+c} + \frac{2c^2}{8b\cdot b+c\cdot b+2c} = m$;

IV. $s = m + \frac{6c^3}{16b\cdot b+c\cdot b+2c\cdot b+3c} + \frac{2\cdot 3\cdot 4\cdot 5\cdot 6c^5}{16}\left[\frac{1\,(A_3)}{b\cdot b+c\cdot b+2c\cdot b+3c\cdot b+4c\cdot b+5c} + \frac{1\,(B_3)}{b+2c\cdot b+3c\cdot b+4c\cdot b+5c\cdot b+6c\cdot b+7c} + \ldots\right]$ fourth approximation. Put $m + \frac{6c^3}{16b\cdot \ldots\ldots\ b+3c} = n$;

V. $s = n + \frac{24c^4}{32b\cdot \ldots\ldots\ b+4c} + \frac{2\cdot 3\cdot 4\cdot 5\cdot 6c^6}{32}\left[\frac{1\,(A_4)}{b\cdot b+c\cdot \ldots\ldots\ldots\ \cdot b+5c\cdot b+6c} + \ldots\ldots\right]$ a fifth approximation, &c. &c.

$$\begin{aligned}
\text{Now } (\text{A}) &= \frac{1}{b \cdot b+c \cdot b+2c} \\
(\text{B}) &= \frac{b \cdot b+c}{b+3c \cdot b+4c} \cdot (\text{A}) \\
(\text{C}) &= (\text{B}) \cdot \frac{b+2c \cdot b+3c}{b+5c \cdot b+6c} \\
(\text{D}) &= (\text{C}) \cdot \frac{b+4c \cdot b+5c}{b+7c \cdot b+8c}
\end{aligned}$$

&c.

$$\begin{aligned}
(\text{A}_1) &= \frac{(\text{A})}{b+3c} \\
(\text{B}_1) &= (\text{A}_1) \cdot \frac{b \cdot b+e}{b+4c \cdot b+5c} \\
(\text{C}_1) &= (\text{B}_1) \cdot \frac{b+2c \cdot b+3c}{b+6c \cdot b+7c} \\
(\text{D}_1) &= (\text{C}_1) \cdot \frac{b+4c \cdot b+5c}{b+8c \cdot b+9c}
\end{aligned}$$

&c.

$$\begin{aligned}
(\text{A}_2) &= \frac{(\text{A}_1)}{b+4c} \\
(\text{B}_2) &= (\text{A}_2) \cdot \frac{b \cdot b+c}{b+5c \cdot b+6c} \\
(\text{C}_2) &= (\text{B}_2) \cdot \frac{b+2c \cdot b+3c}{b+7c \cdot b+8c} \\
(\text{D}_2) &= (\text{C}_2) \cdot \frac{b+4c \cdot b+5c}{b+9c \cdot b+10c}
\end{aligned}$$

&c.

$$\begin{aligned}
(\text{A}_3) &= \frac{(\text{A}_2)}{b+5c} \\
(\text{B}_3) &= (\text{A}_3) \cdot \frac{b \cdot b+c}{b+6c \cdot b+7c} \\
(\text{C}_3) &= (\text{B}_3) \cdot \frac{b+2c \cdot b+3c}{b+8c \cdot b+9c} \\
(\text{D}_3) &= (\text{C}_3) \cdot \frac{b+4c \cdot b+5c}{b+10c \cdot b+11c}
\end{aligned}$$

&c.

$$\begin{aligned}
(\text{A}_4) &= \frac{(\text{A}_3)}{b+6c} \\
(\text{B}_4) &= (\text{A}_4) \cdot \frac{b \cdot b+c}{b+7c \cdot b+8c} \\
(\text{C}_4) &= (\text{B}_4) \cdot \frac{b+2c \cdot b+3c}{b+9c \cdot b+10c} \\
(\text{D}_4) &= (\text{C}_4) \cdot \frac{b+4c \cdot b+5c}{b+11c \cdot b+12c}
\end{aligned}$$

&c.

If $b = 1$ &c. $= 1$, we have $s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$ …… a series of such sluggish convergency that, it has occupied some of the most celebrated Mathematicians; and, as far as I recollect, none of their results have the property of such rapid convergency as those above given: But your limits do not admit of my exemplifying.

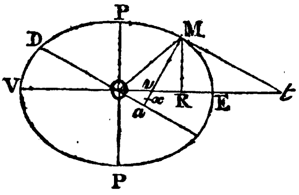
XIII. QUESTION 463, *by Mr.* ADLINGFLEET.

If D and D′ be the lengths of a degree of a meridian at the equator and at the latitude λ respectively; (*a*) and (*b*) the equatorial and polar diameters:

$$\text{Then } \frac{a}{b} = \frac{\text{sine } \lambda}{\sqrt{\left\{\left(\frac{D}{D'}\right)^{\frac{2}{3}} - \cos^2 \lambda\right\}}}.$$

FIRST SOLUTION, *by Mr.* ADLINGFLEET, *the Proposer.*

Suppose P*p*, VE, the polar and equatorial diameters respectively, M a place on the meridian whose latitude = λ, M*t* a tangent, MR an ordinate, OD parallel to M*t*. Take M*xa* perpendicular to M*t*, and let M*x* = rad. of curvature of meridian at M.



Then by conics, $\text{M}x = \frac{\text{OD}^2}{\text{M}a}$ ∴

$\text{M}x \propto \frac{\text{OD}^2 \times \text{M}a^2}{\text{M}a^3} \propto \frac{1}{\text{M}a^3} \propto \text{M}v^3$ ∴ rad. of curvature ∝ $\text{M}v^3$

∴ length of a degree ∝ $\text{M}v^3$.

Now $\text{R}t = \frac{\text{OE}^2}{\text{OP}^2} \times \frac{\text{RM}^2}{\text{OR}}$ and $\text{R}t : \text{RM} :: \text{RM} : \text{R}v = \frac{\text{RM}^2}{\text{R}t}$

$= \text{RM}^2 \div \frac{\text{OE}^2}{\text{OP}^2} \times \frac{\text{RM}^2}{\text{OR}} = \frac{\text{OP}^2}{\text{OE}^2} \times \text{OR}$ ∴ $\text{R}v^2 = \frac{\text{OP}^4}{\text{OE}^4} \times \text{OR}^2$

$= \frac{\text{OP}^4}{\text{OE}^4} \times \frac{\text{OE}^2}{\text{OP}^2} \cdot (\text{PO}^2 - \text{RM}^2) = \frac{\text{OP}^4}{\text{OE}^2} - \frac{\text{OP}^2}{\text{OE}^2} \cdot \text{RM}^2.$

Now $\text{M}v : \text{MR} :: 1 : \sin\lambda$ ∴ $\text{MR} = \text{M}v \cdot \sin\lambda,$

$$\therefore \text{R}v^2 = \frac{\text{OP}^4}{\text{OE}^2} - \frac{\text{OP}^2}{\text{OE}^2} \cdot \text{M}v^2 \cdot \sin^2\lambda,$$

$$\therefore \text{M}v^2 = \frac{\text{OP}^4}{\text{OE}^2} - \frac{\text{OP}^2}{\text{OE}^2} \cdot \text{M}v^2 \cdot \sin^2\lambda + \text{M}v^2 \cdot \sin^2\lambda;$$

∴, by tranposition, $\text{M}v^2 = \frac{\text{OP}^4}{(1 - \sin^2\lambda) \cdot \text{OE}^2 + \text{OP}^2 \sin^2\lambda}.$

∴ length of a degree ∝ $\text{M}v^3 \propto \left\{\frac{1}{(1 - \sin^2\lambda)\,\text{OE}^2 + \text{OP}^2 \cdot \sin^2\lambda}\right\}^{\frac{3}{2}}$

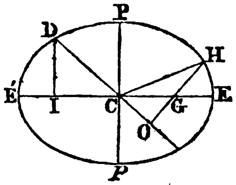
$$\therefore \text{D} : \text{D}' :: \left\{(1 - \sin^2\lambda) \cdot a^2 + b^2 \cdot \sin^2\lambda\right\}^{\frac{3}{2}} : (a)^{\frac{3}{2}},$$

$$\therefore \left(\frac{D}{D'}\right)^{\frac{2}{3}} = \frac{(1-\sin^2\lambda).a^2 + b^2.\sin^2\lambda}{a^2} = \cos^2\lambda + \frac{b^2}{a^2}\sin^2\lambda;$$

$$\therefore \frac{b^2}{a^2}.\sin^2\lambda = \left(\frac{D}{D'}\right)^{\frac{2}{3}} - \cos^2\lambda \therefore \frac{a}{b} = \frac{\sin.\lambda}{\sqrt{\left\{\left(\frac{D}{D'}\right)^{\frac{2}{3}} - \cos^2\lambda\right\}}}.$$

SECOND SOLUTION, *by* L.

Let P*p* represent the polar and E′E the equatorial diameters, and H a place on the meridian, whose latitude is λ. Join HC and draw DC semiconjugate to H meeting the radius of curvature at H in O, and draw the ordinate DI.



The lengths of the degrees of which the middle points are E and H, are, as is well known, as the radii of curvature at E and H, and these radii are as the cubes of the semidiameters CP, CD, (Hamilton's Conic Sections, V. B.) therefore D : D′ :: CP^3 : DP^3, and consequently

$$\left(\frac{D}{D'}\right)^{\frac{2}{3}} = \frac{CP^2}{CD^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, because the semidiameter CD is parallel to the tangent at H it is perpendicular to HO; wherefore the right angled triangles IDC, COG are equiangular, and consequently the angle IDC = DCP = CGO = HGC = λ, the latitude of H; hence CI = CD sin λ, DI = CD cos λ, EI = $\frac{a}{2}$ + CD cos λ, E′I = $\frac{a}{2}$ — CD cos λ, and, by the ellipse,

$$a^2 : b^2 :: EI \times E'I : DI^2,$$

$$\text{that is, } a^2 : b^2 :: \frac{a^2}{4} - CD^2\sin^2\lambda : CD^2\cos^2\lambda,$$

$$\text{and } a^2\,CD^2\cos^2\lambda = \frac{b^2a^2}{4} - b^2\,CD^2\sin^2\lambda.$$

Whence, by reduction,

$$\cos^2\lambda + \frac{b^2}{a^2}\sin^2\lambda = \frac{b^2}{4CD^2} = \frac{CP^2}{CD^2}.$$

But $\left(\frac{D}{D'}\right)^{\frac{2}{3}} = \frac{CP^2}{CD^2}$, (1), ... therefore

$$\cos^2\lambda + \frac{b^2}{a^2}\sin^2\lambda = \left(\frac{D}{D'}\right)^{\frac{2}{3}} \ldots\ldots\ldots\ldots\ldots (2).$$

And, by reduction,

$$\frac{a}{b} = \frac{\sin\lambda}{\sqrt{\left\{\left(\frac{D}{D'}\right)^{\frac{2}{3}} - \cos^2\lambda\right\}}} \text{ as required.}$$

In like manner if d be the length of a degree in latitude δ, we have

$$\cos^2\delta + \frac{b^2}{a^2}\sin^2\delta = \left(\frac{D}{d}\right)^{\frac{2}{3}} \ldots\ldots\ldots\ldots\ldots (3).$$

And eliminating D from equations (2), (3) we get

$$\left(\frac{d}{D'}\right)^{\frac{2}{3}} = \frac{\cos^2\lambda + \frac{b^2}{a^2}\sin^2\lambda}{\cos^2\delta + \frac{b^2}{a^2}\sin^2\delta},$$

or by reduction,

$$\left(\frac{d}{D'}\right)^{\frac{2}{3}} \times \cos^2\delta + \left(\frac{d}{D'}\right)^{\frac{2}{3}} \times \frac{b^2}{a^2}\sin^2\delta = \cos^2\lambda + \frac{b^2}{a^2}\sin^2\lambda.$$

$$\text{Whence } \frac{a}{b} = \sqrt{\left(\frac{\sin^2\lambda - \left(\frac{d}{D'}\right)^{\frac{2}{3}}\sin^2\delta}{\left(\frac{d}{D'}\right)^{\frac{2}{3}}\cos^2\delta - \cos^2\lambda}\right)}$$

the general expression for the ratio of the diameters where λ and δ are any given latitudes.

This question was also answered by Mr. JOHN BAINES *and the late Mr.* ANTHONY COOKE.

XIV. QUESTION 464, *by* A. D.

If any number of planes as P, Q, R, &c. are projected on the same plane, the sum of the projections, when a maximum, is equal to the square root of the following expression:

$P^2 + Q^2 + R^2 + \ldots + 2PQ\cos\widehat{PQ} + 2PR\cos\widehat{PR} + \ldots + 2QR\cos\widehat{QR} + \ldots\ldots$ Required proof.

FIRST SOLUTION, *by* A. D. *the Proposer*.

Let X be the plane of projection, then the sum required

$$S = P\cos\widehat{PX} + Q\cos\widehat{QX} + R\cos\widehat{RX} + \ldots,$$

Take any point, and through it draw three planes at right angles to each other, making with X, P, Q, R, ... the angles, x, y, z; a, b, c; a', b', c'; a'', b'', c''; .... respectively.

Then

$$\cos \widehat{PX} = \cos a \ \cos x + \cos b \ \cos y + \cos c \ \cos z$$
$$\cos \widehat{QX} = \cos a' \ \cos x + \cos b' \ \cos y + \cos c' \ \cos z$$
$$\cos \widehat{RX} = \cos a'' \cos x + \cos b'' \cos y + \cos c'' \cos z$$
&c.

hence

$$S = L \cos x + M \cos y + N \cos z \ldots\ldots(1), \text{ assuming}$$
$$\begin{aligned} L &= P \cos a + Q \cos a' + R \cos a'' + \ldots\ldots \\ M &= P \cos b + Q \cos b' + R \cos b'' + \ldots \qquad (2) \\ N &= P \cos c + Q \cos c' + R \cos c'' + \ldots \end{aligned}$$

and by differentiation

$$dS = - L \sin x dx - M \sin y dy - N \sin z dz.$$

We have also

$$\cos x^2 + \cos y^2 + \cos z^2 = 1$$

whence

$$2 \cos x \sin x dx + 2 \cos y \sin y dy + 2 \cos z \sin z dz = 0.$$

Eliminating dz, the conditions for a maximum value of S give $\frac{dS}{dx} = 0$, $\frac{dS}{dy} = 0$, that is

$$L \cos z = N \cos x$$
$$M \cos z = N \cos y.$$

Hence

$$(L^2 + M^2 + N^2) \cos z^2 = N^2 (\cos x^2 + \cos y^2 + \cos z^2) = N^2.$$

And

$$\cos z = \frac{N}{\sqrt{(L^2 + M^2 + N^2)}}$$
$$\cos y = \frac{M}{\sqrt{(L^2 + M^2 + N^2)}}$$
$$\cos x = \frac{L}{\sqrt{(L^2 + M^2 + N^2)}}.$$

Now

$$S = L \cos x + M \cos y + N \cos z$$
$$= \frac{L^2 + M^2 + N^2}{\sqrt{(L^2 + M^2 + N^2)}} = \sqrt{(L^2 + M^2 + N^2)}.$$

If we now revert to equations (2), observing that

$$\cos a^2 + \cos b^2 + \cos c^2 = 1$$
$$\cos a'^2 + \cos b'^2 + \cos c'^2 = 1$$
&c.

$$\cos a \cos a' + \cos b \cos b' + \cos c \cos c' = \cos \widehat{PQ}$$

$$\cos a \cos a'' + \cos b \cos b'' + \cos c \cos c'' = \cos \widehat{PR}$$

&c.

we shall find

$$L^2 + M^2 + N^2 \text{ or } S^2 = P^2 + Q^2 + R^2 + 2PQ \cos \widehat{PQ} + 2PR \cos \widehat{PR} + 2QR \cos \widehat{QR} + \&c.$$

SECOND SOLUTION, *by Mr.* BONNYCASTLE.

Let the sum of the areas when referred to three rectangular planes, be respectively A, B, and C; and, let also, the angles which these co-ordinate planes make with a fourth plane be a, b, c; then the projections of the areas upon this last will be equal to

$$A \,.\, \cos a + B \,.\, \cos b + C \,.\, \cos c. \;\ldots\ldots (1).$$

And since this expression is to be a maximum, we have the equation

$$A \,.\, d \,.\, \cos a + B \,.\, d \,.\, \cos b + C \,.\, d \,.\, \cos c = 0 \;\ldots. (2).$$

But as the quantities a, b, and c are connected by the equation

$$\cos^2 a + \cos^2 b + \cos^2 c = 1,$$

it will be necessary to eliminate, by means of this expression, one of the variables from (2); let us fix upon c, and, the operation being performed, there arises

$$(A \cos c - C \cos a)\, d \,.\, \cos a + (B \cos c - C \cos b)\, d \,.\, \cos b = 0.$$

From which, and the identity $C \cos c = C \cos c$, we obtain the three equations

$$A \cos c - C \cos a = 0$$

$$B \cos c - C \cos b = 0$$

$$C \cos c - C \cos c = 0.$$

By squaring which expressions, and adding them together, here will be deduced the equation

$$C^2 = (A^2 + B^2 + C^2) \cos^2 c;$$

or

$$\cos c = \frac{C}{\sqrt{(A^2 + B^2 + C^2)}}.$$

And consequently,

$$\cos b = \frac{B}{\sqrt{(A^2 + B^2 + C^2)}}.$$

$$\cos a = \frac{A}{\sqrt{(A^2 + B^2 + C^2)}}.$$

Substituting which in the expression for the total sum of the areas, (1), it becomes

$$\surd(\text{A}^2 + \text{B}^2 + \text{C}^2).$$

That is to say the property of the plane we have found is such, that the square of the sum of the projections which the given areas make upon it, is equal to the sum of the squares of the total projections upon the co-ordinate planes: from which very elegant property we readily deduce the required theorem; for if the planes which contain the areas, make with the co-ordinate planes the angles $\alpha, \alpha', \alpha'', \beta, \beta', \beta'', \gamma, \gamma', \gamma''$, respectively; the values of A, B, C, will be

$$\begin{aligned} \text{A} &= \text{P}\cos\alpha + \text{Q}\cos\beta + \text{R}\cos\gamma, \\ \text{B} &= \text{P}\cos\alpha' + \text{Q}\cos\beta' + \text{R}\cos\gamma', \\ \text{C} &= \text{P}\cos\alpha'' + \text{Q}\cos\beta'' + \text{R}\cos\gamma''. \end{aligned}$$

The sum of the squares of which are

$$\begin{aligned} \text{A}^2+\text{B}^2+\text{C}^2 = \text{P}^2+\text{Q}^2+\text{R}^2 &+ 2\text{PQ}(\cos\alpha\cos\beta + \cos\alpha'\cos\beta' + \cos\alpha''\cos\beta'') \\ &+ 2\text{PR}(\cos\alpha\cos\gamma + \cos\alpha'\cos\gamma' + \cos\alpha''\cos\gamma'') \\ &+ \&\text{c}. \end{aligned}$$

And as the multiples of PQ, PR, &c. are the well known expressions for the cosines of the angles which the planes containing the areas P and Q, P and R, &c. make with each other; the above equation will reduce to

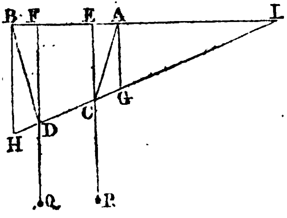
$$\text{A}^2 + \text{B}^2 + \text{C}^2 = \text{P}^2 + \text{Q}^2 + \text{R}^2 + 2\text{PQ}\cos\widehat{\text{PQ}} + 2\text{PR}\cos\widehat{\text{PR}} + \&\text{c}.$$

S. F: *answered it.*

XV. QUESTION 465, *by Mr.* CUNLIFFE.

Let two given weights P and Q, be fastened to the given points C and D in the string ACD; and suppose one end of the string to be fastened to a given point A, and another part of the string to pass over the pulley B, fixed at a given point B, in the horizontal line AB; it is required to find the nature or equation of the curve, which is the locus of D, while the part of the string DB, is slowly drawn over the pulley, the weights P and Q all the time hanging freely in the vertical plane ACDB.

SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*



Produce QD and PC, to meet AB, in F and E; also produce DC to meet BA in L; and draw BH and AG, parallel to DF.

Put AC $=a$, CD $=b$, AB $=c$, BF $=x$, FD $=y$, BL $=v$; then FL $=v-x$, DL $=\sqrt{(v-x)^2+y^2}=z$, CL $=x-b$, LA $=v-c$.

By reason of parallels

$$\text{LF} : \text{FD} :: \text{LA} : \text{AG} = y \times (v - c) \div (v - x),$$
$$\text{LF} : \text{FD} :: \text{LB} : \text{BH} = vy \div (v - x),$$
$$\text{LF} : \text{LD} :: \text{BF} : \text{HD} = xz \div (v - x),$$
$$\text{LD} : \text{DF} :: \text{LC} : \text{CE} = y \times (z - b) \div z,$$
$$\text{LD} : \text{LF} :: \text{DC} : \text{FE} = b \times (v - x) \div z, \text{ whence}$$
$$\text{BE} = x + b \times \frac{(v - x)}{z} = \frac{xz + b \times (v - x)}{z},$$
$$\text{EA} = \text{AB} - \text{BE} = \frac{z \times (c - x) - b \times (v - x)}{z}.$$

Moreover $\text{AC}^2 = \text{EA}^2 + \text{FE}^2$, that is,

$$a^2 = \frac{\{z \times (c - x) - b \times (v - x)\}^2 + y^2 \times (z - b)^2}{z^2};$$ which

expression after proper reduction gives

$$z = \sqrt{(v - x)^2 + y^2} = \frac{2b \times \{(c - x) \times (v - x) + y^2\}}{b^2 - a^2 + (c - x)^2 + y^2} \ldots (1).$$

Again, by reason of parallels,

$$\text{LF} : \text{LD} :: \text{EA} : \text{CG} = \frac{z \times (c - x) - b \times (v - x)}{v - x}.$$

The string, by the hypothesis, being drawn, very slowly, over the pulley B, the weights may be supposed to be in equilibrio, in every possible position.

By the principles of statics, and what has been deduced,

$$\text{AG} : \text{CG} :: p : p \times \frac{z \times (c - x) - b \times (v - x)}{y \times (v - c)} =$$ the force of P, in the direction CG.

Also $\text{BH} : \text{HD} :: q : \frac{qxz}{vy} =$ the force of Q, in the direction DH.

And these two forces, being in opposite directions, when the weights are in equilibrio, will be equal to each other,,

$$\text{that is, } p \times \frac{z \times (c - x) - b \times (v - x)}{y \times (v - c)} = \frac{qxz}{vy};$$

$$\text{whence } z = \frac{pbv \times (v - x)}{pv \times (c - x) - qx \times (v - c)} = \sqrt{(v - x)^2 + y^2} \ldots (2)$$

From the equations (1), and (2), taking away (exterminating) v, we shall then have the required equation of the curve, which is the locus of D, in terms of its own abscissa and ordinate.

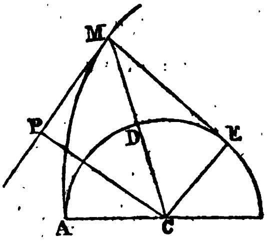
Mr. JOHN BAINES *answered it.*

XVI. QUESTION 466, *by Mr.* CUNLIFFE.

To find the nature of such a curve, that a tangent being drawn to any point thereof, and a perpendicular to the tangent, from a given point in the axis, the part of the tangent intercepted between the point of contact and the perpendicular, shall always be of the same given length.

SOLUTION, *by* L.

Let C be the given point, MA a portion of the required curve, and CP a perpendicular from C upon the tangent at M. Put $\varphi =$ the measure of the angle ACM, radius 1; $y =$ CM; $z =$ the arc AM and $a =$ PM.



Then $\dot{z} = \sqrt{(\dot{y}^2 + y^2\dot{\varphi}^2)}$, and $\dot{z} = \frac{y\dot{y}}{a}$;

wherefore $\sqrt{(\dot{y}^2 + y^2\dot{\varphi}^2)} = \frac{y\dot{y}}{a}$,

and by reduction

$$a\dot{\varphi} = \frac{\dot{y}}{y}\sqrt{(y^2 - a^2)}$$

or

$$a\dot{\varphi} + \frac{a^2\dot{y}}{y\sqrt{(y^2 - a^2)}} = \frac{y\dot{y}}{\sqrt{(y^2 - a^2)}}.$$

And taking the fluents

$$a\varphi + \text{arc, sec.}\, y\,(\text{rad}: a) = \sqrt{(y^2 - a^2)}.$$

Now when $\varphi = 0$, y is a; hence if CA be taken $= a$, the curve passes through A. From the centre C, with the radius CA, describe a circle ADE, and from M draw the tangent ME, then $a\varphi$ is $=$ the arc AD; arc whose secant is y and radius a is $=$ DE, and $\sqrt{(y^2 - a^2)} =$ ME; whence from the above equation,

arc AD + arc DE = ME,

or arc AE = tangent ME;

therefore the required curve MA is the involute of the circle ADE.

Messrs. COOKE *and* BAINES *answered this question.*

XVII. QUESTION 467, *by Mr.* CUNLIFFE.

Prove that the series

$$\frac{1}{(1+x)^2} \times \left\{1 + \frac{2x}{(1+x)^2} + \frac{4x^2}{(1+x)^4} + \frac{8x^3}{(1+x)^6} + \&c.\right\}$$

continued in infinitum is equal to the series

$$\frac{1}{(1-x)^2} \times \left\{1 - \frac{2x}{(1-x)^2} + \frac{4x^2}{(1-x)^4} - \frac{8x^3}{(1-x)^6} + \&c.\right\}$$

continued in infinitum: and find the sum of n leading terms of each series.

FIRST SOLUTION, *by the late Mr.* A. COOKE.

The series $1 + \frac{2x}{(1+x)^2} + \frac{4x^2}{(1+x)^4} + \frac{8x^3}{(1+x)^6} + \&c.$ is evidently in geometrical proportion, the first term being 1, and the common ratio $= \frac{2x}{(1+x)^2}$; consequently it is well known that the sum of n terms of the said series will be =

$$\left\{1 - \left(\frac{2x}{(1+x)^2}\right)^n\right\} \div \left\{1 - \frac{2x}{(1+x)^2}\right\}$$

and the sum of the same continued *in infinitum*

$$= (1+x)^2 \times \frac{1}{1+x^2}.$$

Again, the series $1 - \frac{2x}{(1-x)^2} + \frac{4x^2}{(1-x)^4} - \frac{8x^3}{(1-x)^6} + \&c.$ is also evidently in geometrical proportion, the common ratio being $= -\frac{2x}{(1-x)^2}$ and therefore the sum to n terms =

$$\left\{1 - \left(-\frac{2x}{(1-x)^2}\right)^n\right\} \div \left\{1 + \frac{2x}{(1-x)^2}\right\}$$

and the sum of the same continued *in infinitum* =

$$(1-x)^2 \times \frac{1}{1+x^2}.$$

Hence we evidently have

$$\frac{1}{(1+x)^2} \times \left\{1 + \frac{2x}{(1+x)^2} + \frac{4x^2}{(1+x)^4} + \&c. \text{ to } n \text{ terms}\right\} =$$

$$\frac{1}{(1+x)^2} \times \frac{1 - \left\{\frac{2x}{(1+x)^2}\right\}^n}{1 - \frac{2x}{(1+x)^2}} = \frac{1 - \left\{\frac{2x}{(1+x)^2}\right\}^n}{1+x^2}, \text{ and}$$

when n is infinite, this becomes $= \frac{1}{1+x^2}$.

Also, $\frac{1}{(1-x)^2} \times \left\{1 - \frac{2x}{(1-x)^2} + \frac{4x^2}{(1-x)^4} - \text{\&c. to } n \text{ terms}\right\} =$

$$\frac{1}{(1-x)^2} \times \left\{\frac{1 - \left\{-\frac{2x}{(1-x)^2}\right\}^n}{1 + \frac{2x}{(1-x)^2}}\right\} = \frac{1 - \left\{-\frac{2x}{(1-x)^2}\right\}^n}{1+x^2} \left(= \frac{1}{1+x^2}.\right.$$

when n is infinite).

Consequently the two series continued *in infinitum* are equal to each other.

SECOND SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

It is evident that $1+x^2=(1+x)^2-2x=(1+x)^2\left(1-\frac{2x}{(1+x)^2}\right)$.

$$\therefore \frac{1}{1+x^2} = \frac{1}{(1+x)^2} \cdot \frac{1}{1-\frac{2x}{(1+x)^2}} = \frac{1}{(1+x)^2} \cdot \left\{1 + \frac{2x}{(1+x)^2} + \frac{4x^2}{(1+x)^4} + \frac{8x^3}{(1+x)^6} + \text{\&c. in infinitum}\right\}.$$

And writing $-x$ for x,

$$\frac{1}{1+x^2} = \frac{1}{(1-x^2)} \cdot \left\{1 - \frac{2x}{(1-x)^2} + \frac{4x^2}{(1-x)^4} - \frac{8x^3}{(1-x)^6} + \text{\&c. } \textit{in infinit.}\right\}$$

Therefore the two expressions in the question are equal to each other.

We have found

$$\frac{1}{1+x^2} = \frac{1}{(1+x)^2} \cdot \left\{1 + \frac{2x}{(1+x)^2} + \frac{4x^2}{(1+x)^4} + \frac{8x^3}{(1+x)^6} + \text{\&c. } \textit{in infin.}\right\}$$

Whence $\frac{(1+x)^2}{1+x^2} = 1 + \frac{2x}{(1+x)^2} + \frac{4x^2}{(1+x)^4} + \frac{8x^3}{(1+x)^6}$ &c. *in infinit.* which put $= s$. Then we shall obviously have

$$s = 1 + \frac{2x}{(1+x)^2} + \frac{2^2x^2}{(1+x)^4} + \frac{2^3x^3}{(1+x)^6} + \frac{2^4x^4}{(1+x)^8} \text{ \&c. to } n \text{ terms,}$$

$$+ \frac{2^nx^n}{(1+x)^{2n}} \cdot \left\{1 + \frac{2x}{(1+x)^2} + \frac{2^2x^2}{(1+x)^4} + \frac{2^3x^3}{(1+x)^6} \text{\&c. } \textit{in infinitum}\right\}$$

$$= 1 + \frac{2x}{(1+x)^2} + \frac{2^2x^2}{(1+x)^4} + \frac{2^3x^3}{(1+x)^6} \text{ \&c. to } n \text{ terms} + \frac{2^nx^n}{(1+x)^{2n}} \times s;$$

whence, by transposition, $s \times \left\{1 - \frac{2^nx^n}{(1+x)^{2n}}\right\} = \frac{(1+x)^2}{1+x^2}$

$$\cdot \left\{1 - \frac{2^nx^n}{(1+x)^{2n}}\right\} = 1 + \frac{2x}{(1+x)^2} + \frac{2^2x^2}{(1+x)^4} + \frac{2^3x^3}{(1+x)^6} \text{ to } n \text{ terms,}$$

and therefore $\frac{1}{1+x^2} \cdot \left\{1 - \frac{2^n x^n}{(1+x)^{2n}}\right\}$

$= \frac{1}{(1+x)^2} \cdot \left\{1 + \frac{2x}{(1+x)^2} + \frac{2^2x^2}{(1+x)^4} + \frac{2^3x^3}{(1+x)^6} \text{ \&c.}\right\}$ to n terms.

Writing $-x$ for x in the preceding expression, it becomes

$$\frac{1}{1+x^2} \cdot \left\{1 \mp \frac{2^n x^n}{(1+x)^{2n}}\right\}$$

$$= \frac{1}{(1-x)^2} \cdot \left\{1 - \frac{2x}{(1-x)^2} + \frac{2^2x^2}{(1-x)^4} - \frac{2^3x^3}{(1-x)^6} + \frac{2^4x^4}{(1-x)^8} \text{ \&c.}\right\}$$

to n terms. The upper or lower sign takes place accordingly as n is even or odd.

Messrs. BAINES, BONNYCASTLE, *and* S. F. *answered this Question.*

XVIII. QUESTION 468, *by Mr.* NOSYT.

Required the ratio of the time of describing the parabolic arc AP to the time of falling down PS, in terms of SP and SA. A being the vertex and S the centre of force.

SOLUTION, *by the Proposer.*

Let PG be a normal and PO an ordinate. Then from the nature of the parabola SP = SG and GO = 2SA. Let v = vel. at A in the parabola; $m = 16\frac{1}{12}$ feet; T = time of describing arc AP; F = central force; π = circumference of a circle to radius 1.

Then $\frac{\text{SA} \times \text{T} \times v}{2}$ = area ASP. (Newt. Prin. Sect. 8.)

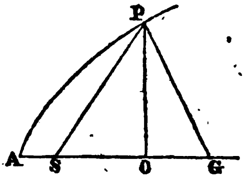
Now vel. at A : vel. in a circle, rad. SA :: $\sqrt{2} : 1$. (Sect 7.).

or vel. at A : $\sqrt{2m\text{F} \times \text{SA}} :: \sqrt{2} : 1$

∴ vel. at A in parab. $= 2\sqrt{m\text{F} \times \text{SA}}$,

∴ by substitution area ASP $= \text{SA}^{\frac{3}{2}} \times \text{T}\sqrt{m\text{F}}$;

$$\therefore \text{T} = \frac{\text{area ASP}}{\text{SA}^{\frac{3}{2}} \times \sqrt{m\text{F}}}.$$



Now, area ASP $= \text{PO} \times \frac{\text{AO} + 3\text{AS}}{6}$. (Newt. Prin. Sect. 6. lib. 1.)

$$= \sqrt{4\text{AS} \times \text{AO}} \times \frac{\text{AO} + 3\text{AS}}{6}, \text{ and}$$

$$\text{AO} = \text{AS} + \text{SO} = \text{AS} + \text{SG} - \text{GO} = \text{AS} + \text{SP} - 2\text{AS} = \text{SP} - \text{AS};$$

$$\therefore \text{area ASP} = \sqrt{4\text{SA} \times (\text{SP} - \text{AS})} \times \frac{\text{SP} + 2\text{AS}}{6};$$

$$\therefore \text{T} = \frac{\sqrt{4\text{SA} \times (\text{SP} - \text{AS})}}{\text{SA}^{\frac{3}{2}}} \times \frac{\text{SP} + 2\text{AS}}{6} \times \frac{1}{\sqrt{m\text{F}}}.$$

Again, time down PS $= \frac{1}{2}$ period. in a circle whose diam. = SP, (Sect. 7.). Now force $\propto \frac{1}{\text{D}^2}$

$$\therefore \text{F} : \text{force at P} :: \text{SP}^2 : \text{SA}^2; \therefore \text{force at P} = \text{F} \times \frac{\text{SA}^2}{\text{PS}^2}$$

$$\therefore \left\{\begin{matrix}\text{velocity}\\ \text{in circle}\end{matrix}\right\}^2 \text{rad. SP} = 4m\text{F} \cdot \frac{\text{SA}^2}{\text{SP}^2} \cdot \frac{\text{SP}}{4} (\text{sect. 2}) = m\text{F} \times \frac{\text{SA}^2}{\text{SP}};$$

$$\therefore \text{velocity in a circle, rad. SP} = \sqrt{\frac{m\text{F}}{\text{SP}}} \times \text{SA}.$$

$$\therefore \text{per. time in a} \odot\text{, rad. SP} = \frac{\text{circumf. of} \odot}{\text{velocity}} = \frac{\pi \cdot \text{SP}^{\frac{3}{2}}}{\text{SA}\sqrt{m\text{F}}};$$

$$\therefore \text{per. time, or time down PS} = \frac{\pi \cdot \text{SP}^{\frac{3}{2}}}{2\text{SA}\sqrt{m\text{F}}};$$

∴ $\frac{1}{2}$ time of describing parabolic arc AP : time of descent through PS $:: \frac{\sqrt{4\text{SA} \cdot (\text{PS} - \text{AS})}}{\text{SA}^{\frac{3}{2}}} \times \frac{\text{SP} + 2\text{AS}}{6} \times \frac{1}{\sqrt{m\text{F}}} :$

$$\frac{\pi \cdot \text{SP}^{\frac{3}{2}}}{2\text{SA}\sqrt{m\text{F}}} :: (\text{SP} + 2\text{AS}) \times \sqrt{\text{SP} - \text{SA}} : \frac{3\pi}{2} \times \text{SP}^{\frac{3}{2}}.$$

Mr. BAINES *answered it.*

XIX. QUESTION 469, *by Mr.* NOSYT.

Sum (n) terms of the series

$$\cos\theta + 2^2 \cdot \cos 2\theta + 3^2 \cdot \cos 3\theta + \text{\&c.}$$

FIRST SOLUTION, *by the Rev. Mr.* E. C. TYSON.

Now $\Sigma(wu) = w\Sigma u - \Delta w \Sigma^2 u_1 + \Delta^2 \Sigma^3 u_2 - \text{\&c.}$
Let $u = \cos z$ and $w = x^2$.
Then $\Delta w = (x + \Delta x)^2 - x^2 = 2x\Delta x + (\Delta x)^2$
$\Delta^2 w = 2(\Delta x)^2$ and $\Delta^3 w = 0$, —

Again, $\Sigma u = \Sigma \cos z = \dfrac{\sin\left(\dfrac{2z - \Delta z}{2}\right)}{2 \sin \dfrac{\Delta z}{2}}$ *

also $u_1 = \cos(z + \Delta z)$ ∴ $\Sigma u_1 = \dfrac{\sin\left(\dfrac{2z + \Delta z}{2}\right)}{2 \sin \dfrac{\Delta z}{2}}$

and $\Sigma^2 u_1 = \dfrac{1}{2 \sin \dfrac{\Delta z}{2}} \times \Sigma \left(\dfrac{2z + \Delta z}{2}\right) = \dfrac{-1}{4 \sin^2 \dfrac{\Delta z}{2}} \times \cos z$

and $u_2 = \cos(z + 2\Delta z)$

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* A concise method of obtaining Σ sin θ and cos θ.

$$\Delta \sin z = \sin(z - \Delta z) - \sin z = 2 \cos \frac{2z + \Delta z}{2} \sin \frac{\Delta z}{2}$$

$$\therefore \cos\left(z + \frac{\Delta z}{2}\right) = \frac{\Delta \sin z}{2 \sin \frac{\Delta z}{2}}.$$

Let $z + \frac{\Delta z}{2} = \theta$ ∴ $\Delta z = \Delta \theta$, (Δz) being constant.

$$\therefore z = \theta - \frac{\Delta \theta}{2}, \therefore \text{ we have } \cos \theta = \frac{\Delta \sin\left(\theta - \frac{\Delta \theta}{2}\right)}{2 \sin \frac{\Delta \theta}{2}}$$

$$\therefore \Sigma \cos \theta = \frac{\sin\left(\theta - \frac{\Delta \theta}{2}\right)}{2 \sin \frac{\Delta \theta}{2}} + c.$$

Again

$$\Delta \cos z = \cos(z + \Delta z) - \cos z = -2 \sin\left(z + \frac{\Delta z}{2}\right) \sin \frac{\Delta z}{2}$$

$$\therefore \sin\left(z + \frac{\Delta z}{2}\right) = \frac{-\Delta \cos z}{2 \sin \frac{\Delta z}{2}};$$

hence the same supposition being made as above,

$$\text{we have } \sin \theta = \frac{-\Delta \cos\left(\theta - \frac{\Delta \theta}{2}\right)}{2 \sin \frac{\Delta \theta}{2}}$$

$$\therefore \Sigma \sin \theta = \frac{-\cos\left(\theta - \frac{\Delta \theta}{2}\right)}{2 \sin \frac{\Delta \theta}{2}}.$$

$$\therefore \Sigma u_2 = \frac{\sin\frac{z+3\Delta z}{2}}{2\sin\frac{z}{2}} \text{ and } \Sigma^2 u_2 = \frac{-1}{4\sin^2\frac{\Delta z}{2}}\cdot\cos(z+\Delta z)$$

and $\Sigma^3 u_2 = \dfrac{-1}{8\sin^3\frac{\Delta z}{2}}\cdot\sin\left(\dfrac{2z+\Delta z}{2}\right)$

$$\therefore \Sigma x^2\cdot\cos z = \frac{x^2,\sin\left(\frac{2z-\Delta z}{2}\right)}{2\sin\frac{\Delta z}{2}} + \frac{2x\Delta x+(\Delta x)^2}{4}\cdot\frac{\cos z}{\sin^2\frac{\Delta z}{2}}$$

$$-\frac{(\Delta x)^2}{4}\cdot\frac{\sin\left(\frac{2z+\Delta x}{2}\right)}{\sin^3\frac{\Delta z}{2}} + c.$$

Now in the series proposed the general term $= n^2\cdot\cos n\theta$; ∴ the increment of the sum $= \overline{n+1}|^2\cdot\cos(n+1)\theta$, this will coincide with $\Sigma x^2\cdot\cos z$ by making $n+1 = x$ and $\overline{n+1}\,\theta = z$ ∴ $\Delta z = \theta$ and $\Delta x = 1$

$$\therefore \Sigma(n+1)^2\cdot\cos n\theta = \frac{(n+1)^2\sin\frac{2n+1}{2}\theta}{2\sin\frac{\theta}{2}} + \frac{2n+3}{4}\cdot\frac{\cos(n+1)\theta}{\sin^2\frac{\theta}{2}}$$

$$-\frac{\sin\frac{2n+3}{2}\theta}{4\sin^3\frac{\theta}{2}} + \text{corr.}$$

$$\therefore \Sigma(n+1)^2\cdot\cos(n\theta) = \frac{(n+1)^2\sin\frac{2n+1}{2}\theta - \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} + \frac{(2n+3)\cdot\cos(n+1)\theta - 3\cos\theta}{4\sin^2\frac{\theta}{2}}$$

$$-\frac{\cos\frac{u+3}{2}\theta\cdot\sin\frac{n\theta}{2}}{2\cdot\sin^3\frac{\theta}{2}}.$$

The correct sum of (n) terms,

SECOND SOLUTION, *by Mr.* JOHN BAINES.

First, by Gregory's Trigonometry; note, at page 45,

$$\cos\theta + \cos 2\theta + \cos 3\theta \ldots\ldots \cos n\theta = \frac{\sin \frac{1}{2}n\theta \cos \frac{n+1}{2}\theta}{\sin \frac{1}{2}\theta}$$

$$= \tfrac{1}{2}\sin n\theta \cot \tfrac{1}{2}\theta - \sin^2 \tfrac{1}{2}n\theta.$$

Taking the differentials on both sides, substituting $-d\theta \sin\theta$ for $d\cos\theta$, $-\frac{1}{2}d\theta \operatorname{cosec}^2 \frac{1}{2}\theta$ for $d\cot\frac{1}{2}\theta$, &c., and dividing by $d\theta$, we have

$$-(\sin\theta + 2\sin 2\theta + 3\sin 3\theta \ldots\ldots n\sin n\theta)$$

$$= \frac{n}{2}\cos n\theta \cot \tfrac{1}{2}\theta - \tfrac{1}{4}\sin n\theta \operatorname{cosec}^2 \tfrac{1}{2}\theta - \tfrac{1}{2}n \sin n\theta.$$

Taking the differentials on both sides again, substituting, dividing, &c. and changing the signs of every term,

$$\cos\theta + 2^2\cos 2\theta + 3^2\cos 3\theta \ldots n^2\cos n\theta + \frac{n^2}{2}(\cos n\theta + \sin n\theta \cot\tfrac{1}{2}\theta)$$

$+ \frac{1}{2}n(\cos n\theta \operatorname{cosec}^2 \frac{1}{2}\theta) - \frac{1}{2}\sin n\theta \cot \frac{1}{2}\theta \operatorname{cosec}^2 \frac{1}{2}\theta$, which, vanishing with n, is the sum of the series required.

Mr. COOKE *answered this question.*

XX. PRIZE QUESTION 470, *by* C. B.

Two persons A and B play at "*heads and tails.*" A has the privilege of regulating the bets at each throw, which he does thus: To the constant sum u he adds the nth part of all his winnings by all the previous throws, or if he have lost he subtracts the nth part of his loss from the same sum u. After $p + q$ throws, out of which p have been won and q lost, what is the amount of his profit?

FIRST SOLUTION, *by* C. B. *the Proposer.*

The first stake is u and whether he is successful or otherwise his profit may be expressed thus, $u(-1)^a$ for if he lose a is an odd but if he wins it is an even number. The second stake is $u + (-1)^a \frac{u}{n}$ and his profit on this may be similarly expressed by

$$\left\{u + (-1)\frac{u}{n}\right\}(-1)^b$$

without deciding which it is:

The third stake is

$$u + (-1)^a \frac{u}{n} + (-1)^b \frac{u}{n} + (-1)^a (-1)^b \frac{u}{n^2}$$

and the profit

$$\left\{ u + (-1)^a \frac{u}{n} + (-1)^b \frac{u}{n} + (-1)^a (-1)^b \frac{u}{n^2} \right\} (-1)^c$$

Continuing in this manner and collecting the profits on each, we have for its amount

$$\mathrm{W} =$$

$$u(-1)^a$$

$$u(-1)^b + \frac{u}{n}(-1)^a (-1)^b$$

$$u(-1)^c + \frac{u}{n}\left\{(-1)^a(-1)^c + (-1)^b(-1)^c\right\} + \frac{u}{n^2}\left\{(-1)^a(-1)^b(-1)^c\right\}$$

$$u(-1)^d + \frac{u}{n}\left\{(-1)^a + (-1)^b + (-1)^c\right\}(-1)^d$$

$$+ \frac{u}{n^2}\left\{(-1)^a(-1)^b + (-1)^a(-1)^c + (-1)^b(-1)^c\right\}(-1)^d$$

$$+ \&c. \ldots\ldots\ldots \frac{u}{n^3}\left\{(-1)^a(-1)^b(-1)^c(-1)^d\right\}.$$

The right side of this equation consists of u multiplied by the sum of all the quantities $(-1)^a$, $(-1)^b$, $(-1)^c$ &c. plus $\frac{u}{n}$ multiplied by the sum of all their products two by two, plus, $\frac{u}{n^2}$ multiplied by the sum of all their products three by three, plus &c. &c.

From this consideration it appears that if the right side were divided by n it would be equal to the sum of all the coefficients except the first of the equation

$$\left\{x + \frac{(-1)^a}{n}\right\}\left\{x + \frac{(-1)^b}{n}\right\}\left\{x + \frac{(-1)^c}{n}\right\} \ldots\ldots\ldots = 0$$

or

$$\frac{1}{n}\mathrm{W} = u\left\{\left(1 + \frac{(-1)^a}{n}\right)\left(1 + \frac{(-1)^b}{n}\right)\left(1 + \frac{(-1)^c}{n}\right) \ldots\ldots - 1\right\}$$

but p of the quantities a, b, c, &c. are by the question even numbers and q are odd, or

$$\frac{1}{n}\mathrm{W} = \left\{\left(1 + \frac{1}{n}\right)^p \left(1 - \frac{1}{n}\right)^q - 1\right\} u$$

hence the whole profit is

$$W = n\left(\frac{n+1}{n}\right)^{p}\left(\frac{n-1}{n}\right)^{q}u - nu.$$

Cor. If $p = q$

$$W = \frac{(n^2-1)^p}{n^{2p-1}}u - nu.$$

SECOND SOLUTION, *by the late Mr.* ANTHONY COOKE.

It is obvious that if A win the first throw his gain will be u, and if he lose it his loss will be u, or his profit $-u$; so that we may represent his profit at the first throw by ua_1, where a_1 will be $= +1$ if A win, or $= -1$ if he lose. The second stake will then be $= u\left(1 + \frac{a_1}{n}\right)$, and A's profit by the second throw $= ua_2 \,.\, \left(1 + \frac{a_1}{n}\right)$, where a_2 will be $= +1$ if A win the second throw, and $= -1$ if he lose it; also the sum of the profits by the first and second throws =

$$ua_1 + ua_2\left(1+\frac{a_1}{n}\right) = u\left(a_1 + a_2 + \frac{a_1a_2}{n}\right) = un \,.\, \left\{\left(1+\frac{a_1}{n}\right).\left(1+\frac{a_2}{n}\right)-1\right\}.$$

Therefore the third stake will be =

$$u + u\left\{\left(1+\frac{a_1}{n}\right).\left(1+\frac{a_2}{n}\right)-1\right\} = u \,.\, \left(1+\frac{a_1}{n}\right).\left(1+\frac{a_2}{n}\right);$$

and A's profit by the third throw $= ua_3\left(1+\frac{a_1}{n}\right).\left(1+\frac{a_2}{n}\right)$, a_3 representing $+1$ or -1 according as A wins or loses, as before; the sum of the profits by the three first throws =

$$un \,.\, \left\{\left(1+\frac{a_1}{n}\right).\left(1+\frac{a_2}{n}\right)-1\right\} + ua_3 \,.\, \left(1+\frac{a_1}{n}\right).\left(1+\frac{a_2}{n}\right) =$$

$$u \,.\, \left\{\left(1+\frac{a_1}{n}\right).\left(1+\frac{a_2}{n}\right)\right\}.(n+a_3) - un =$$

$$un \,.\, \left\{\left(1+\frac{a_1}{n}\right).\left(1+\frac{a_2}{n}\right).\left(1+\frac{a_3}{n}\right)-1\right\}.$$

Wherefore the fourth stake =

$$u + u\left\{\left(1+\frac{a_1}{n}\right).\left(1+\frac{a_2}{n}\right).\left(1+\frac{a_3}{n}\right), -1\right\} =$$

$$u \,.\, \left(1+\frac{a_1}{n}\right).\left(1+\frac{a_2}{n}\right).\left(1+\frac{a_3}{n}\right),$$

A's profit by the 4th throw $= ua_4\left(1+\frac{a_1}{n}\right).\left(1+\frac{a_2}{n}\right).\left(1+\frac{a_3}{n}\right)$, and the sum of his profits by the 4 first throws =

$$u.\left\{(1+\frac{a_1}{n}).1+\frac{a_2}{n}).1+\frac{a_3}{n})-1\right\}+ua_4(1+\frac{a_1}{n}).(1+\frac{a_2}{n}).(1+\frac{a_3}{n})=$$

$$u.\left\{\left(1+\frac{a_1}{n}\right).\left(1+\frac{a_2}{n}\right).\left(1+\frac{a_3}{n}\right)\right\}.(n+a_4)-un=$$

$$un.\left\{\left(1+\frac{a_1}{n}\right).\left(1+\frac{a_2}{n}\right).\left(1+\frac{a_3}{n}\right).\left(1+\frac{a_4}{n}\right)-1\right\},$$

where the law of contiuation is manifest. Hence, if a_m represent + 1 or — 1 according as A wins or loses the mth throw, it is evident that the sum of A's profits by the first m throws will be

$$=un.\left\{\left(1+\frac{a_1}{n}\right).\left(1+\frac{a_2}{n}\right).\left(1+\frac{a_3}{n}\right)\ldots\ldots\left(1+\frac{a_m}{n}\right),-1\right\}.$$

Now, since A wins p and loses q throws, it is obvious that p of the quantities a_1, a_2, a_3, a_4, &c. will be = + 1, and q of them = — 1: whence the whole amount of A's profit in $p+q$ throws will be $= un.\left\{\left(1+\frac{1}{n}\right)^p.\left(1-\frac{1}{n}\right)^q-1\right\}$, as required.

THIRD SOLUTION, *by Mr.* BONNYCASTLE.

If the person who has the management of the stakes, bets at first the sum w; and after winning for x throws has gained Fx; we shall have from the conditions of the question

$$\text{F}.x=u+\left(1+\frac{1}{n}\right)\text{F}.(x-1);\ldots\ldots\ldots(1)$$

or by expanding the right hand member

$$\text{F}.x=u\left\{1+\frac{n+1}{n}+\left(\frac{n+1}{n}\right)^2+\&\text{c.}\right\}+\left(\frac{1+n}{n}\right)^{x-1}\text{F}(x-(x-1));$$

$$\text{or}\quad \text{F}.x=nu\left\{\left(\frac{n+1}{n}\right)^{x-1}-1\right\}+\left(\frac{n+1}{n}\right)^{x-1}w;\ldots\ldots(2)$$

the function F $(x-(x-1))$ being equal to w, or the sum first betted.

Also if this person had constantly lost for the first x throws, and his loss was expressed by F_1x; we should have had

$$\text{F}_1x=u+\left(1-\frac{1}{n}\right)\text{F}_1(x-1).$$

But as this expression is similar to (1) with the sign of n changed, we conclude that the equation (2) will give the sum won or lost in either of these cases; taking n positive when there is a gain, and negative when there is a loss.

Now let us conceive that A first wins x throws, and then loses x'; then, as in the question w is restricted to be equal to u, his winnings at first will be found by substituting u for w in the expression (2); which reduces it to

$$\text{F}x=nu\left\{\left(\frac{n+1}{n}\right)^x-1\right\}:\ldots\ldots\ldots(3)$$

But at every succeeding throw of the next x' throws, he loses $u + \frac{1}{n}$ of what he had won before; wherefore, we may represent the sum that remains in this case by

$$F_2 x' = \frac{n-1}{n} \left\{ F_2 (x-1) \right\} - u : \ldots\ldots\ldots (4)$$

which being the same as (1) with the signs of both n and u changed, its value may be found by changing the signs of those quantities in the equation (2); which will then become

$$F_2 x' = nu \left\{ \left(\frac{n-1}{n}\right)^{x'} - 1 \right\} + w \left(\frac{n-1}{n}\right)^{x'} \ldots\ldots\ldots (5)$$

x' being put instead of $x' - 1$ because in this case it is manifest that the series must be carried one place further back: but as this loss began after having won for x throws, the value of w must be the same as that of Fx in the expression (3); substituting which in our last equation it becomes

$$F_2 x'' = nu \left\{ \left(\frac{n+1}{n}\right)^{x} . \left(\frac{n-1}{n}\right)^{x'} - 1 \right\} :$$

which is the expression for the sum won after winning x throws successively, and then losing x'.

Again had A lost x' throws and won the next x, the formula from which we should have had to compute the sum won or lost, would have been

$$F_3 x = \frac{n+1}{n} F_3 (x-1) - u.$$

Which being the same as (4) with the sign of n changed, its value may be found by changing the sign of n in the expression (5), or

$$F_3 x = w \left(\frac{n+1}{n}\right)^{x} - nu \left\{ \left(\frac{n+1}{n}\right)^{x} - 1 \right\}.$$

Where substituting for w its value $F_1 x' = -nu \left\{ \left(\frac{n-1}{n}\right)^{x'} - 1 \right\}$,

we have $$F_3 x'' = -nu \left\{ \left(\frac{n+1}{n}\right)^{x} . \left(\frac{n-1}{n}\right)^{x'} - 1 \right\};$$

which is the expression for the sum lost after losing x' throws successively, and then winning x.

Wherefore it appears that the result is the same whether A wins x throws, and then loses x', or loses x' throws, and then wins x; from which it will readily be perceived that however the order of winning or losing be varied if p is the whole number of times of winning, and q of losing; the sum won or lost will be

$$F(p_1 q_1) = nu \left\{ \left(\frac{n+1}{n}\right)^{p} . \left(\frac{n-1}{n}\right)^{q} - 1 \right\}.$$

ARTICLE IV.

Obituary.

JOHN PLAYFAIR, Esq. F.R.S. L. and E.

Late Professor of Natural Philosophy in the University of Edinburgh.

John Playfair, the eldest son of the Reverend James Playfair, was born at Bervie, in Forfarshire, on the 10th of March, 1748. He received the rudiments of his education from his father, and at the age of fourteen was sent to the University of St. Andrew's, being intended for the Church. At the University Mr. Playfair evinced those habits of industry and steady application to study which distinguished him through life; and such was the opinion entertained of his character and acquirements, that before his philosophical course was finished, he was selected to read the Public Lectures on Natural Philosophy, during the illness of Professor Wilkie. This incident probably contributed to confirm him in the preference he had already shewn for mathematical studies, in which his proficiency was such, that he was induced by the advice of his friends, to offer himself, in 1766, as a candidate for the Professorship of Mathematics in the Marischal College of Aberdeen, an office which the patrons had determined to fill up by a comparative trial. Although Mr. Playfair was at this time only eighteen years of age, he sustained, with credit to himself, an examination which lasted eleven days; and, of six competitors two only were preferred, namely, the Reverend Dr. Trail, who was then appointed to the office, and Dr. Hamilton, who has since creditably filled it, and who has distinguished himself by his profound calculations respecting finance.

After finishing his course at the University, Mr. Playfair passed several years in Edinburgh, where he enjoyed the friendship and familiar society of most of the illustrious men who at that period adorned the Scottish metropolis.—Dr. Robertson, the historian, Dr. Adam Smith, Dr. Matthew Stewart, Dr. Black, Dr. James Hutton, &c. In 1772 he was an unsuccessful candidate for the chair of Natural Philosophy in St. Andrew's, vacant by the death of his friend Mr. Wilkie, private influence in this case interfering to prevent his appointment. In the course of the following year he was inducted to the living which had been held by his father, who had died some time previously. To a mind stored with the truths of natural science, and ambitious of literary honours and distinction, the monotonous duties of a parish minister are not calculated to afford much pleasure; yet Mr. Playfair is said to have entered upon them with considerable zeal, and to have devoted the greater portion of his time to the religious instruction of the peasantry. His leisure hours were employed in the education of his brothers, and when these double claims upon his time were satisfied, few moments remained for the prosecution of his favorite studies.

While Mr. Playfair filled this situation he had an opportunity of paying a visit to Dr. Maskelyne, the Astronomer Royal, when engaged in making experiments on Schehallien, to determine the effect of the attraction of mountains in altering the direction of the plumb-line. Mr. Playfair passed some days in the interesting society of this celebrated observer, and the friendship thus commenced among the mountains of Perthshire, continued without interruption till the death of Dr. Maskelyne.

Mr. Playfair continued in his living of Bervie till 1782, when he was induced to resign it in order to superintend the education of the sons of a private gentleman in Fifeshire. Three years after, in 1785, in consequence of an arrangement entered into between Dr. Adam Ferguson and Mr. Dugald Stewart, he obtained a chair in the University of Edinburgh. Dr. Ferguson's health rendering him unfit to discharge his academical duties, he resigned his own chair of Moral Philosophy, to Mr. Stewart, and accepted that of Mathematics; Mr. Playfair being appointed joint professor, and charged with the care of instructing the class. This was the most agreeable office which could have been selected for him. His residence in Edinburgh afforded him the means of enjoying that literary society of which he was so distinguished an ornament; and the long vacation of seven months allowed him ample leisure for the prosecution of those scientific researches which he had already commenced with such happy success.

The fruits of the leisure which this appointment afforded him soon appeared in a series of excellent papers, which he now began to contribute to the Transactions of the Royal Society of Edinburgh. A Memoir on the Life of Dr. Matthew Stewart,—On the causes which effect the accuracy of Barometrical Measurements,—Remarks on the Astronomy of the Brahmins,—On the Origin and Investigation of Porisms, &c. Before his appointment he had contributed a Paper to the London Transactions, on the Arithmetic of Impossible quantities. In 1795, he published an edition of the First Six Books of Euclid, accompanied with an investigation of the Rectification and Quadrature of the Circle, the Intersections of Planes, and the Geometry of Solids; also a Treatise on Plane and Spherical Trigonometry with the Arithmetic of Sines. This work, which is well known, went through five editions during the life time of the author, and has gone through one or two editions since, under the learned care of his distinguished successor Professor Wallace. It differs from Simson's chiefly by the introduction of Algebraic symbols in the second and fifth Books; a plan which has not been universally followed in this country, although by rendering the demonstrations more compact, without interfering in any respect with their spirit, it unquestionably tends to facilitate the acquisition of these Books to the young student. The notes which accompany the volume are extremely interesting and valuable.

While Mr. Playfair occupied the Mathematical chair, a great portion of his time, during many years, was devoted to the composition of his "Illustrations of the Huttonian Theory of the Earth." This work is an exquisite model of elegant and perspicuous writing; indeed it is doubtful, if, at the time of its publication, there existed in the English language, any scientific disquisition at all to be compared with it in point of style and composition. Notwithstanding, however, its great merits in this respect, and its admirable developement of the theory in question, it is to be regretted that Mr. Playfair spent so much time on a subject, which, however seductive to a mind delighting to trace the general laws of the Universe through the endless variety of terrestrial phenomena, leaves, after all, so much to hypothesis and fanciful speculation.

In the year 1805 Mr. Playfair was appointed by the patrons of the University to succeed Professor John Robison in the chair of Natural Philosophy. For this department he was admirably qualified. The wider range of the course afforded ample opportunity for bringing forward the vast stores of information which he had accumulated on every subject connected with physical science; and his clear, impressive, and eloquent expositions, by pouring a flood of light over the most abstruse topics, rendered his lectures equally interesting and instructive.

In 1814, he published his *Outlines of Natural Philosophy*. This work was intended for the use of the students in his class, and contains merely the heads of the lectures, the principal propositions on which the different theories are founded being given without the demonstrations. The second volume, however, contains an admirable outline of Physical Astronomy, and is invaluable to the student as a guide to direct him in the study of that difficult science. It points out the order in which the investigations can be undertaken with the greatest advantage, and refers to the best sources of information. A third volume was projected, to contain Optics, Electricity, and Magnetism; unfortunately it has not been executed.

In the summer of 1815, Mr. Playfair proceeded on a tour to the Continent, passed the following winter in Italy, and before his return visited the most interesting parts of that country, as well as of France and Switzerland. The opportunity of examining in person the geological features of these countries, with which he had long been familiar from the descriptions of Naturalists, afforded him unbounded gratification, and he intended to employ the observations he might be able to make in correcting and improving the *Huttonian Theory*, of which he contemplated a new and extended edition. But before this design could be carried into execution, the power which had been so frequently and so sccessfully exerted in tracing the mutual connexion, and common dependance, of multitudes of isolated facts,—which was able to spread a charm over the driest and most uninviting subjects,—was extinguished by death. The account of the slide at Alpnach is all that has been gained from this scientific tour. After his return from the Continent, the short remainder of his life was employed in the composition of the *Dissertation on the Progress of the Mathematical and Physical Sciences*, which appeared in the Supplement to the Encyclopœdia Britannica, and which has, perhaps, contributed more to extend his reputation than any of his other writings. He died on the 19th of July 1819, and the civic authorities of Edinburgh did themselves honour by granting to his remains a public funeral. A monument has been erected to his memory by private subscription; it forms an angle of the wall which now surrounds the Observatory. A selection of his works has been published since his death, in 4 vols. 8vo.

Although when compared with his illustrious contemporaries Lagrange and Laplace, Mr. Playfair can scarcely be regarded as a Mathematician of the first order, it must be admitted that he contributed more than perhaps any other individual of his day in Britain, to propagate a taste for Mathematical studies. He was early acquainted with the most recent improvements made in the science on the Continent, and was mainly instrumental in introducing them into this country, not only by explaining their advantages and utility within the circumscribed sphere of his class, but also by his eloquent recommendations of them in the Edinburgh Review, and other publications. As an instructive public lecturer he has seldom been equalled. His knowledge of the subject was so complete, and his ideas so matured and methodically arranged, that his illustrations never failed to be accompanied with a degree of luminous simplicity, which rendered them perfectly intelligible and highly instructive, even to the least informed of his auditory. Few who have enjoyed the benefit of hearing his lectures will soon forget the earnest, impressive, and dignified manner in which he inculcated the truths of science, the felicity of his illustrations, and the enthusiasm with which he seemed to be inspired when explaining the grand and magnificent phenomena of nature. or the sublime harmony of the material universe. As an individual he possessed great influence in society, less on account of his splendid scientific acquirements than by reason of the virtues which adorned his private character. Kind and benevolent in his disposition, perfectly disinterested in his conduct, and of spotless purity of manners, he seemed to look down from an eminence on the contemptible jealousies and passions which infest ordinary society, and are but too apt to disturb the repose even of philosophers. " Independent of his high attainments," says his eloquent panegyrist, Mr. Jeffery, " Mr. Playfair was " one of the most amiable and estimable of men,—delightful in " his manners, inflexible in his principles, and generous in his " affections, he had all that could charm in society or attach " in private; and while his friends enjoyed the free and unstudied conversation of an easy and intelligent associate, " they had at all times the proud and inward assurance that " he was a being on whose perfect honour and generosity they " might rely with the most implicit confidence, in life and in " death,—and of whom it was equally impossible, that, under " any circumstances, he should ever perform a mean, a selfish, " or a *questionable* action, as that his body should cease to gravitate or his soul to live."

CHARLES HUTTON, LL.D. F.R S.

Dr. Charles Hutton was born at Newcastle-upon-Tyne, on the 14th of August, 1737, He was descended from a family in Westmoreland, one branch of which had removed into Northumberland, and another branch into Lincolnshire, where a female of the family married into that of Sir Isaac Newton, being indeed the *aunt* of that illustrious philosopher. Dr. Hutton's father, though not a man of theoretical science, had considerable knowledge and skill in practical mechanics, and had extensive employment as a viewer of mines; being also, if I am rightly informed, for some years land-steward to the then Lord Ravensworth. He intended devoting his son to his own profession; and, to that end, procured for him the best instruction which could then be obtained at Newcastle, and from a clergyman in a neighbouring village; the knowledge he thus acquired extended simply to the rudiments of the English and Latin languages, and the leading principles of mathematics. The youth, however, had an injury done to one of his arms in very early life, which, by unfitting him for such active pursuits as his father had proposed for him, rendered it necessary that he should devote himself still more sedulously to study. From the earliest age, he evinced an extreme fondness for reading of every description; but especially delighted in the "Border," songs, legends, and stories, of which, while a mere child, he made a large collection. But no sooner did it become necessary for him to devote himself to study, with a view to a profession, than the natural bent of his inclinations led him to that of a mathematical tutor; to prepare himself for which, he laboured day and night with unwearied vigilance and assiduity.

At about the age of 18, having long before lost his father, he commenced the employment of a teacher of mathematics, at the then delightfully sequesterd village of Jesmond, near Newcastle: but, young as he was, such was his success, and such the proficiency of many of his pupils, that he was soon invited to remove to Newcastle itself, that noble capital of the northern mining districts. Here, although he received no pupils but upon about double the terms that had previously been charged in that quarter of the kingdom, his pupils soon became numerous. He did not, however, *confine* himself to the mere business of their instruction, arduous as it was; but, being fired by the laudable ambition of reaching the acmè of his profession, he determined to go through an extensive course of mathematical reading. This he pursued chronologically, going cautiously over the principal mathematical productions of the Greeks, Romans, Spaniards, French, and Germans, as well as those which had appeared in Britain. Such an extensive course at once proceeded from a genuine love of science, and increased it: and this, together with his unfailing success as a preceptor, greatly augmented his well-earned reputation. His character became accurately appreciated at Newcastle: so that he was frequently consulted and employed in reference to other important points, than those which related immediately to his profession.

Notwithstanding the multiplicity of his avocations, he found time for investigation and composition; and made himself known to the world as an author. Before the year 1760, he was a frequent contributor to those well-known publications in England, the *Ladies' and Gentleman's Diaries*, and *Martin's Magazine;* in all of which he proposed and solved many mathematical problems of considerable difficulty as well as utility. His first avowed separate publication was a small treatise on *Arithmetic* and *Book-keeping*, for the use of schools. It made its appearance in the year 1764; and has since gone through numerous editions.

Soon after this, he began to employ his leisure in the composition of a much more elaborate and recondite work, viz. *A Treatise on Mensuration, in Theory and Practice.* At the time when Mr. Hutton commenced this undertaking, the books on mensuration, that were generally adopted in seminaries of education, were those of *Hawney* and *Robertson.* Of these, the first contained some attempts at theory, but exhibited in so inelegant and inaccurate a manner, as to render the volume altogether useless. Robertson's work was neat and correct, but limited in its nature, being confined altogether to the exhibition of practical rules and examples. There had been, it is true, from the time of Wallis and Huygens, and especially since the invention of the fluxional analysis, a variety of disquisitions and investigations relative to rectifications, quadratures, cubatures, &c. inserted in the works of eminent mathematicians, and in the transactions of different societies and academies both at home and abroad. But there needed some masterly hand to seize and collect these scattered fragments, to reduce them to method and order, to correct what was erroneous, curtail what was too protracted, expand and elucidate what was incomplete and obscure, and develope with perspicuity the practical results and applications. All this is attempted with complete success in this treatise on mensuration, which was first published in periodical numbers, and then altogether in a quarto volume, in the year 1770. A second, and improved edition, was published in a thick octavo volume in 1788; since which time there have appeared two more editions. There can be no question that this is by far the best treatise on mensuration, in its several branches, which has yet been published in any country. Indeed, the subject is so exhausted in this performance, that subsequent writers upon it, at least in England, have attempted little else than mere abridgments.

The overthrow of Newcastle Bridge, and other bridges across the river Tyne, in November, 1771, by reason of a very high and rapid flood in that river, drew our author's attention to another subject, that is to say, the theory of the equilibration of arches and piers. The result of his investigations was laid before the public in 1772, in an octavo tract, on "*The Principles of Bridges;* containing the mathematical demonstrations of the properties of the arches, the thickness of the piers, the force of the water against them, &c. together with practical observations and directions drawn from the whole." This performance was entirely out of print, when on occasion of Telford and Douglas's project for erecting an iron bridge over the Thames at London, it was republished in 1801, verbatim from the Newcastle edition. The author had for several years been collecting materials for an enlarged and improved edition of this treatise, among which were theoretical and practical observations on several of the most celebrated bridges in Italy and other parts of continental Europe; but these, unfortunately for the world, are lost; the author, however, has made several improvements, though by no means to the extent which he had contemplated, in the edition of this essay, which appears in his Collection of Tracts in 1812.

About the year 1772, also, our indefatigable author commenced the republication of the "*Ladies' Diaries,*" from the origin of the series. From a full persuasion of the utility of such an undertaking, Mr. Hutton determined to republish the whole of the poetical and mathematical departments periodically. As he proceeded, he gave new and improved solutions to many of the problems. He also inserted in each successive number of his publication, a series of new and curious problems; while, in several numbers, there appeared valuable disquisitions, by himself and his correspondents, on a variety of subjects, connected both with pure and mixed mathematics. These were afterwards published separately, under the title of *The Mathematical Miscellany*, in a single volume.

Labours like those which we have already enumerated, soon led our author into a most extensive correspondence, and procured for him a very exalted reputation; such, indeed, as occasioned his removal to a situation of great importance, as well as respectability. The professorship of Mathematics in the Royal Military Academy at Woolwich, became vacant, in consequence of the resignation of Mr. J. L. Cowley; and the Marquis of Townsend, at that time Master General of the Ordnance, formed the laudable determination of giving the appointment to the individual, who, by a public examination, should prove himself the best qualified to discharge the duties of a mathematical professor. Mr. Hutton was persuaded by his friend Colonel Edward Williams, of the Royal Artillery, himself a mathematician of considerable acquirements, to become a candidate for the situation. His natural diffidence, which has at all times been as remarkable as his talents, caused him at first to shrink from all desire to obtain a professorial chair, which had previously been occupied by one so deservedly eminent as *Thomas Simpson*: but his friends at length overcame his scruples, and he travelled from Newcastle to Woolwich, a distance of 300 miles, to propose himself a candidate. He had no less than *ten* competitors, among whom were *Mr. Benjamin Donn*, the author of *The Geometrician*, and other well-known works; *Mr. Hugh Brown*, the able translator and commentator upon *Euler's Gunnery*. The gentlemen appointed to conduct the examination, were *Dr. Maskeyline*, then astronomer Royal; *Bishop Horsley*, the learned editor of Newton's works; *Col. Henry Watson*, afterwards Chief Engineer in Bengal; and *Mr. Landen*. To all these gentlemen Mr. Hutton was entirely unknown, except by character, The examination continued for several days, and was directed to—the manner of communicating instruction, the best treatises to be employed in different branches of science, the history of the principal departments of mathematics, the solutions of a variety of problems, &c. More than half the candidates passed through this ordeal in such a way as gave entire satisfaction to the examiners; but the superiority of Mr. Hutton was, in every respect, so marked and decisive, that they unanimously recommended him as peculiarly qualified to fill and adorn the situation; to which he was accordingly, appointed on the 24th of May, 1773.

About the same time he was appointed, by the Stationers' Company, the editor of *The Ladies' Diary*; and shortly afterwards that truly respectable body entrusted to him the astronomical computation and management of the principal Almanacs which they publish. These he continued to conduct, with great honour to himself, and advantage to the Company, until the year 1818.

Shortly after Mr. Hutton's removal to Woolwich, he was elected a fellow of the Royal Soceity; and about the year 1779 he received the degree of LL. D, from the University of Edinburgh. In January, 1779, he was appointed Foreign Secretary to the Royal Society, an office which he held till the end of the year 1783; when in consequence of the dissensions which unhappily prevailed in the Society, and the aversion and disrelish which were then evinced with respect to the mathematical sciences, by some of the leading officers and members, he retired from the Society, together with Dr. Maskelyne, Dr. Horsley, and other eminent mathematicians.

Dr. Hutton devoted himself very seduously to the discharge of his academical duties; yet found time, notwithstanding, to pursue a variety of interesting analytical investigations, as well as to plan some extensive and important experimental inquiries. arising naturally out of his appointment at Woolwich. Among the papers which about this period he sent to the Royal Society, was one published in the Philosophical Transactions for 1776, entitled "*A new and general method of finding Simple and Quickly Converging Series;* by which the proportion of the diameter of a circle to its circumference may be easily computed to a great number of figures.

In the Transactions for 1778 appeared Dr. Hutton's first paper on *the force of exploded gunpowder*, and *the velocities of balls* projected from artillery. The experiments detailed in this paper commenced in the year 1773, and were carried on by a consecutive series till a short time before the preparation of this account of them. This paper was no sooner laid before the Royal Society than its ingenuity and value were acknowledged, by the Council awarding to Dr. Hutton the Copleian medal for that year.

In the same year our author laid before the Royal Society, his "*Account of the Calculations made from the Survey and Measures taken at Mount Shichallien, in Perthshire, in order to ascertain the mean Density of the Earth.*" This is a truly excellent disquisition, and the calculations of which it exhibits the results, were more laborious, and at the same time called for more ingenuity, than has probably been brought into action in any computation undertaken by a single person since the preparation of logarithmic tables. It is but justice here to state that Dr. Hutton was, unquestionably, the first person who made a tolerably correct appreciation of the mean density of the earth by elaborate computations applied to actual experiment.

While the Doctor was engaged in these researches, he turned his attention to some analogous inquiries, calculated to be useful to those who might in other places repeat similar experiments and observations to those at Shichallien. These were read to the Royal Society in November, 1779, and afterwards published in their Transactions under the title of "*Calculations to determine at what point in the side of a Hill its attraction will be the greatest.*

In the year 1780, Dr. Hutton presented to the Royal Society an Essay on Cubic equations and Infinite Series; and in the year 1783 he laid before the same learned body, his project for a New Division of the Quadrant.*

In 1784, appeared the first edition of the Compendious Measurer; and in the succeeding year that highly useful and laborious performance, the Mathematical Tables was published.

In 1786 Dr. Hutton published a quarto volume of Tracts, "Mathematical and Philosophical;" and in 1787, "Elements of Conic Sections, with Select Exercises in various branches of Mathematics and Philosophy, for the use of the Royal Military Academy at Woolwich."

For several years after the publication of the last-mentioned volume, Dr. Hutton employed his leisure from academical duties in the composition of his "*Mathematical and Philosophical Dictionary*," which made its appearance in 1796, in two large quarto volumes. This was a work of great labour, and and has been regarded by the British public as of considerable value. A new and greatly enlarged edition, with many improvements, was published in 1815.

In 1798, appeared the first edition of Dr. Hutton's *Course of Mathematics*, in two octavo volumes, for the use of the Gentlemen Cadets in the Royal Military Academy. This has gone through several editions. In 1811, a third volume of this Course was published: it was written by the Doctor, in conjunction with Dr. Gregory.

From 1803 to 1809 our author was employed, in conjunction with Doctors Pearson and Shaw, in laying before the world an *Abridgment of the Philosophical Transactions* of the Royal Society of London, from their commencement in 1665, to the end of the last century. The departments of pure and mixed mathematics, including the practice, as well as the theory, of mechanics, hydrodynamics, optics, astronomy, electricity, and magnetism, were undertaken by Dr. Hutton, in addition to the general editorship and correction of the press, of the whole. The greater portion of the biographical articles, as well as of the translations, were from the Doctor's pen; and every competent judge will admit that they do him considerable honour.

In July, 1807, Dr. Hutton having suffered much from a pulmonary complaint during the preceding winter and spring, resigned his Professorship at Woolwich, after having most honourably filled it for upwards of 34 years. On his retirement, the Board of Ordnance assigned him a pension of £500 per annum, in testimony of regard for his long and faithful services.

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* In the year 1781, Dr. Hutton also prepared "Tables of Powers and Products," which were published by the Board of Longitude.

In the year 1812, Dr. Hutton published, in three volumes octavo, a collection entitled "*Tracts on Mathematical and Philosophical subjects*; comprising among numerous important articles, the Theory of Bridges, with several plans of recent improvements: also, the results of numerous experiments on the force of Gunpowder, with applications to the modern practice of Artillery." These volumes contain, with improvements and corrections, several of the detached papers and essays of which mention has already been made: they also include the History of the Writings and Investigations in Trigonometry and Logarithms, as published in the Introduction to the Doctor's Mathematical Tables, and the History of the Discoveries and Inventions in Algebra, which was first published under the word ALGEBRA, in the Doctor's Mathematical and Philosophical Dictionary. But several of these "Tracts" are altogether new, and contain a methodical and perspicuous account of Dr. Hutton's valuable experiments on gunnery, and with the whirling machine; together with a copious exposition and application of the principal scientific deductions which have been made from these numerous and cautious experiments.

As a Preceptor, Dr. Hutton was characterized by mildness, kindness, promptness in discovering the difficulties which his pupils experienced, patience in labouring to remove those difficulties, unwearied perseverance, and a never-failing *love of the act of communicating knowledge by oral instruction.* His *patience*, indeed, was perfectly invincible. No dulness of apprehension, no forgetfulness in the pupil, ever induced him to yield to irascible emotions: to forfeit his astonishing power of self-control.

As a Lecturer, his manner was deliberate and perspicuous, his illustrations happy and convincing, and his experiments usually performed with neatness and success.

As an Author, Dr. Hutton has long been the most popular of English mathematical writers; and there are obvious reasons for this popularity; which promises to be as permanent as it is extensive. His grand objects were *utility* in the topics of investigation, *simplicity* in the mode of their attainment or advancement. He had a constitutional, and, I believe, a conscientious, aversion from the pedantry and parade of science. He loved science, and he promoted it for its own sake, and that of its tendencies. He never, by affecting to be abstruse, became obscure; nor did he ever slide into digressions, for the purpose of shewing how much he knew of other things besides those that were immediately under discussion. Hence, he was at once concise and perspicuous; and though he evidently wrote rather to be *useful* than to obtain celebrity, he has procured for himself a reputation, such as hundreds, who have written for reputation alone, will never attain.

During the last six or seven years, the Doctor's increasing infirmities led to rather more quiet and recluse habits than he had formerly adopted; though he became neither indolent, nor estranged from his friends.

Much of his time was occupied in carrying through the press, new editions of some or other of his works. Thus, during the last year of his life, he published the 15th edition of his *Arithmetic*, the 8th edition of his *Compendious Measurer*, and the 6th of his *Mathematical Tables.*

Dr. H. had often, and at distant intervals, expressed his conviction that many of the computations in *Mr. Henry Cavendish's* paper on the Earth's mean Density were erroneous; a circumstance which he ascribed, not to any mistake in Mr. Cavendish, who was an admirable mathematician, but to errors committed by the individual whom that gentleman employed in the subordinate calculations. After in vain inviting different friends to go through the computations *de novo*, this Nestor of science, *in his eighty-fourth year*, undertook and completed the labour. As he anticipated, he found several errors of no trifling magnitude, which, with their correction, may be seen in a paper sent by the Doctor to the Royal Society in 1821, and inserted in the Philosophical Transactions for that year. This, we believe, was the last of his important scientific labours.*

Towards the end of the same year, 1821, a meeting was held by several of the Doctor's friends, with the intention of paying him a tribute of respect, and also with the desire of obtaining a correct likeness of him. They accordingly appointed a committee, who agreed to employ a sculptor of the first eminence to execute a Bust in marble, from which casts or copies could be taken in any number that might be required. The Committee resolved to commence a subscription for this purpose, on such a plan as to afford room for numerous friends and admirers of Dr. Hutton to participate in the pleasure of thus manifesting their esteem. A very ample list of subscribers was soon collected, at the head of which stood the name of the *Lord Chancellor*, who was one of the Doctor's early pupils at Newcastle.

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* The extensive course of Lectures on Natural Philosophy, delivered by Dr. Hutton, at the Royal Military Academy, was quite ready for publication; but the manuscript was lost in a very extraordinary manner about twenty years ago.

The artist, *Gahagan*, having completed the Bust; on the 21st of September, 1822, the Committee for conducting the Subscription presented it to Dr. Hutton, at his house in Bedford Row, according to the original intention.

On this gratifying occasion, the Doctor's spirits evinced no ordinary flow: his memory, his reason, his science, and his wit, were excited into unusual activity; and the select few who were then present, will long remember the peculiar display of intellectual vigour, as well as of generous and grateful emotion, which they then witnessed.

The infirmities of age now stole upon him incessantly, yet almost imperceptibly: time, that most mighty, though most silent of innovators, was making effectual inroads. In October following, the Doctor, by some unavoidable exposure to the effects of a chilling atmosphere caught a severe cold. This issued in a pulmonary complaint, which soon made rapid encroaches upon his constitution. His physical strength visibly declined; and many of his actions, and not a little of his conversation, evinced that anticipated approaching dissolution. He retained, however, the entire possession of his faculties till very near his death, which took place on the morning of Monday, January the 27th, 1823, when he expired without a groan.

For a more detailed account of the Character and Works of Dr. Hutton, we must refer to a Memoir written by his friend, Dr. O. Gregory, and published in the Imperial Magazine, for March, 1823, and of which the preceding is only a brief abstract.

MR. DALBY,

Late Professor of Mathematics in the Senior Department of the Royal Military College.

ISAAC DALBY was born at a village in Gloucestershire, in the year 1744; and died at Farnham, in Surrey, (to which place the senior department of the Royal Military College was transferred from High Wycombe,) October 14th, 1824, in his eighty-first year. Some years previously to his death, he sent to a friend an account of the principal events which had occurred since he had reached maturity; and from this we are permitted to present the following narration.—

"So, my good friend, you request me to inform you where, and how, I have passed my time since we met in London, which I think was in 1775 or 1776. To satisfy you in that respect, would be something like undertaking to write my own life—a task I could not possibly execute to my own satisfaction, if I omitted nothing, and stuck to the truth, and 'nothing but the truth.' You shall, however, have something like a biographical sketch from the time I left Gloucestershire.

"You cannot but remember when we were wont to meet in the evenings and puzzle ourselves with *x plus y minus z*, &c. You were under the tuition of Mr. Davis, who was reckoned a conjurer in the mathematics;—I had nothing to help me out but Gordon's Young Man's Companion, an old edition of Wingate's Arithmetic, and Stone's Euclid. Simpson's Algebra, and Martin's Trigonometry, however, soon increased my library; and in a short time I was qualified for an usher. I then left the cloth-working business, for which I was designed, and engaged myself to Mr. James, who kept a very respectable boarding-school at Stoke-Bishop, near Bristol. I staid there about three years, and left it with an intention of commencing school-master in another part of the country: but I had only just made the attempt, when I found there was no chance of succeeding. Thus disappointed, I left the country rather in disgust, and arrived in London in the summer of 1772.

"My first lodging was at Mr. Cook's carpet-weaver, at Hoxton. During the first two or three months I did nothing but ramble about London. I was taken ill; and while that obliged me to keep at home, I had many opportunities, of examining the process of carpet-weaving; and by frequent trials, by way of amusement, became so dexterous, that Mr. Cook offered to engage me as a workman. An advertisement, however, for an usher to teach arithmetic, brought me towards the west end of the town, and I was engaged at Archbishop Tennison's grammar-school, near the King's mews. The master's name was Applegarth, a very fine penman, but not equal to Chinnery or Tomkins. You know my name had appeared in some of the magazines, and the Ladies' Diary, among those of other correspondents who sometimes answered mathematical questions; I therefore made myself known to Mr. Thomas Todd, by whom I was introduced to most of the mathematicians in and near London, This brings to my recollection a letter you sent me full thirty years ago, in which you inquired about the Philomaths who figured away in the periodical publications at that time: this letter, I believe, was not answered. Those, however, with whom I was particularly acquainted were, Jos.

Keech, a clerk and attorney in the Lord Mayor's court office,—Reuben Robbins, bricklayer, and surveyor,—Mr, Todd, salesman by commission in Smithfield market,—G. Sanderson, tailor, but latterly a private teacher of mathematics,—Mich. Taylor, who published a large book of Logarithms,—Mr. Wales, mathematical master of Christ's Hospital,—Mr. Moss, supervisor in the excise,—Dr. Maskelyne, the astronomer royal,—Dr. Hutton, professor of mathematics in the royal academy, Woolwich, but lately retired,—Mr, Bonnycastle, who succeeded Dr. Hutton, at Woolwich,—The Rev. Messrs. Crakelt and Lawson,—Mr. Landen, many years agent to Earl Fitzwilliam; he usually came to London once a year,—Mr. Burrow, who published the Diary for Mr. Carnan, in opposition to those published by the Stationers' Company,—Mr. Witchel, some time master of the royal naval academy at Portsmouth. He was bred a watchmaker; but had great genius as a mathematician.—The greater number of these ingenious men have solved their *last* problems; and the expressions for life, like vanishing fractions, are become equal *to zero*.

"We had a sort of club, or society, which met once a fortnight; but it frequently became of the convivial kind. Keech, who was a *bon vivant*, and sung an excellent song, usually took the chair. The company, however was not exclusively confined to mathematicians; for among the visitors we sometimes had the Rev. G. Crabbe the poet.

"I had not been twelve months with Mr. Applegarth when I met Mr. Bayly, who, for a short time had been at Muswell-hill, near Highgate, making astronomical observations at an observatory which the Hon. Topham Beauclerk had lately erected in his pleasure grounds: Bayly, however, was about to be engaged as an astronomer to sail with Captain Cook; and on his recommendation I obtained his situation at Muswell-hill. The observatory was furnished with all the necessary instruments of the best kind, except the mural arc by *Sisson*, which was badly divided. The observatory was built under the direction of Dr. Bevis; but unfortunately the situation was a declivity over a stratum of clay, and in less than four years the stone pillars gave way, which rendered the mural-arc and transit useless. Mr. Beauclerk had a very complete optical and philosophical apparatus, of which I had the management to make experiments. He also built a laboratory; but Dr. G. Fordyce was the chemical operator; he usually came to Muswell-hill early on the Sundays: I was *Whackum*; but frequently in the Doctor's absence got busily at work with the retorts, salts, and acids; this often cost me a pair of ruffles; for, as you may recollect, ruffles were so universally the fashion at that time, that it was not uncommon to see a pair hanging at the knuckles when the shirt was wanting.

"The library, a remarkably fine room, was just finished when I went to Muswell-hill: it contained upwards of 30,000 volumes; and I frequently assisted in arranging them; I next undertook to write the catalogue:—thus from an astronomer I was insensibly, as it were metamorphosed into a librarian: and instead of stargazing, spent my time in poring over De Bure's *Biblographie Instructive*, and the *Bibliotheque Curieuse* of David Clement. I now found that I had been wholly inattentive to Latin since I had left the grammar school, (where, by the bye, I never got much) however, by dint of application, aided by dictionaries and translations, I soon acquired more than I had lost, but have no pretensions to a critical knowledge of that language. I was not long in getting French enough to translate any book of science, but could never speak it fluently. My time was now divided between Muswell-hill and London; but I had little to do besides rummaging old book stalls, and attending book auctions when I was in town.

"While I resided at Muswell-hill I had a sort of dispute (by letter) with the Hon. *Daines Barrington* about an old watch. The honourable gentleman had a paper in the Antiquarian Society's Tracts on the invention of pocket-watches, and to prove their antiquity, he described an old silver watch, said to have been lately found at Bruce castle. On the dial plate was engraved *Robertus B. Rex Scottorum*, in the old English or German character; this led the antiquarian to conclude, and maintain, that the watch must have belonged to Robert Bruce, king of Scotland. The last of the Roberts died about 1406, which induced me to suspect that the engraving was an imposition. A friend of mine who corresponded with his brother in Scotland, undertook to make enquiry about the watch. The result was a letter from a Mr. Jamieson, who lived at Bruce castle, then the residence of Lady Clackmannan. He informed us the people at the castle had never heard of any watch being found; but after much enquiry he met with an ingenious watch maker of Glasgow, who told him that an old silver watch had fallen into his hands two or three years ago, and for a 'piece of fun' *he* added the engraving *Robertus B. Rex Scottorum*, in the old character. All this I communicated to the honourable gentleman;—but an antiquarian never recants: besides, his paper was evidently written for the purpose of bringing forward this watch; he therefore removed all doubts and difficulties by telling me *he was certain* the Glasgow watch-maker had heard of the discovery, and wanted to put off his old watch for the genuine one found at

Bruce castle. The watch, it appears, was brought from Scotland, and belonged to a Mr. Campbell, upholsterer to his Majesty at that time. Mr. Hatton, a watch-maker in the Old Bailey, and author of a valuable treatise on watch-making, who had examined the watch, assured me that it differed little in the construction from those made in the time of Hooke and Huygens. In 1780 I saw the watch among other curiosities in a sort of museum, near the Observatory in Kew Gardens, and immediately perceived that the engraved letters were not shaped like those of the 14th century, but appeared to be much more modern.

"In 1780 Mr. Beauclerk's instruments, chemical apparatus, &c. were sold by auction. The sale, which I attended for the purpose of receiving the ready money, finished just as the riots began. During this commotion I usually slept in the day-time, and rambled about London at night, and was present at most of the conflagrations; all this time I had upwards of £600 in a side-pocket;—a fine prize I should have been for some of the mobility, had they known the circumstance. In 1781 Mr. B.'s library was sold, the sale lasted 50 days.

"My next engagement was with Lord Beauchamp, who wanted a person to arrange and make a catalogue of his books: I was, however, not long in his lordship's service; for in 1782 I was appointed a mathematical master at the maritime school, Chelsea. This institution, for maintaining and educating forty boys, the sons of naval officers, was supported by voluntary subscriptions. The 'great and good' Jonas Hanway, Esq. (as he was usually denominated) was treasurer; and no person could have been found more proper. He wrote books with a view to its encouragement, and was indefatigable in soliciting subscriptions. But this individual, like every other mortal, had failings; and *vanity* was the most prominent: he conceived himself born to rule; and *aut Cæsar aut nullus* should have been his motto. About 1784 the Duke of Cumberland, brother to his Majesty George III. became president; this was too much for Mr. Hanway.—President ranks higher than treasurer: and it was soon remarked that the zeal of the latter began to cool, and that he did not exert himself as before. But whatever might have been the cause, the subscriptions fell off; and in about two years the Committee transferred the concern to the French master and myself. We carried it on till the spring of 1787, when it was finally given up.

"I shall now answer your inquiries about the Trigonometrical Survey. An uncertainty of three or four seconds in time had constantly existed respecting the difference of longitude between the observatories of Greenwich and Paris, which had always been determined from astronomical observation. To settle this point, the French mathematicians proposed to find the difference of meridians by geodetical measurement, for which purpose they extended a series of triangles from Paris to the coast opposite Dover; and in 1783 they presented a memorial, through their ambassador, requesting our co-operation. This memorial was laid before the Royal Society; and *Mr. Ramsden* was immediately desired to contrive an instrument by which horizontal angles could be taken to a sufficient degree of accuracy: and *General Roy* selected to execute the business on our side of the channel. In 1784 he measured a base between five and six miles in length on Hounslow Heath: but Ramsden's instrument was not finished till July, 1787. I had been intimately acquainted with Ramsden from the time I went to Muswell Hill, and he recommended me as an assistant to the General.

"In 1787 and 1788, the triangles were extended from Hounslow Heath to Greenwich observatory, and thence through Kent, and part of Sussex, to the coast opposite France; and by reciprocal observations across the channel, connected with those of the French. The difference of longitude deduced from this operation was found to be very nearly the same as that obtained from the best astronomical observations.

"General Roy's account of this measurement is in the Philosophical Transactions; but it is not alogether what it ought to have been. His description of the apparatus, detail of occurrences, &c. are well enough; but he should not have meddled with the mathematical part, *for his knowledge did not extend beyond Plane Trigonometry.* I drew up the computations in that form which I thought the most proper for publication, but he was continually making alterations. *He did not even understand the rule I made use of for finding the excess of the sum of the three angles of a spherical triangle above 180°, (which since that time has been quoted as General Roy's theorem*) and would not insert it until he had consulted the Hon. Henry Cavendish.* For conducting the business in the field, however, few persons could have been found better qualified than the General. I believe he was the best topographer in England; and knew the situation of every barrow, cairn, and hillock in Great Britain. He had something of an observatory in the upper part of his dwelling, and could regulate a clock or watch by means of transits. In fact, he was ready enough at calculations which depended merely on the use of tables. *But the rules which he published for measuring heights by the barometer, all came from Mr. Ramsden.*

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* It is not until very recently that Mr. Dalby has had justice done him with regard to this ingenious rule. At p. 138, of the new edition of vol. 3, of *Hutton's Course of Mathematics*, published by Dr Gregory, in 1827, we find this note:—"This is commonly called 'General Roy's rule,' and given by him "in the Philosophical Transactions for 1790, p. 171; it is, however, due to "the late Mr. Isaac Dalby, who was then General Roy's assistant in the Trigo-"nometrical Survey, and for several years the entire conductor of the mathe-"matical department."

" Gen. Roy died in 1790, and soon afterwards the President and Secretary of the Royal Society requested me to examine his account of our operations. My corrections with the errata, were printed in their Transactions. About the same time I had another paper in the same publication, on the figure of the earth, deduced from measurement.

" In 1791, Mr. Ramsden finished another instrument like that which he had made for Gen. Roy. This was intended for the East India Company, but some dispute having taken place about the price, he sold it to the *Dnke of Richmond*, then Master General of the Ordnance. His Grace then determined principally on the earnest solicitation of Dr. Hutton, Professor of Mathematics at Woolwich, that a general survey of England should be commenced without delay; and I was immediately engaged, together with Colonel Williams, and Capt. (now Col.) Mudge, of the Royal Artillery, (the latter being strongly recommended by Dr. Hutton,) to execute the business. Ten or twelve artillery men, mostly artificers, were selected to assist in the operations.

" We began our operations by re-measuring the base on Hounslow Heath, with a steel chain one hundred feet long; General Roy measured it with glass rods: the difference, however, in the two results was less than three inches; a truly remarkable circumstance!

" The details of our progress were printed in the Philosophical Transactions. But in 1798, Col. Mudge and myself undertook to revise Gen. Roy's paper, and connect it with the subsequent accounts to the end of 1796. The whole was reprinted and published in a quarto volume, by *Mr, Faden*, the geographer, Charing Cross.

" Colonel Williams, our commandant, had now very suddenly taken his departure to the next world;—an event from which the survey received a solitary benefit, in consequence of Colonel Mudge's appointment to the higher situation.—But on this I shall not enlarge, recollecting the old maxim, *de mortuis nil nisi bonum.*

" We usually left London in March, and seldom returned before November; and consequently led a sort of rambling life during seven months of the year: as our encampments were necessarily confined to the highest situations, except when measuring the base, we sometimes found the tents to be very uncomfortable lodgings in boisterous wet weather. But the time had now arrived for another change in my occupations.

"General Jarry and Colonel Le Marchant, by order of the Commander-in-chief, the Duke of York, had established the Military College, at High Wycombe, in 1799; and I got my present situation, in consequence of a letter which the Colonel wrote to Captain Mudge, informing him that a mathematical master was wanting.—I was afraid of being *worn out* in the service, being then more than fifty years old, and thought this a fair opportunity of securing a retreat from the fatigues of the field.

"The survey still goes on under the direction of Col. Mudge. We had carried the triangles through the southern counties to the Land's End, before I quitted—but they are now extended over the greater part of England."

Here Mr. Dalby's letter suddenly terminates. He proposed writing another, in which he might pourtray the characters of *Mr. Reuben Burrow*, and of the celebrated *Ramsden*, in their respective departments; but his increasing infirmities prevented him from executing that task.

Of his own scientific character it remains for another to speak. Considered as a mathematician, he cannot be regarded as profound; but his knowledge was extensive, and his application of that knowledge to practical purposes usually successful. Like his eminent contemporary, Hutton, he hated the mere parade of science, and aimed constantly at simplicity and perspicuity.

Besides the valuable papers on the trigonometrical survey, the figure of the earth, &c. published in the Philosophical Transactions, Mr. Dalby proposed and resolved a number of curious and interesting problems at different times in the Ladies' and Burrow's Diaries, and other periodical publications of that kind; and soon after his appointment to the Royal Military College he published a Course of Mathematics for the use of the Students of that Institution, a work of great utility, and which has now reached a 5th edition.

In his various transactions and intercourse with mankind, the subject of this sketch was regarded as honourable and upright; he discharged attentively, and for many years with great activity, the duties of his successive appointments; and they who knew him intimately, were charmed with his society,

LAPLACE.

LAPLACE, (PIERRE SIMON MARQUIS DE), one of the greatest mathematicians of modern times, was born at Beaumont-en-Auge, in the department of Calvados, on the 23d of March, 1749. The son of a labourer, who could only, by his daily toil, procure for himself and family the strict necessaries of life, Laplace did not appear destined to take his place among the learned; but the extraordinary talents which he manifested from his earliest years, procured him the patronage of some rich individuals, who undertook the charge of his education, and facilitated to him the means of studying the mathematics, for which he evinced an uncommon aptitude. His progress was so rapid, that while yet very young, he was appointed Professor of that science in the College of his native town; but in this situation he remained only a very short time. Instigated by the desire of acquiring a more profound knowledge of the sciences, and of obtaining greater facilities for study, he removed to Paris, where the great proficiency he had already made soon acquired him friends and protectors; among others, the illustrious d'Alembert, who, in some measure, directed his first steps in the career of science, and who did not fail to recognize in the young geometer, one whom he would shortly have for a rival. Laplace profited so well by the lessons of this great master, that in a short time he signalized himself by the capital discovery of the invariability of the mean distances of the planets from the sun, and by various memoirs which he began to publish on subjects of the highest importance. These essays were made under the auspices of the president Saron, to whom he dedicated his works, and who defrayed the expence of their publication. Laplace, already a skilful courtier, had the address to draw the utmost advantages from this high patronage; he appeared frequently before the public, and soon acquired a reputation for a knowledge of the transcendental mathematics, equalled only by that of Lagrange. With so much activity he could not remain long without employment. By some dexterous manœuvres he contrived to succeed Bezout in the situation of examiner of the Royal Corps of Artillery, an office which prepared the way for farther preferment. Laplace, who studied much, and was as ambitious of literary honours as of popular applause, was only twenty-four years of age when he was admitted into the Academy of Sciences; the members of which were eager to appropriate the splendid talents by which they had already reason to expect so much lustre would be reflected on their own body. Besides, his admission was no more than an act of justice. It would have been impossible to have preferred a geometer of superior merit; and the zeal with which he discharged the duties of an academician is attested by the long series of important Essays with which he enriched the memoirs during a period of sixty years. From this time he directed his meditations to the research of the laws which govern the universe, and occupied himself with the establishment of many analytical theorems which lead to important applications; and, at the same time, contemplated a review of the theory of Probabilities, of which he has greatly elucidated the doctrines, extended the applications, and approximated the calculations to certainty. In 1796, Laplace, whose indefatigable zeal never relaxed, did homage to the Council of Five Hundred, by the work which was the fruit of his nights of labour. The Exposition of the System of the World, which he now published, was received with admiration by the learned, and caused a sensation throughout Europe. A few years before this, he had made one of a deputation who presented themselves at the bar of the National Convention to swear eternal hatred against royalty: the times had now become calm, and these revolutionary parades being no longer in season, it was necessary to have recourse to other methods of acquiring notoriety; he therefore proposed to his colleagues of the Institute to offer to the Representives of the People an annual account of their labours; and he appeared at the head of the members who were charged with this mission. In the discourse which he pronounced on this occasion, he recapitulated the names of all the great men who had done honour to France by their genius or their learning, and rendered a touching homage to the memory of his benefactor, the president Saron. After the 18th *Brumaire* (9th November), 1799, Laplace was appointed, by the Consuls, to the ministry of the interior, but he was obliged, in the course of a few weeks, to resign this high office into the hands of Lucien Bonaparte. It was not without regret, and some attempts to retain it, that he surrendered a portfolio so flattering to his ambition; but although he considered himself as accomplished a statesman as he was universally allowed to be a profound mathematician, he could never inculcate this opinion on Bonaparte, who, in other respects, was always disposed to shew him favour, and do justice to his rare endowments. "A geometer of the first order" wrote the latter, "Laplace quickly appeared a minister below mediocrity. His very first official act convinced us that we had deceived ourselves with regard to him. Laplace never seized any question in it's true point of view; he sought after subtleties every where; his ideas were problematical, and he carried the spirit of infinitesimals into the administration." After a ministry of six weeks, in which he performed no useful service, he was called to the Senate, by a decree of the month of December, 1799. Vice-president of this body, in July, 1803, he became Chancellor of it the following month, and received the Grand Cross of the Legion of Honour. In 1805, when the imperial government proposed to efface the last traces of the Republic, Laplace, who never lost an opportunity of paying court to the ruling powers, hastened to make a report to the Senate on the necessity of re-establing the Gregorian calendar, and of abandoning that of the revolution. In 1811, he became President of the *Maternal Society;* two years after, he was nominated Grand-officer of the order of *Re-union;* he had been Count of the Empire since 1806. Loaded with all these favours by the kindness of Bonaparte, Laplace, in 1814, actively contributed to the overthrow of the man to whom he owed them; he was one of the first to subscribe to the subversion of the imperial throne, and to vote for the re-establishment of a provisional government. Although the charge of ingratitude is not extremely rare, yet on the restoration of the Bourbon family, he was rewarded for the facility with which he had abandoned his ancient master by the title of Marquis and a seat in the Chamber of Peers. On this occasion he maintained the engagements he had contracted in accepting a seat in the Aristocratic Chamber of the kingdom, and during the hundred days he did not make his appearance at the Tuilleries. In 1816 he was chosen a member of the French Academy, of which he was president in 1817. He was also one of the founders of the *Societé d'Arcueil,* composed of persons who consecrated their labours, and even their fortunes, to the promotion of science. He died on the 27th of March, 1827; and his last illness was attributed to the chagrin which he experienced on account of the pasquinades of which he had become the object in the public journals.

As a mathematician and a philosopher Laplace must be regarded as one of the most highly gifted men who ever lived. In the course of his long and brilliant career he had the glory of completing what had been begun by the genius of Newton; and like that great man, whose sublime meditations on the laws of the heavenly motions would have been sufficient to absorb any inferior intelligence, he could descend, so to speak, to the phenomena of our own earth, and give a new character to the study of that nature with which we are so intimately surrounded. To his refined and comprehensive views, seconded by the most ingenious methods of the calculus,—to the persevering sagacity which formed a distinguished feature of his character—we are indebted for the first germs of that mathematical system of corpuscular philosophy, to the perfection of which the highest efforts of the human mind will henceforth be directed. Newton comprehended in a single thought the constant laws which act on matter, and what is not less worthy of admiration, he indicated the greater part of the consequences of his principle, which have been developed by time and assiduous observation. But there is a vast distance between the anticipative views of that great genius, and the perfect appreciation of phenomena, and their comparison with experience which constitutes the astronomy of the present age. The labours of Clairaut, d'Alembert, Euler, Lagrange, and Laplace were necessary to bring it to this perfect state; and the appearance of the Mécanique Céléste in 1799, formed the complete development of the book of Natural Philosophy. Posterity will regard Laplace as a great philosopher, who endeavoured to explain nature by the application of geometry;—an application by which he has given the theory of capillary attraction; determined the degrees of probability of different methods of the calculus applied to a great number of observations;—arrived at the laws of the flux and reflux of the sea. which, in spite of all the arbitrary elements on which they depend, he has expressed by formulæ which represent, with singular accuracy, observations separated by an interval of centuries. By the same means he discovered the cause and the measure of the secular inequalities of the moon, and those of Jupiter and Saturn—problems which had resisted the efforts of all preceding geometers: detected among the numerous periodical inequalities of the moon, those which depend on the solar parallax, and afford the means of determining the figure and compression of the earth, so that an astronomer without leaving his observatory may determine the form of our planet, as well as its distance from the sun; and to shorten this very incomplete enumeration of the great results with which he has benefitted astronomy, he has unravelled the complicated laws of Jupiter's satellites—a problem of which the peculiar difficulty consists in a circumstance regarding the motions of the three first satellites unique in the system of the world, and which he has seized with a prodigious perspicacity. But it was not enough for Laplace to have traversed the heaven of Newton, and to have there inscribed his name beside that of his illustrious predecessor; he carried his calculus into regions less known, and guided by a genius no less penetrating than vast, descried in the molecular construction of material bodies, so many new systems of worlds still obeying the laws of mechanics. The introduction of mathematical considerations into this class of phenomena is of vast importance to natural philosophy and chemistry. Descartes caught a glimpse of this application of mechanics, Newton partially introduced it, but it was reserved for Laplace to establish it on irrevocable foundations, and open up a wide field for the researches of his successors.

The labours of Laplace were continued without interruption during the long period of sixty years. His scientific life is one of the fullest that can be cited; and if fecundity were not itself one of the miracles of genius, it would be difficult to believe in the almost marvellous extent and variety of his services to science. His activity and ardour continued unabated to the very close of his life; fifteen days before his last illness he read to the *Bureau* of Longitudes a memoir on the oscillations of the atmosphere, which has been published in the *Connoisance des Temps*. He had begun the impression of a new edition of the *Système du Monde;* was preparing a new Supplement to the Mécanique Céléste; and the 7th volume of the Memoirs of the Academy still contains a paper by him. The influence he exercised on the progress of mathematical and physical knowledge was immense. The crowd of active and enthusiastic youths by whom he was constantly surrounded were taught to seek glory by perfecting methods of calculation, instruments, and means of research; and sent forth to conquer nature by new facts, and new methods of observation. In this sense he may be said to have founded a school. The Aragos, the Guy-Lussacs, the Poissons, the Biots, are among the disciples he formed. Among foreigners he has always been honoured with a sustained admiration; and France, notwithstanding some weaknesses in his character, will long be proud to reckon him among those of her sons who have contributed the most to illustrate their country. Laplace is one of the few men of science who have been distinguished in the belles-lettres. His works are written with equal perspicuity, elegance, and purity, and afford an illustrious proof that scientific disquisitions do not exclude the merit of style. His *Exposition du Système du Monde* is a chef d'œuvre of lucidity and precision.

The works which we have of Laplace are:—First, Theory of the motions and of the elliptic figures of the Planets, 1784, in 4to.;—Second, Theory of the attractions of Spheroids, and of the figure of the Planets, 1785, in 4to.;—Third, Exposition of the System of the World, 1796, 2 vols. 8vo., 1799, in 4to., 4th edition, 1813, in 4to. or 2 vols. 8vo., 5th edition, revised and augmented by the author, 1824, in 4to., with portrait, or 2 vols. 8vo.;—Fourth, Treatise of the Mécanique Céléste, 1798, 2 vols. in 4to., tom. III. 1803, in 4to., tom. IV. 1805, in 4to. tom. V. 1825:—Fifth, Analytical Theory of Probabilities, 1812, in 4to., 3rd edition, 1820, in 4to.;—Sixth, Philosophical Essays on Probabilities, 1814, in 8vo., 5th edition, 1825, in 8vo.; Seventh, *Precis* of the History of Astronomy, 1821, in 8vo. (This is now printed with the Exposition of the System of the World);—Eighth, Fourth Supplement to the Theory of Probabilities, 1825, in 4to. He has besides given a great number of memoirs in the Journal of the Polytechnic School, and essays in the Connoisance des Temps, &c.

From the Biographie Universelle et Portative des Contemporains.

ARTICLE V.

Solutions to Questions proposed in No. XIX.

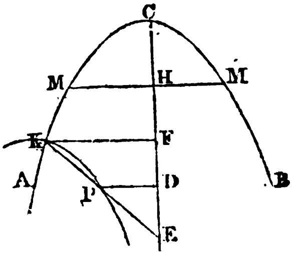
◆

I. QUESTION 471, *by* G V.

Draw a normal to a given parabola from a given point.

First Solution, *by* L.

Let ACB be the given parabola, CE its axis, P the given point, KPE the required normal; draw the ordinate KF and the line PD perpendicular to CE. By similar triangles KF : FE :: DP : DE; wherefore KF × DE = FE × DP. But, by the property of the parabola, FE is equal to half the parameter, and DP is a given line; therefore KF × DE = a given space. Take DH = FE, and through H draw MHM′ perpendicular to CE, then FH = DE, and KF × DE = KF × FH a given space. Wherefore the locus of the point K is a rectangular hyperbola passing through P the asymptotes of which are the straight lines CE and MM′.



When the point P is in the axis, the locus is the straight line MM′.

Second Solution, *by* R. N.

Let p and q be the ordinates of the given point, x and y the co-ordinates of the required point in the curve. Then $q - y = -\frac{dx}{dy}(p - x)$ is the equation to the normal at the point xy which also passes through the point pq. Also if $y^{m+n} = a^m x^n$ be the equation to the parabola, then $-\frac{dx}{dy} = \frac{m+n}{n} \cdot \frac{x}{y}$. Therefore $q - y = \frac{m+n}{n} \cdot \frac{x}{y}(x - p)$; from this equation and the equation of the curve, the values of x and y may be found, and the normal may therefore be drawn.

If $m = 1$ and $n = 1$, the parabola is the common one, and $2y^3 + (a^2 - 2ap)\,y = a^2 q$: From which y and thence x may be found.

This question was also answered by Messrs. **Baines**, **Davies**, *and* **Godward**.

II. QUESTION 472, *by* Phantom.

In a plane triangle let r denote the radius of the inscribed circle, R, R′, and R″ the radii of the three circles touching the sides of the triangle externally; then will $\frac{1}{r} = \frac{1}{R} + \frac{1}{R'} + \frac{1}{R''}$; required a demonstration.

First Solution, *by Mr.* T. S. Davies, *Bath.*

$r \cdot R \cdot R' \cdot R'' = r^2 \times$ semiperimeter (Gent. Mathematical Companion, No. xxii. or Phil. Mag. vol. ii. N. S. page 28.) $= r^2\{R \cdot R' + R \cdot R'' + R' \cdot R''\}$ (Repository, xvi. p. 5. Comp. xxvii. or Phil. Mag. ubi supra.)

Hence, dividing by $r^2 \cdot R \cdot R' \cdot R''$ we get

$$\frac{1}{r} = \frac{1}{R} + \frac{1}{R'} + \frac{1}{R''}. \quad Q.\,E.\,D.$$

Second Solution, *by Mr.* John Baines, *Thornhill, near Wakefield.*

Denoting the sides of the triangle by a, b, c, we have, by a well known property, twice the area $= r(a + b + c) = R(b + c - a) = R'(a - b + c) = R''(a + b - c)$; hence $\frac{1}{R} = \frac{b + e - a}{r(a + b + c)}$, $\frac{1}{R'} = \frac{a - b + c}{r(a + b + c)}$; and $\frac{1}{R''} = \frac{a + b - c}{r(a + b + c)}$, and consequently $\frac{1}{R} + \frac{1}{R'} + \frac{1}{R''} = \frac{b + c - a}{r(a + b + c)} + \frac{a - b + c}{r(a + b + c)} + \frac{a + b - c}{r(a + b + c)} = \frac{a + b + c}{r(a + b + c)} = \frac{1}{r}$. *Q. E. D.*

Mr. **Godward** *answered it.*

III. QUESTION 473, *by* G. V.

Let the sides of a triangle be a, b, c and its angles A, B. C. Prove that $\dfrac{(a+b+c)^2}{4\,.\,\text{area}} = \dfrac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C - 1}$.

First Solution, *by* R. N.

If r be the radius of the inscribed circle, then $\dfrac{a}{r} =$ $\cot\frac{1}{2}B + \cot\frac{1}{2}C = \dfrac{\sin\frac{1}{2}(B+C)}{\sin\frac{1}{2}B\,\sin\frac{1}{2}C} = \dfrac{\sin\frac{1}{2}(B+C)\,\sin\frac{1}{2}A}{\sin\frac{1}{2}A\,\sin\frac{1}{2}B\,\sin\frac{1}{2}C} =$ $\dfrac{2\cos\frac{1}{2}A\,\sin\frac{1}{2}A}{2\sin\frac{1}{2}A\,\sin\frac{1}{2}B\,\sin\frac{1}{2}C} = \dfrac{\sin A}{2\sin\frac{1}{2}A\,\sin\frac{1}{2}B\,\sin\frac{1}{2}C}$. Hence $\dfrac{a+b+c}{r} = \dfrac{\sin A + \sin B + \sin C}{2\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}$. Also we have $(a+b+c)r$ $= 2$ area of the $\triangle$; therefore $\dfrac{a+b+c}{4\text{ area}} = \dfrac{1}{2r}$. Therefore

$$\frac{(a+b+c)^2}{4\text{ area}} = \frac{\sin A + \sin B + \sin C}{4\sin\frac{1}{2}A\,\sin\frac{1}{2}B\,\sin\frac{1}{2}C} =$$

$$\frac{\sin A + \sin B + \sin C}{2\left\{\cos\frac{1}{2}(A-B) - \cos\frac{1}{2}(A+B)\right\}.\sin\frac{1}{2}C} =$$

$$\frac{\sin A + \sin B + \sin C}{\sin\frac{1}{2}(A+C-B) + \sin\frac{1}{2}(C+B-A) + \sin\frac{1}{2}(A+B-C) + \sin\frac{1}{2}(A+B+C)}$$

$= \dfrac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C - 1}$, since $\frac{1}{2}(A+B-C) = \frac{1}{2}(A+B+C)$ $- C = 90^\circ - C$; and $\therefore \sin\frac{1}{2}(A+B-C) = \cos C$.

Second Solution, *by Mr.* Wm. Godward, *Wakefield.*

Put $s = \frac{1}{2}(a+b+c)$; then by trigonometry, $\sin A = \dfrac{2\text{ area}}{bc}$; $\sin B = \dfrac{2\text{ area}}{ac}$, and $\sin C = \dfrac{2\text{ area}}{ab}$; hence $\sin A + \sin B + \sin C$ $= \dfrac{2\text{ area}}{abc}(a+b+c) = \dfrac{4s}{abc} \times \text{area}$.

Also $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$, and

$= \frac{a^2+b^2-c^2}{2ab}$; therefore $\cos A + \cos B + \cos C - 1 =$

$\frac{1}{2abc}\left\{a(b^2+c^2-a^2) + b(a^2+c^2-b^2) + c(a^2+b^2-c^2) - 2abc\right\}$.

Whence by decomposing this expression into factors we have

$\cos A + \cos B + \cos C - 1 = \frac{1}{abc} . (b+c-a) . (a+c-b)$

$. (a+b-c) = \frac{4}{abc} . (s-a) . (s-b) . (s-c) = \frac{4}{abcs} \times \text{area}^2$.

Therefore $\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C - 1} = \frac{s^2}{\text{area}} = \frac{(a+b+c)^2}{4\ \text{area}}$.

Messrs. BAINES *and* DAVIES *answered this Question.*

IV. QUESTION 474, *by* X.

Determine x, y and z from the three equations

$$x^2 + xy + y^2 = a$$
$$x^2 + xz + z^2 = b$$
$$y^2 + yz + z^2 = c$$

a, b, c being given numbers.

FIRST SOLUTION, *by Mr.* DAVIES, *Bath.*

$$x^2 + xy + y^2 = a = 28 \ldots \ldots (1)$$
$$x^2 + xz + z^2 = b = 21 \ldots \ldots (2)$$
$$y^2 + yx + z^2 = c = 7 \ldots \ldots (3)$$

Subtract (2) from (1), and (3) from (2), and we get

$x(y-z) + y^2 - z^2 = \overline{x+y+z} . \overline{y-z} = a - b \ldots . (4)$

$z(x-y) + x^2 - y^2 = \overline{x+y+z} . \overline{x-y} = b - c \ldots (5)$.

Divide (4) by (5) and we get

$\frac{y-z}{x-y} = \frac{a-b}{b-c}$ or $x = \frac{\overline{a-c} . y - \overline{b-c} . z}{a-b}$, or putting

$\frac{a-c}{a-b} = A$, $-\frac{b-c}{a-b} = B$, it is $x = Ay + Bz \ldots . (6)$.

Insert this value of x, (from 6), in equation (2) and we obtain the homogenous quadratic

$$A^2 . y^2 + \overline{2B+1} . A . yz + \overline{B^2+B+1} . z^2 = b \ldots (7)$$

The equations (3) and (7) combined in any of the methods applicable to the case will give the values of y and z; and by means of these values and equation (6) we find the value of x.

A single example adapted to the numbers which I have annexed to the original statement will be sufficient.

In this case we find $A = 3$, $B = -2$, and our equations become

$$9y^2 - 9yz + 3z^2 = 21 \ldots\ldots\ldots(7')$$
$$y^2 + yz + z^2 = 7 \ldots\ldots\ldots(3').$$

Put $y = vz$ in these equations and divide $(7')$ by $(3')$ which gives

$$\frac{9v^2 - 9v + 3}{v^2 + v + 1} = 3, \text{ or } v = 2.$$

In equation $(3')$ write $2z$ for y, and we obtain

$z = 1$; then, also $y = 2z = 2$, and, $x = 3y - 2z = 4$.

These are the equations which express the lines drawn from the angles of a triangle to a point within it such that their sum is a *minimum*. See Peacock's Examples, where it is stated that the elimination leads to a *fourth power*.

SECOND SOLUTION, *by Mr.* WM. GODWARD, *Wakefield.*

Making $y = mx$ and $z = nx$ the given equations become

$$x^2(m^2 + m + 1) = a$$
$$x^2(n^2 + n + 1) = b$$
$$x^2(m^2 + mn + n^2) = c.$$

Hence

$$x^2 = \frac{a}{m^2 + m + 1} = \frac{b}{n^2 + n + 1} = \frac{c}{m^2 + mn + n^2}.$$

These equalities give

$$m^2 + mn + n^2 = \frac{c}{a}(m^2 + m + 1) \ldots\ldots(\mathrm{A})$$

$$n^2 + n + 1 = \frac{b}{a}(m^2 + m + 1) \ldots\ldots(\mathrm{B}(.$$

And by taking (B) from (A) we obtain

$$(m - 1)n + m^2 - 1 = \frac{c - b}{a}(m^2 + m + 1).$$

Hence, by putting $\frac{c - b}{a} = d$, we have

$$n = \frac{d}{m - 1}(m^2 + m + 1) - (m + 1).$$

This value of n substituted in equation (B), and the result divided by the trinomial m^2+m+1, gives

$$\left(\frac{d}{m-1}\right)^2\cdot(m^2+m+1)-\frac{d(2m+1)}{m-1}=\frac{b}{a}-1.$$

This equation reduces to the quadratic

$$(d^2-2d-\frac{b}{a}+1)m^2+(d^2+d+\frac{2b}{a}-2)m=\frac{b}{a}-1-d^2-d.$$

Let us now take an example in numbers, *viz.* $a=28$, $b=172$, and $c=208$; then $d=\dfrac{c-b}{a}=\dfrac{2}{7}$. These values being substituted in the final equation give, by reduction, $m^2-\dfrac{24}{11}m=-\dfrac{4}{11}$, the resolution of which determines $m=2$ and $n=\dfrac{d}{m-1}(m^2+m+1)-(m+1)=6$. Hence $x=\left(\dfrac{a}{m^2+m+1}\right)^{\frac{1}{2}}=2$, $y=mx=4$, and $z=nx=12$.

THIRD SOLUTION, *by* Mr. JOHN BAINES.

Put $y=mx$, and $z=nx$, then

$$x^2(1+m+m^2)=a\ldots.(1)$$
$$x^2(1+n+n^2)=b\ldots.(2)$$
$$\text{and }x^2(m^2+mn+n^2)=c\ldots.(3).$$

Dividing (1) by (2) $\dfrac{m^2+m+1}{n^2+n+1}=\dfrac{a}{b}\ldots.(4)$

and (3) by (2) $\dfrac{m^2+mn+n^2}{n^2+n+1}=\dfrac{c}{b}\ldots.(5)$

Subtract (4) from (5) and $m(n-1)=\dfrac{c-a}{b}(n^2+n+1)-(n^2-1)$, which gives

$$m=\frac{c-a}{b}\left(\frac{n^2+n+1}{n-1}\right)-(n+1)\ldots.(6).$$

Substitute this value of m in (4), or in its equal, $m^2+m+1=\dfrac{a}{b}(n^2+n+1)$, and

$$\left(\frac{c-a}{b}\right)^2\cdot\frac{(n^2+n+1)^2}{(n-1)^2}-\frac{2(c-a)}{b}\cdot\frac{(n+1)\cdot(n^2+n+1)}{n-1}+(n+1)^2+\frac{c-a}{b}\left(\frac{n^2+n+1}{n-1}\right)-(n+1)+1=\frac{a}{b}(n^2+n+1).$$

Clearing this last of its denominator $(n-1)^2$, and reducing, it is observable that the whole is divisible by n^2+n+1, and hence we have

$$\left(\frac{c-a}{b}\right)^2 \cdot (n^2+n+1) - \frac{c}{b} \cdot (2n^2-n-2) + \frac{a}{b} \cdot (n^2+n-2) +$$

$n^2-2n+1=0$, and this by farther reduction becomes

$$n^2+n \cdot \frac{(c-a)^2+b(a-2b+c)}{(c-a^2)+b(a+b-2c)} = -\frac{(c-a)^2+b(-2a+b+c)}{(c-a)^2+b(a+b-2c)};$$

Consequently, completing the square, &c.

$$n = \frac{\pm(c-a)\sqrt{\{3b(a+b+2c)-3(c-a)^2\}}-(c-a)^2-b(a-2b+c)}{2(c-a)^2+2b(a+b-2c)}.$$

From this value of n, m may be obtained by substituting it in (6), and thence x, y, and z from (1) and the assumptions at the beginning.

Ex. If $a=1$, $b=2$, and $c=4$, we have $n=\frac{3\sqrt{21}+11}{2}$, $m=\sqrt{21}+4$, $x=\sqrt{\frac{2}{3}-\frac{1}{7}\sqrt{21}}=\cdot109604$, $y=\sqrt{\frac{14+\sqrt{21}}{21}}=\text{'}940683$, and $z=\sqrt{\frac{91-3\sqrt{21}}{42}}=1\text{'}356222$, &c.

V. QUESTION 475, *by* G. V.

Shew how a straight line may be drawn to touch a given circle from a given point either in the circumference or without the circle by means of a ruler only, without employing compasses.

SOLUTION, *by Mr.* DAVIES, *Bath.*

Let A be the given point, and situated (*Fig.* 1.) *without* the circle. Draw any three lines from A to cut the circle in B, C; D, E; F, G, respectively. Join D, C and B, E intersecting in H, and BG, and CF intersecting in I. The line HI being drawn will cut the circle in K; L, the

Fig. 1.

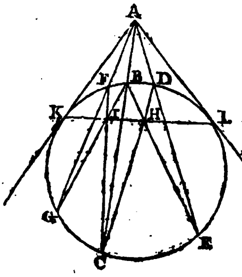
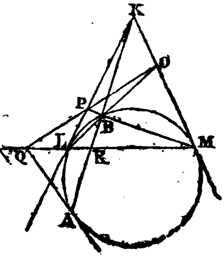


Fig. 2.



points of contact of the tangents from A. This has been repeatedly proved.

Fig. 2. When the point A is *in* the circumference. Take any point K without it, and find the tangents KL, KM (case 1.). Join KA cutting the circle again in B; through B draw LB, MB meeting the tangents in O and P respectively. Finally draw OP, LM intersecting in Q and QA will be the tangent sought.

For, draw QK. Then since AB, LM are harmonically divided in K, R and Q, R, the line QK is the *polar* of the *pole* R. Whence A, B are the points of contact of tangents from Q to the circle.

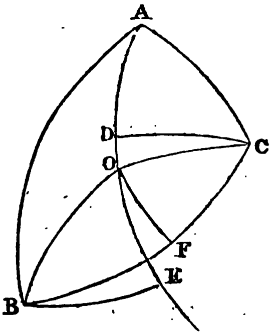
Schol. The same construction applies to all the conic sections as to the circle, and resting on the same investigation.

This question was answered by Messrs. BAINES *and* GODWARD.

VI. QUESTION 476, *by* A. B.

If from the extremities of the base of a spherical triangle perpendiculars be drawn to the arc which bisects the vertical angle: Prove that the rectangle contained by the sines of the perpendiculars is equal to the rectangle of the sines of the segments of the base made by the contact of the inscribed circle.

FIRST SOLUTION, *by* R. N.



Let ABC be the triangle, O the centre of the inscribed circle which is situated in the great circle ADE bisecting the vertical angle A. Draw OF perpendicular to BC; CD and BE perpendiculars to DE; join CO, BO. Now, since it is well known that the angle COF = BOE and that BOF = COE = supplement of COA, we have by Napier's Rules

$$\sin \mathrm{BE} = \sin \mathrm{BO} \,.\, \sin \mathrm{BOE} = \sin \mathrm{BO} \,.\, \sin \mathrm{COF}$$

$$\text{and } \sin \mathrm{CD} = \sin \mathrm{CO} \,.\, \sin \mathrm{COD} = \sin \mathrm{CO} \,.\, \sin \mathrm{BOF}$$

$$\therefore \sin \mathrm{BE} \,.\, \sin \mathrm{CD} = \sin \mathrm{BO} \,.\, \sin \mathrm{CO} \,.\, \sin \mathrm{CQF} \,.\, \sin \mathrm{BOF} \ldots. (1)$$

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* See my Paper on Solid Geometry, for the definition of this term.

Again by the same rules

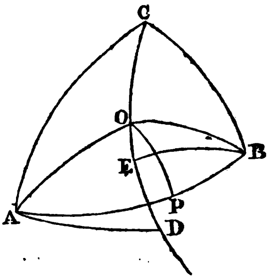
$\sin BF = \sin BO \,.\, \sin BOF$

and $\sin CF = \sin CO \,.\, \sin COF$. Therefore

$\sin BF \,.\, \sin CF = \sin BO \,.\, \sin CO \,.\, \sin BOF \,.\, \sin COF$

$= \sin BE \,.\, \sin CD$ by (1).

SECOND SOLUTION, *by* A. B., *the Proposer.*



Let O be the centre of the inscribed circle, P the point of contact with the base; AD, BE the perpendicular arcs on the line bisecting the vertical angle. Then it is well known that the angle AOP is = the angle BOE and consequently the angle AOD = the angle BOP. Now in the right angled triangles AOD, BOP,

$\sin AO : \sin AD :: (\text{rad} : \sin AOD\,(BOP)) :: \sin OB : \sin BP$,

and in the right angled triangles AOP, BOE,

$\sin AP : \sin AO :: (\text{rad.} : \sin AOP\,(BOE)) :: \sin BE : \sin OB$;

therefore ex equali

$\sin AP : \sin AD :: \sin BE : \sin BP$

and $\sin AP \,.\, \sin BP = \sin AD \,.\, \sin BE$. *Q. E. D.*

From this theorem we immediately deduce the well known rule in spherical trigonometry for finding an angle when the three sides are known. For $\sin AD = \sin AC \times \sin \frac{1}{2} \angle C$, and $\sin BE = \sin CB \times \sin \frac{1}{2} \angle C$, therefore $\sin AD \times \sin BE = \sin AC \sin BC \sin^2 \frac{1}{2} C$,

$$\text{and } \sin \tfrac{1}{2}C = \sqrt{\frac{\sin AD \sin BE}{\sin AC \sin BC}} = \sqrt{\frac{\sin AP \sin BP}{\sin AC \sin BC}} =$$

$$\sqrt{\frac{\sin \frac{1}{2}(AC + AB - BC) \sin \frac{1}{2}(AB + BC - AC)}{\sin AC \sin BC}}.$$

Cor. $\tan^2 OP = \tan OD \tan OE$.

For, $\tan AO : \tan OP :: (r : \cos AOP\,(BOE) ::) \tan OB : \tan OE$

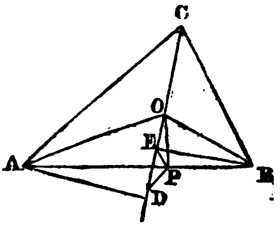
$\tan OD : \tan AO :: (r : \cos AOD\,(BOP) ::) \tan OP : \tan OB$.

Therefore $\tan OD : \tan OP :: \tan OP : \tan OE$,

or $\tan^2 OP = \tan OD \tan OE$.

In a similar manner it may be proved in a plane triangle that $AD \times BE = AP \times PB$.

Because the angle $AOD = BOP$, and $AOP = BOE$; and the angles at D, P, and E right angles, the triangle ADO is similar to BOP, and AOP similar to BOE;



Therefore $AD : AO :: BP : BO$,

and $AO : AP :: BO : BE$;

therefore, ex equali

$AD : AP :: BP : BE$,

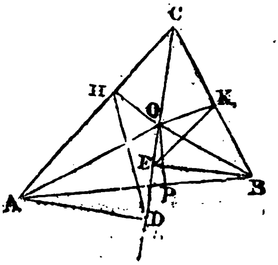
and $AD \times BE = AP \times BP$.

Or, it may be proved otherwise, as follows:

Because the points A, O, D, B are in a circle, the angle $APD = AOD = BOP$. Also, because the points O, E, P, B are in a circle, the angle $PEB = BOP$; therefore $APD = PEB$, and BAD is $= PBE$; therefore the triangles PDA, BEP are equi-angular. Hence $AD : AP :: BP : BE$ and $AD \times BE = AP \times BP$.

The same may also be proved in this manner:

Draw the radii OH, OK, and join HD, EK. Because the points A, H, O, D are in a circle, the angle $AHD = AOD = BOP$; and because the points B, P, E, O, K are in a circle, the angle $KEB = KOB = BOP$, therefore $AHD = KEB$, and the angle DAH is $= EBK$ (each being the complement of $\frac{1}{2}C$); therefore the triangles AHD, EBK are equi-angular; wherefore $AD : AH :: BK : BE$, and $AD \times BE = AH \times BK = AP \times BP$.



Cor. $OP^2 = OD \times OE$.

Answered also by Messrs. **Baines**, **Davies**, *and* **Godward**.

VII. QUESTION 477, *by* A—B.

To find an expression for the area of a spherical quadrilateral inscribed in a small circle of the sphere in terms of the four sides of the quadrilateral.

First Solution, *by* M. Guenea D'Aumont.

Taken from the Annales de Mathematiques; Tome XII.

Let a, b, c, d be the consecutive sides of a spherical quadrilateral inscribed in a circle of the sphere; x and y the two diagonals of this quadrilateral; the first being that which terminates in the summit of the angles (a, b), (c, d); and the latter that which terminates in the summit of the angles (b, c), (a, d).

The chords of the four sides and of the two diagonals, which are

$$2\sin\tfrac{1}{2}a,\ 2\sin\tfrac{1}{2}b,\ 2\sin\tfrac{1}{2}c,\ 2\sin\tfrac{1}{2}d,\ 2\sin\tfrac{1}{2}x,\ 2\sin\tfrac{1}{2}y,$$

are the four sides and the two diagonals of the rectilineal quadrilateral inscribed in the same circle; and substituting these values in the well known formulæ for the diagonals (in the plane quadrilateral) in terms of the sides, we have

$$\left.\begin{array}{l}\sin^2\tfrac{1}{2}x = \\ \dfrac{\left(\sin\frac{a}{2}\sin\frac{c}{2}+\sin\frac{b}{2}\sin\frac{d}{2}\right)\left(\sin\frac{a}{2}\sin\frac{b}{2}+\sin\frac{c}{2}\sin\frac{d}{2}\right)}{\sin\frac{b}{2}\sin\frac{c}{2}+\sin\frac{a}{2}\sin\frac{d}{2}} \\ \sin^2\tfrac{1}{2}y = \\ \dfrac{\left(\sin\frac{a}{2}\sin\frac{c}{2}+\sin\frac{b}{2}\sin\frac{d}{2}\right)\left(\sin\frac{a}{2}\sin\frac{b}{2}+\sin\frac{c}{2}\sin\frac{d}{2}\right)}{\sin\frac{d}{2}\sin\frac{b}{2}+\sin\frac{c}{2}\sin\frac{d}{2}}\end{array}\right\}\text{(I.)}$$

whence it follows that

$$\left.\begin{array}{l}\sin\tfrac{1}{2}x\sin\tfrac{1}{2}y = \sin\tfrac{1}{2}a\sin\tfrac{1}{2}c+\sin\tfrac{1}{2}b\sin\tfrac{1}{2}d \\ \dfrac{\sin\frac{1}{2}x}{\sin\frac{1}{2}y} = \dfrac{\sin\frac{1}{2}a\sin\frac{1}{2}b\sin\frac{1}{2}c\sin\frac{1}{2}d}{\sin\frac{1}{2}b\sin\frac{1}{2}c\sin\frac{1}{2}a\sin\frac{1}{2}d}\end{array}\right\}\ldots\ldots\text{(II.)}$$

In the spherical triangle of which the three sides are a, d, x, we have

$$\cos(a, d) = \frac{\cos x - \cos a\cos d}{\sin a\sin d},$$

whence

$$2\sin\tfrac{1}{2}(a, d) = 1 - \cos(a, d) = \frac{\cos(a-d) - \cos x}{\sin a\sin d},$$

$$2\cos\tfrac{1}{2}(a, d) = 1 + \cos(a, d) = \frac{\cos x - \cos(a+d)}{\sin a\sin d};$$

But, in general,

$$\cos k = 1 - 2\sin^2\tfrac{1}{2}k,$$

therefore

$$\sin^2\tfrac{1}{2}(a, d) = \frac{\sin^2\frac{1}{2}x - \sin^2\frac{1}{2}(a-d)}{\sin a\sin d},$$

$$\cos^2\tfrac{1}{2}(a, d) = \frac{\sin^2\frac{1}{2}(a+d) - \sin^2\frac{1}{2}x}{\sin a\sin d}.$$

Putting for $\sin^2\frac{1}{2}x$, in these two formulæ, its value just found (I) they become, by collecting into one part all the terms multiplied by sin $\frac{1}{2}b$ sin $\frac{1}{2}c$, and into another all the terms multiplied by sin $\frac{1}{2}a$ sin $\frac{1}{2}d$,

$$\sin^2\tfrac{1}{2}(a, d) = \frac{\left.\begin{array}{l}\sin\frac{1}{2}b\sin\frac{1}{2}d\,\{\sin^2\frac{1}{2}b + \sin^2\frac{1}{2}c - \sin^2\frac{1}{2}(a-d)\}\\ +\sin\frac{1}{2}b\sin\frac{1}{2}a\,\{\sin^2\frac{1}{2}a + \sin^2\frac{1}{2}d - \sin^2\frac{1}{2}(a-d)\}\end{array}\right\}}{(\sin\frac{1}{2}b\sin\frac{1}{2}c + \sin\frac{1}{2}a\sin\frac{1}{2}d)\sin a\sin d}.$$

$$\cos^2\tfrac{1}{2}(a, d) = \frac{\left.\begin{array}{l}\sin\frac{1}{2}a\sin\frac{1}{2}d\,\{\sin^2(a+d) - \sin^2\frac{1}{4}b - \sin^2\frac{1}{2}c\}\\ +\sin\frac{1}{2}b\sin\frac{1}{2}c\,\{\sin^2\frac{1}{2}(a+d) - \sin^2\frac{1}{2}a - \sin^2\frac{1}{2}d\}\end{array}\right\}}{(\sin\frac{1}{2}b\sin\frac{1}{2}c + \sin\frac{1}{2}a\sin\frac{1}{2}d)\sin a\sin d};$$

but it is easily found that

$$\sin^2\tfrac{1}{2}a + \sin^2\tfrac{1}{2}d - \sin^2\tfrac{1}{2}(a-d) = 2\sin\tfrac{1}{2}a\sin\tfrac{1}{2}d\cos\tfrac{1}{2}(a+d),$$

$$\sin^2\tfrac{1}{2}(a+d) - \sin^2\tfrac{1}{2}a - \sin^2\tfrac{1}{2}d = 2\sin\tfrac{1}{2}a\sin\tfrac{1}{2}d\cos\tfrac{1}{2}(a+d);$$

by substituting therefore, and putting in the denominators for sin a and sin d their values 2 sin $\frac{1}{2}a$ cos $\frac{1}{2}a$ and 2 sin $\frac{1}{2}d$ cos $\frac{1}{2}d$, and suppressing the common factor sin $\frac{1}{2}a$ sin $\frac{1}{2}d$, we find

$$\sin^2\tfrac{1}{2}(a, d) =$$

$$\frac{\sin^2\frac{h}{2} + \sin^2\frac{c}{2} - \sin^2\frac{1}{2}(a-d) + 2\sin\frac{b}{2}\sin\frac{c}{2}\cos\frac{1}{2}(a-d)}{4(\sin\frac{1}{2}b\sin\frac{1}{2}c + \sin\frac{1}{2}a\sin\frac{1}{2}b\cos\frac{1}{2}a\cos\frac{1}{2}d}$$

$$\cos^2\tfrac{1}{2}(a, d) =$$

$$\frac{\sin^2\frac{1}{2}(a+d) - \sin^2\frac{b}{2} - \sin^2\frac{c}{2} + 2\sin\frac{b}{2}\sin\frac{c}{2}\cos\frac{1}{2}(a+d)}{4(\sin\frac{1}{2}b\sin\frac{1}{2}c + \sin\frac{1}{2}a\sin\frac{1}{2}d)\cos\frac{1}{2}a\cos\frac{1}{2}d}$$

whence by changing respectively, in the numerators, $\sin^2\frac{1}{2}(a-d)$ and $\sin^2\frac{1}{2}(a+d)$ for $1 - \cos^2\frac{1}{2}(a-d)$ and $1 - \cos\frac{1}{2}(a+d)$

$$\sin^2\tfrac{1}{2}(a, d) = \frac{\left.\begin{array}{c}\cos^2\frac{1}{2}(a-d) + 2\sin\frac{1}{2}b\sin\frac{1}{2}c\cos\frac{1}{2}(a-d)\\ -(1\sin^2\frac{1}{2}b - \sin^2\frac{1}{2}c)\end{array}\right\}}{4(\sin\frac{1}{2}b\sin\frac{1}{2}c + \sin\frac{1}{2}a\sin\frac{1}{2}d)\cos\frac{1}{2}a\cos\frac{1}{2}d}$$

$$\cos^2\tfrac{1}{2}(a, d) = \frac{\left.\begin{array}{l}(1 - \sin^2\frac{1}{2}b - \sin^2\frac{1}{2}c)\\ -\{\cos^2\frac{1}{2}(a+d) - 2\sin\frac{1}{2}b\sin\frac{1}{2}c\cos\frac{1}{2}(a+d)\}\end{array}\right\}}{4(\sin\frac{1}{2}b\sin\frac{1}{2}c + \sin\frac{1}{2}a\sin\frac{1}{2}d)\cos\frac{1}{2}a\cos\frac{1}{2}d}$$

But

$$1 - \sin^2\frac{b}{2} - \sin^2\frac{c}{2} = \left(1 - \sin^2\frac{b}{2}\right)\left(1 - \sin^2\frac{b}{2}\right) - \sin^2\frac{b}{2}\sin^2\frac{c}{2}$$

$$= \cos^2\tfrac{1}{2}b\cos^2\tfrac{1}{2}c - \sin^2\tfrac{1}{2}b\sin^2\tfrac{1}{2}c;$$

therefore by substitution

$$\sin^2\tfrac{1}{2}(a, d) = \frac{\{\cos\frac{1}{2}(a-d) + \sin\frac{1}{2}b\sin\frac{1}{2}c\}^2 - \cos\frac{1}{2}b\cos\frac{1}{2}c}{4(\sin\frac{1}{2}b\sin\frac{1}{2}c + \sin\frac{1}{2}a\sin\frac{1}{2}d)\cos\frac{1}{2}a\cos\frac{1}{2}d}.$$

$$\cos^2\tfrac{1}{2}(a, d) = \frac{\cos^2\frac{1}{2}b\cos^2\frac{1}{2}c - \{\cos\frac{1}{2}(a+d) - \sin\frac{1}{2}b\sin\frac{1}{2}c\}^2}{4(\sin\frac{1}{2}b\sin\frac{1}{2}c + \sin\frac{1}{2}a\sin\frac{1}{2}d)\cos\frac{1}{2}a\cos\frac{1}{2}d}.$$

The numerator of the first fraction is decomposable into the two factors

$$\cos\tfrac{1}{2}(a-d)+\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c+\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c=\cos\tfrac{1}{2}(a-d)+\cos\tfrac{1}{2}(b+c)$$

$$\cos\tfrac{1}{2}(a-d)-\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c+\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c=\cos\tfrac{1}{2}(a-d)-\cos\tfrac{1}{2}'b+c).$$

The numerator of the second into the two

$$\cos\frac{b}{2}\cos\frac{c}{2}-\sin\frac{b}{2}\sin\frac{c}{2}+\cos\tfrac{1}{2}(a+d)=\cos\tfrac{1}{2}b+c)+\cos\tfrac{1}{2}(a+d),$$

$$\cos\frac{b}{2}\cos\frac{c}{2}+\sin\frac{b}{2}\sin\frac{c}{2}\cos\tfrac{1}{2}(a+d)=\cos\tfrac{1}{2}(b-c)-\cos\tfrac{1}{2}(a+d);$$

whence we have

$$\sin^2\tfrac{1}{2}(a,d)=\frac{\{\cos\tfrac{1}{2}(a-d)+\cos\tfrac{1}{2}(b-c)\}\{\cos\tfrac{1}{2}(a-d)-\cos\tfrac{1}{2}(b-c)\}}{4\left(\sin\frac{b}{2}\sin\frac{c}{2}+\sin\frac{a}{2}\sin\frac{d}{2}\right)\cos\frac{a}{2}\cos\frac{d}{2}},$$

$$\cos^2\tfrac{1}{2}(a,d)=\frac{\{\cos\tfrac{1}{2}(b+c)+\cos\tfrac{1}{2}(a+d)\}\{\cos\tfrac{1}{2}(b-c)-\cos\tfrac{1}{2}(a+d)\}}{4\left(\sin\frac{b}{2}\sin\frac{c}{2}+\sin\frac{a}{2}\sin\frac{d}{2}\right)\cos\frac{a}{2}\cos\frac{d}{2}};$$

but, we have

$$\cos\tfrac{1}{2}(a-d)+\cos\tfrac{1}{2}(b-c)=2\cos\tfrac{1}{4}(a+b-c-d)\cos\tfrac{1}{4}(a+c-b-d),$$

$$\cos\tfrac{1}{2}(a-d)-\cos\tfrac{1}{2}(b-c)=2\sin\tfrac{1}{4}(a+b+c-d)\sin\tfrac{1}{4}(b+c+d-a),$$

$$\cos\tfrac{1}{2}(b+c)+\cos\tfrac{1}{2}(a+d)=2\cos\tfrac{1}{4}(a+b+c+d)\cos\tfrac{1}{4}(b+c-a-d),$$

$$\cos\tfrac{1}{2}(b-c)-\cos\tfrac{1}{2}(a+d)=2\sin\tfrac{1}{4}(d+a+b-c)\sin\tfrac{1}{4}(c+d+a-b);$$

therefore finally

$$\sin^2\tfrac{1}{2}(a,d)=\frac{\cos\tfrac{1}{4}(a+b+c+d)\cos\tfrac{1}{4}(a+c-b-d)\sin\tfrac{1}{4}(a+b+c-d)\sin\tfrac{1}{4}(c+d-a-b)}{(\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c+\sin\tfrac{1}{2}o\sin\tfrac{1}{2}d)\cos\tfrac{1}{2}a\cos\tfrac{1}{2}d},$$

$$\cos^2\tfrac{1}{2}(a,d)=\frac{\cos\tfrac{1}{4}(a+b-c-d)\cos\tfrac{1}{4}(b+c-a-d)\sin\tfrac{1}{4}(d+a+b-c)\sin\tfrac{1}{4}(c+d+a-b)}{(\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c+\sin\tfrac{1}{2}a\sin\tfrac{1}{2}d)\cos\tfrac{1}{2}a\cos\tfrac{1}{2}d}.$$

Putting, for the sake of abridging,

$$\cos\tfrac{1}{4}(a+b+c+d)\cos\tfrac{1}{4})a+b-c-d)\cos\tfrac{1}{4}(a+\zeta-b-d)\cos\tfrac{1}{4}(b+c-a-d)=\text{M};$$

$$\sin\tfrac{1}{4}(b+c+d-a)\sin\tfrac{1}{4}(c+d+a-b)\sin\tfrac{1}{4}(d+a+b-c)\sin\tfrac{1}{4}(a+b+c-d)=\text{N};$$

in which case M and N are symmetrical functions of the four sides a, b, c, d; and recollecting that

$$\sin(a,d)=2\sin\tfrac{1}{2}(a,d)\cos\tfrac{1}{2}(a,d),$$

we have

$$\sin(a,d)=\frac{2\text{MN}}{(\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c+\sin\tfrac{1}{2}a\sin\tfrac{1}{2}d)\cos\tfrac{1}{2}a\cos\tfrac{1}{2}d}.\;\ldots\ldots(\text{III})$$

whence we see that here the opposite angles are not supplements to each other as they are in the inscribed rectilineal quadrilateral.

By changing respectively a, b, c, d for c, d, a, b we find

$$\sin^2 \tfrac{1}{2}(b, c) =$$
$$\frac{\cos\frac{1}{4}(a+b-c-d)\cos\frac{1}{4}(a+c-b-d)\sin\frac{1}{4}(c+d+a-b)\sin\frac{1}{4}(d+a+b-c)}{(\sin\frac{1}{2}b\sin\frac{1}{2}c+\sin\frac{1}{2}a\sin\frac{1}{2}d)\cos\frac{1}{2}b\cos\frac{1}{2}c}\,.$$

$$\cos^2 \tfrac{1}{2}(b, c) =$$
$$\frac{\cos\frac{1}{4}(a+b+c+d)\cos\frac{1}{4}(b+c-a-d)\sin\frac{1}{4}(b+c+d-a)\sin\frac{1}{4}(a+b+c-d)}{(\sin\frac{1}{2}b\sin\frac{1}{2}c+\sin\frac{1}{2}a\sin\frac{1}{2}d)\cos\frac{1}{2}b\cos\frac{1}{2}c}\,.$$

but we have

$$\sin\tfrac{1}{2}\{(a,d)+(b,c)\}=\sin\tfrac{1}{2}(a,d)\cos\tfrac{1}{2}(b,c)+\cos\tfrac{1}{2}(a,d)\sin\tfrac{1}{2}(b,c),$$
$$\cos\tfrac{1}{2}\{(a,b)+(b,c)\}=\cos\tfrac{1}{2}(a,d)\cos\tfrac{1}{2}(b,c)-\sin\tfrac{1}{2}(a,d)\sin\tfrac{1}{2}(b,c)\,;$$

substituting, therefore, and putting for abridging

$$\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c\cos\tfrac{1}{2}d=\text{P}$$

we get

$$\sin\tfrac{1}{2}\{(a,d)+(b,c)\}=$$
$$\frac{\left\{\begin{array}{l}\quad\sin\frac{1}{4}(b+c+d-a)\sin\frac{1}{4}(a+b+c-d)\\+\sin\frac{1}{4}(c+d-a-b)\sin\frac{1}{4}(d+a+b-c)\end{array}\right\}}{\sin\frac{1}{2}b\sin\frac{1}{2}c+\sin\frac{1}{2}a\sin\frac{1}{2}d}\sqrt{\frac{\text{M}}{\text{P}}}\,,$$

$$\cos\tfrac{1}{2}\{(a,d)+(b,c)\}=$$
$$\frac{\left\{\begin{array}{l}\quad\cos\frac{1}{4}(a+b+c+d)\cos\frac{1}{4}(b+c-a-d)\\-\cos\frac{1}{4}(a+b-c-d)\cos\frac{1}{4}(a+c-b-d)\end{array}\right\}}{\sin\frac{1}{2}b\sin\frac{1}{2}c+\sin\frac{1}{2}a\sin\frac{1}{2}d}\sqrt{\frac{\text{N}}{\text{P}}}\,;$$

which become

$$\sin\tfrac{1}{2}\{(a,d)+(b,c)\}=\frac{\left\{\begin{array}{l}\quad\{\cos\frac{1}{2}(a-d)-\cos\frac{1}{2}(b+c)\}\\\{+\cos\frac{1}{2}(b-c)-\cos\frac{1}{2}(a+d)\}\end{array}\right\}}{2\sin\frac{1}{2}b\sin\frac{1}{2}c+\sin\frac{1}{2}a\sin\frac{1}{2}d}\sqrt{\frac{\text{M}}{\text{P}}}\,,$$

$$\cos\tfrac{1}{2}\{(a,d)+(b,c)\}=\frac{\left\{\begin{array}{l}\quad\{\cos\frac{1}{2}(a+d)+\cos\frac{1}{2}(b+c)\}\\-\{\cos\frac{1}{2}(a-d)+\cos\frac{1}{2}(b-c)\}\end{array}\right\}}{2\sin\frac{1}{2}b\sin\frac{1}{2}c+\sin\frac{1}{2}a\sin\frac{1}{2}d}\sqrt{\frac{\text{N}}{\text{P}}}\,;$$

whence, by further developement,

$$\sin\tfrac{1}{2}\{(a,d)+(b,c)\}=+\sqrt{\frac{\text{M}}{\text{P}}}\,,$$

$$\cos\tfrac{1}{2}\{(a,d)+(b,c)\}=-\sqrt{\frac{\text{N}}{\text{P}}}\,.$$

These two functions being symmetrical, it results that

$$\sin\tfrac{1}{2}\{(a, d) + (b, c)\} = \sin\tfrac{1}{2}\{(a, b) + (c, d)\} = +\sqrt{\frac{M}{P}},$$

$$\cos\tfrac{1}{2}\{(a, d) + (b, c)\} = \cos\tfrac{1}{2}\{(a, b) + (c, d)\} = +\sqrt{\frac{N}{P}}.$$

and consequently

$$(a, d) + (b, c) = (a, b) + (c, d);$$

that is, that in all inscribed spherical quadrilaterals the sum of two opposite angles is equal to the sum of the other two opposite angles; a property which belongs to the inscribed rectilineal quadrilateral as well as spherical, with this difference only, that in the first the two sums are constant, whereas in the last they are variable.

If we designate by Q the area of the quadrilateral, we have, as is well known

$$Q = (a, d) + (b, c) + (c, d) + (a, d) - 2\varpi,$$

or, from what has been said,

$$Q = 2\{(a, d) + (b, c) - \varpi\};$$

whence

$$\sin\tfrac{1}{4}Q = -\cos\tfrac{1}{2}\{(a, d) + (b, c)\} = \sqrt{\frac{N}{P}},$$

$$\cos\tfrac{1}{4}Q = +\sin\tfrac{1}{2}\{(a, d) + (b, c)\} = \sqrt{\frac{M}{P}},$$

and again,

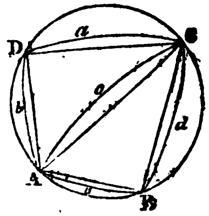
$$\sin\tfrac{1}{2}Q = 2\sin\tfrac{1}{4}Q\cos\tfrac{1}{4}Q = \frac{2\sqrt{MN}}{P},$$

$$\tan\tfrac{1}{4}Q = \frac{\sin\tfrac{1}{4}Q}{\cos\tfrac{1}{2}Q} = \sqrt{\frac{N}{M}}; \quad \ldots\ldots\ldots\ldots\ldots\text{(IV)}$$

functions which are all symmetrical.

SECOND SOLUTION, *by Mr.* JOHN BAINES.

Let ABCD be the small circle of the sphere, circumscribing the spherical quadrilateral DCBA, whose four sides AB, BC CD and DA are arcs of great circles. Let a great circle be drawn on the sphere, passing through the angles BAD, BCD, also, let the chords AB, BC, CD, DA and AC be drawn, then it is evident that the same circle which circumscribes the spherical, will also circumscribe the right-lined quadrilateral ABCD. Denoting the angles of the spherical figures by the letters at the angular points, and the opposite sides by the small letters a, b, c and d, and the arc AC by θ, we have, chord CD $= 2 \sin \frac{1}{2}a$, AD $= 2 \sin \frac{1}{2}b$, AC $= 2 \sin \frac{1}{2}\theta$, AB $= 2 \sin \frac{1}{2}c$, and BC $= 2 \sin \frac{1}{2}d$. Now, because, in the plane figures, $\angle$ ADC + $\angle$ ABC $= 180°$, one of them is the supplement of the other, and



$$\frac{4 \sin^2 \frac{a}{2} + 4 \sin^2 \frac{b}{2} - 4 \sin^2 \frac{\theta}{2}}{4 \sin \frac{1}{2}a \sin \frac{1}{2}b} = -\frac{4 \sin^2 \frac{c}{2} + 4 \sin^2 \frac{d}{2} - 4 \sin^2 \frac{\theta}{2}}{4 \sin \frac{1}{2}c \sin \frac{1}{2}d},$$

which gives $\sin^2 \frac{1}{2}\theta =$

$$\frac{\left\{ \begin{array}{l} \sin \frac{1}{2}c \sin \frac{1}{2}d (\sin^2 \frac{1}{2}a + \sin^2 \frac{1}{2}b) \\ + \sin \frac{1}{2}a \sin \frac{1}{2}b (\sin^2 \frac{1}{2}c + \sin^2 \frac{1}{2}d) \end{array} \right\}}{\sin \frac{1}{2}a \sin \frac{1}{2}b \sin \frac{1}{2}c \sin \frac{1}{2}d};$$

hence the diagonal θ is given in terms of a, b, c, and d.

Again, since the area of a spherical triangle is proportional to the excess of the sum of its three angles above two right angles, technically termed the spherical excess, which (spherical excess) by Simon Lhuillier's Theorem, may be obtained from the three sides only, we can from hence find the area of the quadrilateral in terms of its sides. Putting e and $\epsilon =$ the spherical excess in the triangles ADC, ABC, we have by S. L—'s Theor.

$$\tan \tfrac{1}{4}e = \surd\left(\tan \frac{a+b+\theta}{4} . \tan \frac{a+b-\theta}{4} . \tan \frac{a-b+\theta}{4} . \tan \frac{-a+b+c}{4}\right),$$

and

$$\tan \tfrac{1}{4}\epsilon = \surd\left(\tan \frac{c+d+\theta}{4} . \tan \frac{c+d-\theta}{4} . \tan \frac{c-d+\theta}{4} . \tan \frac{-a+b+\epsilon}{4}\right).$$

consequently, not only θ, but also e and ϵ are known in terms of the sides, and hence by the common formula for the area of a spherical triangle, $\frac{\pi d^2 (e+\epsilon)}{720} =$ area of the quadrilateral required, $\pi = 3·141592$, and $d =$ diameter of the sphere.

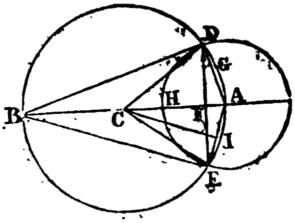
Messrs. **DAVIES** *and* **GODWARD** *answered it.*

VIII. QUESTION 478, *by* G. V.

If on A one end of the diameter of a circle an arc DHE be described meeting the circle in D and E and the chord DE be drawn meeting the diameter in F. Prove that the space between the arcs DBE and DHE are equal to $AC \cdot DF + CF \cdot$ arc BD.

First Solution, *by* Mr. Wm. Godward.

Let C be the centre of the circle and join AD, AE, CD, and CE, then the line $DBEH =$ trap. $ADCE +$ sector $DCEC -$ sect. $ADHR$.



Now the trap. $ADCE = AC \cdot DF$, and the sect. $DBEC = AC \cdot$ arc BD. Also the sect. $ADHE = AD \cdot$ arc DH; but $\frac{1}{2} \angle BCD = BAD$, and therefore $CD : \frac{1}{2}$arc $BD :: AD :$ arc $DH = AD \cdot \frac{1}{2}$ arc $BD \div CD = AD \cdot$ arc $BD \div AB$; hence the sect. $ADHE = AD^2 \cdot$ arc $BD \div AB = AF \cdot$ arc BD. Wherefore the lune $DBEH = AC \cdot DF + (AC - AF) \cdot$ arc $BD = AC \cdot DF + CF \cdot$ arc BD; but this only holds good when CF is positive or AF less than AC. When CF is negative or AF greater than AC, the lune $= AC \cdot DF - CF \cdot$ arc BD. When CF vanishes or $AF = AC$, the lune is simply $= AC^2$.

Second Solution, *by* R. N.

The area $DHEB = 2$ semicircle $- 2$ sector $ADH - 2$ segment $AGD =$ arc $BD \cdot AC +$ arc $DA \cdot AC - DH \cdot AH -$ arc $AD \cdot AC + AC , DF = BD \cdot AC - DH \cdot AH + AC \cdot DF$.

But $\angle DAC = \frac{1}{2} DCB \therefore$ arc $DH : \frac{1}{2}$ arc $DB :: AH : AC \therefore$ arc $DH = \frac{1}{2}$ arc $DB \cdot AH \div AC \therefore$ arc $DH \cdot AH = \frac{1}{2}$ arc $BD \cdot AH^2 \div AC = \frac{1}{2}$ arc $BD \cdot$ (chord $AD)^2 \div AC =$ arc $BD \cdot AF \cdot AB \div 2AC =$ arc $BD \cdot AF \therefore$ area $DHEB =$ arc $BD \cdot AC -$ arc $BD \cdot AF + AC \cdot DF =$ arc $BD \cdot CF + AC \cdot DF$.

Third Solution, *by* Mr. John Baines.

Draw CI parallel to BE. The area of the sector $EBDC = CE \cdot$ arc BE and the area of the triangle $ECD = EF \cdot CF$; hence the area of the segment $EBDE = CE \cdot$ arc $BE + EF \cdot CF$. Also, the area of the segment $EHDE = AE \cdot$ arc $BH - FE \cdot AF$;

when the difference of these two, *viz.* CE . arc BE + FE . CF — AE . arc EH + FE . AF, or AC . FE + AC . arc BE — AH . arc BE = the area of the curvilinear space included between the arcs EBD, EHD. Again ∠ ACE = 180° — BCE, but ∠ ACI = ∠ AEF = ∠ ABE = $\frac{1}{2}$∠ACE = 90° — $\frac{1}{2}$ BCE, and ∠ CIA is a right angle, ∴ the ∠ CAI = $\frac{1}{2}$∠ BCE, and AH . arc EH = $\frac{1}{2}$AH . arc BE × 2 sin ∠ ACI = AE sin ∠ AEF × arc BE = AF . arc BE: hence AC . FE + (AC — AF) × arc BE = AC . FE + CF . arc BE = area of the space EBDHE.

Fourth Solution, *by Mr.* Davies, *Bath.*

The ∠ BCE = 2BAE and EAD = 2EAH = (∴) BCE.
Hence arc BE : BA :: arc EHD : 2AE :: arc EH : BA,
or arc BE : arc EH :: BA : AB :: AE : AF
and ∴ AF . arc EB = AE . arc EH ....(1)
Again, EBDH = BDCE + ECD — EHDA + EBD........(2)
Now, ECD + EAD = AC . EF ....(3)
BDCE = arc EE . CA ....(4)
EADH = arc EH . AA = (by 1) arc BE . AF....(5)

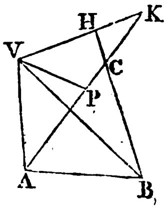
Making in (2) the substitutions furnished by (3), (4), (5), we shall obtain

EBDH = AC . EF + arc BD . CF.

IX. QUESTION 479, *by* G. V.

Let AC, BC be two straight lines given in position, and A, B given points in these lines: If AV, BV be inflected to a point V, so that the tangents of the angles VAC, VBC may have a given ratio; what is the locus of the point V?

First Solution, *by* ———.



Draw VP, VH perpendicular to AC and BC and produce VH to meet AC in K. Put AC = a, BC = b, α = angle ACB, CH = x, VH = y, and the given ratio of tan VAC : tan VBC :: $m : n$.

Then tan VAC = $\frac{\text{VP}}{\text{AP}}$, and tan VBC = $\frac{\text{VH}}{\text{BH}}$.

therefore $\frac{\text{VP}}{\text{AP}} : \frac{\text{VH}}{\text{BH}} :: m : n$. Now HK = $x \tan \alpha$, CK = $x \sec \alpha$, VK = $y + x \tan \alpha$, VP = VK $\cos \alpha = (y + x \tan \alpha) \cos \alpha$; PK

$= \mathrm{VK}\sin\alpha = (y + x\tan\alpha)\sin\alpha$, $\mathrm{AK} = (a + x\sec.\alpha)$, $\mathrm{AP} = (a + x\sec\alpha) - (y + x\tan\alpha)\sin\alpha$; therefore

$$\frac{(y + x\tan\alpha)\cos\alpha}{(a + x\sec\alpha) - (y + x\tan\alpha)\sin\alpha} : \frac{y}{x + b} :: m : n,$$

and by reduction

$$\begin{aligned} &n\sin\alpha x^2 + m\sin\alpha y^2 \\ &+ (n\cos\alpha - m\sec\alpha + m\tan\alpha\sin\alpha)xy \\ &+ nb\sin\alpha x + (nb\cos\alpha - ma)y = 0, \end{aligned}$$

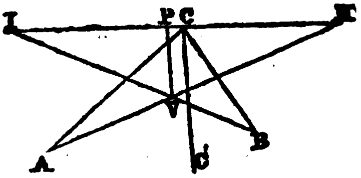
an equation to a conic section.

SECOND SOLUTION, *by Mr.* T. S. DAVIES.

Through C the intersection of AC, BC draw the rectangular co-ordinates CC′, IH. Let $x'y'$ and $x''y''$ be the co-ordinates of A, B, referred to these axes; and the lines themselves be expressed by

$y = a'x$........(1)

and $y = a''x$........(2)



Then let φ = tangent of the variable angle ACV, and $m\varphi$ = tangent of CBV. We shall have the equations of AV, BV, (Hamilton's Anal. Geom. p. 66.) *viz.*

$$y - y' = \frac{a'' - \varphi}{1 + a'\varphi}(x - x') \text{ that of AV,} \ldots\ldots\ldots(3)$$

$$\text{and } y - y'' = \frac{a' - m\varphi}{1 + a''m\varphi}(x - x'') \text{ that of BG} \ldots\ldots(4)$$

As in these two equations we have the variables x, y, φ, if we eliminate one of them, φ, we shall have one equation containing two variables x, y which will represent the locus of V, the point common to (3) and (4). This is best done by finding φ from (3), (4) and equating the values and reducing by means of (1) and (2). We shall thus get

$$\left.\begin{aligned} &(ma' - a'')\,\overline{x - x'}\,.\,\overline{x - x''} + (a - a'')\,\overline{y - y'}\,.\,\overline{y - y''} \\ &+ (1 + ma'a'')\,\overline{x - x'}\,.\,\overline{y - y''} - (m + a'a'')\,\overline{y - y'}\,.\,\overline{x - x''} \end{aligned}\right\} = 0.$$

or (since we have assumed no relation between a', and a'', we may put them $= a$ and $-a$ respectively, and we obtain)

$$\left.\begin{aligned} &\overline{m + 1}\,.\,a\,.\,\overline{x - x'}\,.\,\overline{x - x''} + 2a\,.\,\overline{y - y'}\,.\,\overline{y - y''} \\ &+ (1 - ma^2)\,\overline{y - y''}\,.\,\overline{x - x'} + (a^2 - m)\,\overline{y - y'}\,.\,\overline{x - x''} \end{aligned}\right\} = 0.$$

The species of the curve will depend upon $B^2 - 4AC$, where $A = \overline{m+1}\cdot a$, $C = 2a$, and $B = \overline{1+ma^2} - \overline{m+a^2} = \overline{1-m}\cdot\overline{1-a^2}$, that is upon $\overline{1-m}|^2\cdot\overline{1-a^2}|^2 - 8a^2\cdot\overline{1+m}|$ ....(5).

It is easy to perceive that for all values of a there may be such values of m determined as shall render (5) either positive, negative, or nothing; that is, which shall render the locus an hyperbola, ellipse, or parabola. We shall here notice only one or two varieties.

1. That it may be a parabola, we have $\overline{1-m}|^2\cdot\overline{1-a^2}|^2 = 8a^2\cdot\overline{1+m}|$ or by reduction.

$$m = \left(\frac{1+a^2}{1-a^2}\right)^2 \pm \frac{4a^2}{1-a^2}\text{. that is,}$$

$$m = \frac{1-2a^2+5a^4}{1-2a^2+a^4} \text{ or } m = \frac{1+6a^2-3a^4}{1-2a^2+a^4}.$$

2. For the equilateral hyperbola, we must have

$$\overline{m+1}|\cdot a = -2a, \text{ and } \overline{1-m}|^2\cdot\overline{1-a^2}|^2 = 0.$$

The second of these equations may be fulfilled either by $m = 1$, or $a^2 = 1$: but the first requires $m = -3$, and hence the second must be satisfied by $a^2 = 1$, or $a = \pm 1$. Hence that the locus may be a rectangular hyperbola the tangents must have the ratio of $-3 : 1$ or $3 : 1$, and be reversed in their position with respect to one another and the co-ordinate axes, and the given lines must be perpendicular to one another.

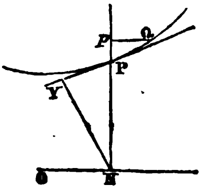
3. For the circle we must have $\overline{m+1}\cdot a = 2a$, and $\overline{1-m}|^2\cdot\overline{1-a^2}|^2 = 0$. Both these are satisfied by the same value of m, and independently of any specific values of a. Hence whatever relation subsist amongst the data so that the angles of inflection be equal the locus will be a circle. Perhaps this is the only case where the locus can be a figure similar to a given figure without some specific relation between a and m.

X. QUESTION 480, *by* IGNOTUS.

Investigate the nature of the curve in which a line drawn from the foot of the ordinate perpendicular on the tangent is an invariable magnitude.

First Solution, *by* R. N.

Let ON $= x$, NP $= y$, PQ a small portion of the curve, Qp perp. to NP, NY perp. to tangent at P; let NY $= a$, a given quantity by the question: Then by the sim. triangles QPp, NPy, PQ : Qp :: NP : NY that is $ds : dx :: y : a$



$$\therefore \frac{dx^2 + dy^2}{dx^2} = \frac{y^2}{a^2}, \text{ or } \frac{dy}{dx} = \frac{\sqrt{(y^2 - a^2)}}{a}$$

$$\therefore \frac{dy}{\sqrt{(y^2 - a^2)}} = \frac{dx}{a}; \text{ hence}$$

hyp. log. $\frac{y + \sqrt{(y^2 - a^2)}}{c} = \frac{x}{a}$: if $y = a$, when $x = 0$, then

hyp. log. $\frac{y + \sqrt{(y^2 - a^3)}}{a} = \frac{x}{a} \therefore \frac{y}{x} + \sqrt{\frac{y^2}{a^2} - 1} = e^{\frac{x}{a}}$, e being the base of the Naperian system of logarithms $\therefore \sqrt{\frac{y^2}{a^2} - 1}$

$= e^{\frac{2x}{a}} - 2\,\frac{y}{a}e^{\frac{x}{a}} + \frac{y^2}{a^2}$ and consequently $y = \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}$

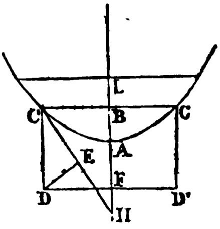
which is the equation to the catenary.

We may observe that the equation, hyp. log. $\frac{y + \sqrt{(y^2 - a^2)}}{c}$

$= \frac{x}{a}$, gives $\sqrt{(y^2 - a^2)} = 0$ for a particular solution, and $\therefore y = \pm a$, which is the equation to two straight lines parallel to the axis of (x), one above and the other below it at the distance (a) but the proper solution is the catenary as above.

Second Solution, *by* L.

On a straight line FL take AF = the given magnitude, and draw DFD′ at right angles to FL. Let CAC′ be the required curve, CD an ordinate to the axis DD′, DE the perpendicular from the foot of the ordinate upon CH the tangent to the curve at C, and draw CB parallel to DF.



Put $a =$ DE $=$ AF, $x =$ AB, $y =$ CB $=$ DF and $z =$ arc AC; then CD $= a + x$, CH $= \frac{y\dot{z}}{\dot{y}}$, and by sim. triangles CH : CB

$:: \text{CD} : \text{DE} = \frac{\text{CB} \times \text{CD}}{\text{CH}} = (a + x)\frac{\dot{y}}{\dot{z}}$, wherefore, by the question,

$$(a + x)\frac{\dot{y}}{\dot{z}} = a$$

$$\text{or } (a + x)^2 \dot{y}^2 = a^2\dot{z}^2.$$

But $$\dot{y}^2 = \dot{z}^2 - \dot{x}^2,$$

therefore by substitution

$$(a + x)^2 (\dot{z}^2 - \dot{x}^2) = a^2\dot{z}^2$$

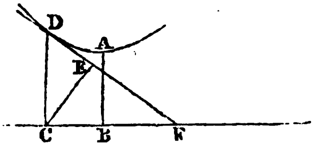
and $$\dot{z} = \frac{(a + x)\dot{x}}{\sqrt{(2ax + x^2)}}.$$

Taking the fluents,

$z = \sqrt{(2ax + x^2)}$, which is an equation to the catenary, the vertex of which is A.

THIRD SOLUTION, *by Mr.* GODWARD.

Having constructed the diagram, let CE = AB = a, BC = x and CD = y; then CF = $\frac{ydx}{dy}$,



and DF = $\frac{y}{dy}(dx^2 + dy^2)^{\frac{1}{2}}$.

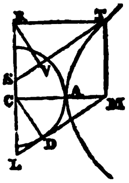
Wherefore by similar triangles

$$\frac{y}{dy}(dx^2 + dy^2)^{\frac{1}{2}} : \frac{ydx}{dy} :: y : a.$$

This analogy reduces to $dx = \frac{ady}{\sqrt{(y^2 - a^2)}}$, ∴ by integration, $x = a \,.\, \text{hyp log.}(y + \sqrt{(y - a^2)}) + c$. But when $x = 0$, $y = a$, and $c = -a \,.\, \text{h. l.}\, a$; hence $x = a \,.\, \text{h. l.}\, \frac{y + \sqrt{(y^2 - a^2)}}{a}$ an equation to the catenary, which is therefore the curve required.

FOURTH SOLUTION, *by Mr.* T. S. DAVIES.

Let CR be the axis, CA the ordinate to the point at which



the tangent is parallel to CR; T, any point in the curve, CR, RT, its co-ordinates, and TM parallel and equal to CR. Draw ML parallel to TS, and CD to RV. Then, obviously, the triangles TRS, MCL are similar and equal, and ∴ RV is parallel and equal to CD. Hence CD is a given line; and C being a given point, the locus of D is a circle, whose radius is the given perpendicular, CD = CA = RV. Hence (Leslie's Geom. of Curve Lines, p. 384.) the point T is in a catenary whose vertex is A and parameter is CA. *Q. E. I.*

Scholium. The straight line parallel to the axis fulfils this condition also: *but it does not fulfil it as a variety of the catenary.* If we consider it as a specific catenary the parameter is = 0: but the locus may be a straight line with *any value of* RV. The truth is, that if we take the differential equation of the curve

$$dx'(y - y') - dy'(x - x') + 0.$$

from which we obtain the value of the perpendicular,

$$p = \frac{y'' + \frac{dy'}{dx'}x'' + y' - \frac{dy'}{dx'}x'}{\sqrt{1 + \frac{dy'^2}{dx'^2}}}$$

where $x''y''$ is the point from which the perpendicular is drawn. Now in the case before us $y'' = 0$, $x'' = x'$, and $p = a$. Then we get the differential equation

$$\frac{y^2}{a^2} = 1 + \frac{dy^2}{dx^2}$$

which belongs to the catenary.

Integrating this we have $y = a \log. \frac{x + a + \sqrt{x^2 + 2ax}}{a}$.

Now it is evident that no value can be given to the constant a, which will reduce this to an algebraic equation, much less to the simple equation which denotes a straight line. This result then agrees with the inference we deduced at the head of this scholium from the *geometrical* character of the curve.

Though the *integral* equation can never represent a straight line, yet if in the *differential* equation we put $y = a$, we obtain a *particular solution*, and that particular solution is the straight line already mentioned.

Mr. BAINES *answered this question.*

XI. QUESTION 481, *by* IGNOTUS.

Find the area and length of a curve whose equation is

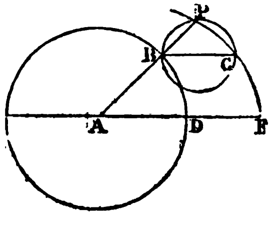
$$(x^2 - bx + y^2)^2 = a^2(x^2 + y^2).$$

FIRST SOLUTION, *by* R. N.

Let $x = r \cos \theta$ and $y = r \sin \theta$, and the equation is transformed to $r = a + b \cos \theta$; hence area $= \int \frac{1}{2} r^2 d\theta = \frac{2a + b^2}{4} \theta + ab \sin \theta + \frac{b^2}{8} \sin 2\theta$, from $\theta = 0$ to any value θ; if $\theta = 2\pi$ the whole oval figure is $= \frac{2a^2 + b^2}{2} \pi$.

Again the length of the curve is $\int \sqrt{(dr^2 + r^2 d\theta^2)} = \int (a^2 + b^2 + 2ab \cos\theta) d\theta = 2(a+b) \cdot \int \frac{d\theta}{2} \sqrt{\left\{1 - \frac{4hb}{(a+b)^2}\left(\sin \frac{\theta}{2}\right)^2\right\}}$, which is known to be the length of an elliptic arc whose semi-axes are $2(a + b)$ and $2(a - b)$, and the abscissa $= 2(a + b) \times \sin \frac{\theta}{2}$, the arcs being measured from the extremity of the axis minor; the whole length is therefore equal to the semi-circumference of the ellipse whose half axes are $2(a + b)$ and $2(a - b)$.

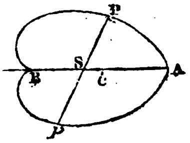
Cor. If $a = b$, the whole length $=$ the axis major of the ellipse $= 4(a + b)$.



Observation. This curve may be traced out by conceiving a circle CPB to move along the circumference of another circle DB so that the diameter BC be always parallel to AD, and the radius AB of the circle DB produced to cut the circle CPB in P, which is the generating point, AD being $= a$ and BC $= b$; for if $\theta = \angle$ DAB $= \angle$ CBP, then $r =$ AP $=$ AB $+$ BP $= a + b \times \cos \theta$.

SECOND SOLUTION, *by the Proposer*.

Consider the curve as a spiral, and take S the origin of the co-ordinates as the pole; SA $= a + b$, SB $= a - b$; the radius vector SP $= r$, $\angle$ ASP $= \theta$.



Since $r^2 = x^2 + y^2$ and $x = r\cos\theta$, the spiral equation is $(r^2 - br\cos\theta)^2 = a^2r^2$ or $r = a + b\cos\theta$.

The curve has this property in common with the circle that all lines drawn through the pole are equal.

For produce PS to meet the curve in p: and let S$p = r'$, then $r' = a - b\cos\theta$ and PS$p = r + r' = 2a$.

To find the area, d. ASP $= \frac{1}{2}r^2d\theta = \frac{1}{2}a^2d\theta + ab\cos\theta d\theta + \frac{1}{2}b^2\cos^2\theta d\theta = \frac{1}{2}a^2d\theta + ab\cos\theta d\theta + \frac{1}{2}b^2\sqrt{(1-\sin^2\theta)}\,d\sin\theta$; $\therefore$ ASP $= \frac{1}{2}a^2\theta + ab\sin\theta + \frac{1}{2}b^2$. cir, area, dist. from centre $= \sin\theta$; for correction $= 0$, $\therefore$ APB $= \frac{1}{2}\pi a^2$; and the whole area $=$ the area of a circle, rad. $= a$.

To rectify it, we have d. AP $= \sqrt{(dr^2 + r^2d\cos\theta)} = d\theta\sqrt{\{b^2\sin^2\theta + (a + b\cos\theta)^2\}} = d\theta\sqrt{(a^2 + b^2 + 2ab\cos\theta)} =$

$d\theta\sqrt{\{a^2 + b^2 + 2ab(1 - 2\sin^2\frac{1}{2}\theta)\}} = 2(a + b)\frac{d\theta}{2}\sqrt{(1 - \frac{4ab}{(a+b)^2}}$ $. \sin^2\frac{1}{2}\theta)$; $\therefore$ AP $= (a + b)$ elliptic arc, semi-axes $= 1$ and $\sqrt{\left(1 - \frac{4ab}{(a+b)^2}\right)}$, and abscissa $= \sin\frac{\theta}{2}$; (Jephson's Flux. Calc. Ch. 9. 11. form (6)). $\therefore$ APB $= 2(a + b)$. elliptic quad. semi-axes $= 1$ and $\frac{a-b}{a+b} =$ semi-ellipse, half axes $= a + b$ and $a - b$; and the whole periphery $=$ that of an ellipse whose semi-axes are SA and SB.

Mr. DAVIES *answered this Question.*

XII. QUESTION 482, *by* PHANTOM.

Find the sum of n terms of the series,

$$\sin^3 z + \sin^3 2z + \sin^3 3z + \sin^3 4z + \&c.$$

where z denotes a circular arc radius 1.

FIRST SOLUTION, *by* R. N.

Since $\sin z^3 = \frac{3}{4}\sin z - \sin 3z$ the sum of the proposed series is $=$

$$\frac{3}{4}\left\{\sin z + \sin 2z + \&c. \ldots\ldots\ldots\ldots + \sin nz\right\}$$

$$-\frac{1}{4}\left\{\sin 3z + \sin 6z + \&c. \ldots\ldots\ldots\ldots \sin 3nz\right\}.$$

But $$\cos\frac{z}{2} - \cos\frac{3z}{2} = 2\sin z \,.\, \sin\frac{z}{2}$$

$$\cos\frac{3z}{2} - \cos\frac{5z}{2} = 2\sin 2z \,.\, \sin\frac{z}{2}$$

$$\cos\frac{5z}{2} - \cos\frac{7z}{2} = 2\sin 3z \,.\, \sin\frac{z}{2}$$

&c. &c. &c.

$$\cos\frac{2n-1}{2}z - \cos\frac{2n+1}{2} = 2\sin nz \,.\, \sin\frac{z}{2}.$$

Adding we have

$$\cos\frac{z}{2} - \cos\frac{2n+1}{2}z = 2\sin\frac{z}{2}\,.\,(\sin z + \sin 2z + \ldots\ldots \sin nz)$$

$$\therefore \frac{\cos\frac{z}{2} - \cos\frac{2n+1}{2}z}{2\sin\frac{z}{2}} = \frac{\sin\frac{4z}{2}\,.\,\sin\frac{n+1}{2}z}{\sin\frac{z}{2}}$$ is the sum of the series $\sin z + \sin 2z + \ldots\ldots\ldots\ldots \sin nz$.

And in like manner

$$\frac{\sin\frac{3z}{2}\,.\,\sin\frac{3(n+1)}{2}z}{\sin\frac{3z}{2}} = \sin 3z + \sin 6z + \ldots\ldots + \sin 3nz.$$

Hence the sum of the $\sin^3 z + \sin^3 2z + \sin^3 3z +$ &c. to n terms $$= \frac{3}{4}\,.\,\frac{\sin\frac{nz}{2}\,.\,\sin\frac{n+1}{2}z}{\sin\frac{z}{2}} - \frac{1}{4}\,.\,\frac{\sin\frac{3nz}{2}\,.\,\sin\frac{3(n+1)}{2}z}{\sin\frac{3z}{2}}.$$

SECOND SOLUTION, *by Mr.* JOHN BAINES.

Here $\sin^3 z = -\frac{1}{4}\sin 3z + \frac{3}{4}\sin z$, $\sin^3 2z = -\frac{1}{4}\sin 6z + \frac{3}{4}\sin 2z$, $\sin^3 3z = -\frac{1}{4}\sin 9z + \frac{3}{4}\sin 3z$ &c ; hence $\sin^3 z + \sin^3 2z + \sin^3 3z + \ldots\ldots \sin^3 nz = \frac{3}{4}(\sin z + \sin 2z + \sin 3z + \ldots\ldots \sin nz) - \frac{1}{4}(\sin 3z + \sin 6z + \sin 9z + \ldots\ldots \sin \frac{1}{3}nz)$.

These series are of the same form as the series at pa. 51, Woodhouse's Trig. 3rd. ed. from which we obtain

$$\frac{3\sin\frac{nz}{2}\times\sin\frac{(n+1)}{2}z}{4\sin\frac{1}{2}z} - \frac{\sin\frac{3nz}{2}\times\sin\frac{3(n+1)}{2}z}{4\sin\frac{3z}{2}} = \text{the}$$

sum of n terms of the given series.

THIRD SOLUTION, *by* MR. DAVIES.

$$\sin^3 nz = \left\{\frac{e^{nz\sqrt{(-1)}} - e^{-nz\sqrt{(-1)}}}{2\sqrt{(-1)}}\right\}^3$$

$$= -\frac{e^{3nz\sqrt{(-1)}} - e^{-3nz\sqrt{(-1)}}}{4\times 2\sqrt{(-1)}} + \frac{3\left(e^{nz\sqrt{(-1)}} - e^{-nz\sqrt{(-1)}}\right)}{4\times 2\sqrt{(-1)}}$$

$$= \frac{3\sin nz}{4} - \frac{\sin 3nz}{4}.$$

If now we substitute $n-1$, $n-2$, &c. ....1 for n, we shall have the difference of the sums of the two following series equal to the given one: viz.

$$s = \frac{3}{4}\cdot\left\{\sin nz + \sin(n-1)z + \ldots\ \sin z\right\}$$

$$-\frac{1}{4}\cdot\left\{\sin 3nz + \sin 3(n-1)z + \ldots\sin 3z\right\}.$$

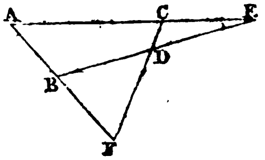
Now these two series, it is well known, (see Lardner's Trig. p. 290) gives us the following result,

$$s = \frac{1}{4}\left\{\frac{3\sin\frac{n+1}{2}z\times\sin\frac{nz}{2}}{\sin\frac{z}{2}} - \frac{\sin 3\cdot\frac{n+1}{2}z\cdot\sin 3\cdot\frac{nz}{2}}{\sin 3\cdot\frac{z}{2}}\right\}.$$

XIII. QUESTION 483, *by Mr.* T. S. DAVIES, *Bath*

Let any three circles (A), (B), (C) be given, and let them be cut by any other circle, the three chords which join the extremities of the segments cut off will form a triangle whose angles are on three given lines: and if any two circles cut these, the two chords which are in (A) intersecting in a, those in (B) intersecting in b, and those in (C) intersecting in c, the points a, b, c are in a straight line.

SOLUTION, *by Mr.* DAVIES, *the Proposer.*



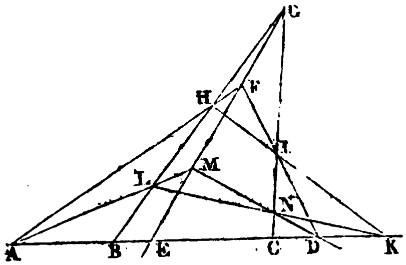
LEMMA 1. Let BCEF be a trapezium whose diagonals intersect in D, and one of its pair of sides FB, FC in A: then CD . DB : ED . DE :: BA . AC : FA . AE.

For the sides of the triangle

BDF cut by AE gives CD . AF : AB . CF :: DE : EB
CDE ...... AF ..... EA . CF : AC . FD :: BE : BD

by compounding which two analogies, the property above stated is obtained.

LEMMA 2. Let two lines AD, GE be given in position, intersecting in E; in the former let four points A, D, B, C be taken, and upon AD, and BC let triangles be constructed having their vertices in EG, and intersecting each other in H, I. Then HI being drawn will cut AD in K, so that



BE . ED : AE . EC :: BK . KD : AC . KC.

For by Lem. 1. the trapezium

GFCD gives GI . ID : FI . IC :: GE . ED : FE . EC,
ABFG .... BH . HF : AH . HG :: BE . EF : AE . EG.

The trian. BGC cut by transv. HI gives HG . IC : BH . GI :: CK : KB
...... AFD .... AH . IF : ID . HF :: AK : KD

and by compounding these four analogies, the theorem in question will be derived.

LEMMA 3. If EG be subjected to the condition of passing through E, and any number of triangles be constituted on AC, BD, the lines HI, LM, &c. in which they intersect will all pass through the same point K.

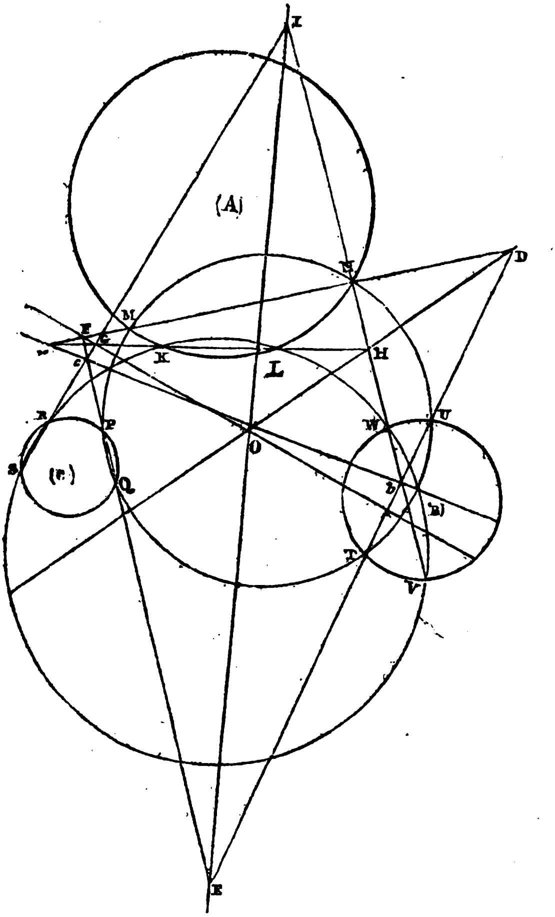
This is an immediate consequence of *Lem.* 2; for the point K being dependant only upon A, B, E, C, D, it will be the same in whatever direction EG is drawn, and in whatever position of it the points M, F, G, &c. are taken.

LEMMA 4. Let three lines MG, MD, MA radiating from the same point M, and two triangles GLN, FAD whose vertices are on these lines; then if the three pairs of sides intersect in H, I, K, these three points will be in the same straight line.

For produce GL, GM, GN to meet AD in B, E, C; and then the lines will be related as in *Lem.* 3. and AI will pass through K.

We come now to the proposition itself.

Let the figure be constructed according to the conditions of the theorem, the chords in (A) being KL, MN, those in (B), TU, VW, and those in (C) PQ, RS: and let DEF be the triangle formed of those in the circle MNUTQP, and GHI the other.



Then, because P, Q, T, U are in the same circle; the rectangle

PE . EQ = UE . ET, and hence as the lines EP, EU are also secants of (B), (C), the tangents to these circles from E are also equal. The locus of E is therefore a straight line, as is shewn in our elementary works. In the same manner the loci of D and F are straight lines. G, H, I are also in these lines inasmuch as they are specimens of the same system of points with D, E, F.

The lines just determined (the *radical axes* of the French Geometers) meet in a point (the *radical centre*) as is well known and simply proved. This is the first part of the proposition.

The second follows immediately, being an instance of Lemma 4. Whence the truth of the theorem is completely established.

XIV. QUESTION 484, *by* G. V.

Find the nature of a curve such that the curvilineal spaces contained between any three ordinates to the axis, and the axis, shall be to one another as the arcs of the curve between these ordinates.

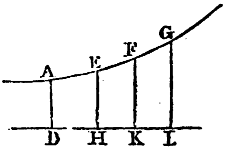
FIRST SOLUTION, *by* L.

It is evident that the conditions of the question will be fulfilled if we find a curve such that the area contained between any two ordinates may be equal to the rectangle contained by the intercepted arc and a given line.

Let AEG be such a curve, DL its axis, and AD, EH two ordinates. Put $x =$ DH, $y =$ HE and $a =$ a given line. Because area ADHE $=$ AE $\times a$, we have $y\dot{x} = a\dot{z}$, or putting for $\dot{x}$ its value $\sqrt{(\dot{z}^2 - \dot{y}^2)}$

$$y\sqrt{(\dot{z}^2 - \dot{y}^2)} = a\dot{x}$$

and by reduction $\dot{z} = \dfrac{y\dot{y}}{\sqrt{(y^2 - a^2)}}$.



Taking the fluents $\dot{z} = \sqrt{(y^2 - a^2)}$, which is an equation to a catenary, the vertex of which is A and ordinate AD $= a$.

Now let EH, KF, GL be any three ordinates to the axis DL.

Because area ADHE $=$ arc AE $\times a$, and similarly, area ADKF $=$ arc AF $\times a$, area ADLG $=$ arc AG $\times a$, we have by subtraction,

$$\text{area EHKF} = \text{arc EF} \times a$$
$$\text{area FKLG} = \text{arc FG} \times a;$$

therefore the areas EHFK, FKLG are to one another as the intercepted arcs EF, FG wherever the ordinates EH, FK, and GL are drawn.

SECOND SOLUTION, *by Mr.* DAVIES.

Let one of the ordinates pass through the origin of x (which will no wise diminish the generality of the solution); and call the areas reckoned from this first ordinate A, A′, and the corresponding portions of the curve S, S′.

Then, per question,

$$\text{A} : \text{A}' - \text{A} :: \text{S} : \text{S}' - \text{S}, \text{ or, componendo, } \text{A} : \text{A}' :: \text{S} : \text{S}'.$$

$$\therefore (d\text{A})^2 : (d\text{A}')^2 :: (d\text{S})^2 : (d\text{S}')^2, \text{ that is}$$

$y^2dx^2 : y'^2dx'^2 :: dx^2 + dy^2 : dx'^2 + dy'^2$ (where xy, $x'y'$ are the co-ordinates of the two points whose ordinates cut off the portions of the area, A, A′.)

$$\text{or,} \quad y^2 : y'^2 :: 1 + \frac{dy^2}{dx^2} : 1 + \frac{dy'^2}{dx'^2}.$$

$$:: \text{cosec}^2\, ty : \text{cosec}^2\, t'y'$$

(ty, $t'y'$ denoting the angles formed by t and y, and t' and y'.

$$\therefore y \sin ty = y' \sin t'y'.$$

But $y \sin ty$ is the perpendicular from the foot of the ordinate upon the tangent and which, by this equation is constant. Whence as was shewn in the solution to Prop. 10, the locus is a catenary, or a straight line parallel to the axis of x, according as we derive our conclusion from the *integral* or *differential* equation.

XV. QUESTION 485, *by* A. B.

Supposing the form of the earth to be perfectly spherical, what curve must a person describe on its surface so that at every point of his journey he may be at the same distance from London that he is from the equator.

FIRST SOLUTION, *by* R. N.

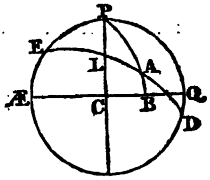
Let l = latitude of London, and x the latitude of the person when his longitude from London is λ, hence x is the distance of the person from London, and in a spherical triangle contained by arcs joining the person's place with London and the pole, the three sides are $90° - l$, x and $90° - x$,

$$\therefore \text{ the } \cos\lambda = \frac{\cos x - \sin l \,.\, \sin x}{\cos l \,,\, \sin x} = \frac{1}{\cos l} - \tan l \,.\, \tan x =$$

$\sec l - \tan l \,.\, \tan x$, radius being 1.

Hence cos. longitude from meridian of London = sec, lat. of London — tan. lat. of London × tan. lat. of observer's place; radius 1.

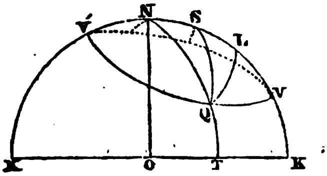
SECOND SOLUTION. *by Mr.* BAINES.



In the annexed figure let ÆQ represent the equator, P the north pole, and L London. Also, let A be a point on a sphere in the required curve; through P, A and L, A describe the great circles PAB, ELAD.

Put PL $= a$, the complement of the latitude of London, PA $= x$, and ∠ APZ $= \theta$, then from the two sides PL, PA and the included angle APL, in the spherical triangle ALP, we have $\cos$ AL $= \cos a \cos x + \sin a \sin x \cos\theta$. Now, the arc PB $= 90°$, and consequently AB $= 90° - x$. This is the shortest distance from A to the equator, and AL, from A to London. Hence, cos AB = cos AL, that is, $\cos(90° - x) = \cos a \cos x + \sin a \sin x \cos\theta$. Divide by $\cos x$ and $\tan x = \cos a + \sin a \tan x \cos\theta$, which gives $\tan x = \dfrac{\cos a}{1 - \sin a \cos\theta}$, or $\cot x = \sec a - \tan a \cos\theta$, and hence the nature of the curve is known.

THIRD SOLUTION, *by Mr.* T. S. DAVIES.



Let Q be one of the positions in the required route, L London, TR the equator, NS the earth's axis. Let NQ $= \lambda$, NL $= \lambda'$. and QNL = N.

Then by spherics,

$$\cos \text{LQ} = \cos\lambda \cos\lambda' + \sin\lambda \sin\lambda \cos \text{N}\,; \ldots\ldots(1).$$

But cos LQ = cos QT (by the quest.) = sin NQ = $\sin\lambda$; hence by substitution and reduction

$$\tan\lambda = \frac{\cos\lambda'}{1 - \sin\lambda' \cos \text{N}} \ldots\ldots\ldots(2).$$

The gnomonic projection substitutes for λ its tangent; therefore, we find at once that the projection on a plane at the pole converts it into this form

$$v = \frac{a}{1 - e \cos \varphi} \ldots\ldots\ldots\ldots (3)$$

in which we recognise the ellipse referred to polar ordinates issuing from the focus, and having that focus coincident with the pole of projection S. We are not, however, from this to infer that the (equation 2.) represents a circle: and in truth we shall find that the figure possesses none of the properties of the circle. All we can infer from it is, that the path in question is in the intersection of two concentric surfaces of the second order, a sphere and an elliptical cone. Curves of this kind are commonly of double curvature, a construction of which is given by Monge in his Descriptive Geometry, § III, art. 78.

Moreover, if we project it orthographically we shall find the locus of the fourth order as it should be for the intersection of two surfaces of the second order: but offering nothing remarkable in its form either in respect to utility or curiosity. The same may be said of the more complicated expression furnished by stereographic projection.

There is one remark which suggests itself upon the first view of the question—the similarity in the generations between this curve and both the parabola and ellipse, when described in plano. It resembles the parabola in as much as every point is equidistant from a certain point L and a great circle directrix CT. It resembles the ellipse, inasmuch as NQ + QL, arcs issuing from N and L, make the same constant sum, a quadrant.

If V, V′ be the points where the path crosses the meridian of London, then VV′ = NQ + QL = 90°. Also, V′N = VL = $\frac{1}{2}\lambda'$. And, lastly, the greatest value of SQ (S being the middle of NL) is when it becomes the prime vertical to VV′ at S: as is evident from the slightest consideration.

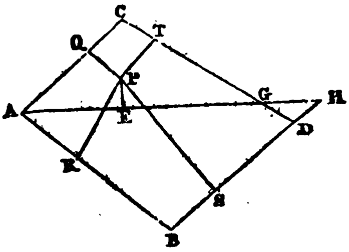
The curve, if any of its details be required, will assume a more convenient form by treating it as a polar curve whose origin is S, and its co-ordinates SQ and QSV.

XVI. QUESTION 486, *by* A. B.

If from a point perpendiculars be drawn to four straight lines given by position, it is required to describe the locus of that point when the rectangle of two of the perpendiculars has to the sum of the squares of the other two a given ratio.

SOLUTION, *by* L.

Let AC, AB, DB, and DC be the four straight lines given by position; PQ, PR, PS, and PT perpendiculars on these lines from a point P so that PQ × PR may be to $PS^2 + PT^2$ in a given ratio $m : n$. Draw AH to bisect the angle CAB and meet DC, BD in G and H. Put $AG = a$, $AH = b$, angle $CAG = BAH = \theta$, angle $AGC = \varphi$, $AHB = \gamma$, $AE = x$, and $EP = y$.



$$\begin{aligned} \text{Then } PQ &= x \sin\theta - y\cos\theta \\ PR &= x \sin\theta + y \cos\theta \\ PT &= (a - x)\sin\varphi - y\cos\varphi \\ PS &= (b - x)\sin\gamma + y\cos\gamma. \end{aligned}$$

$$\begin{aligned} \text{Therefore } PQ \times PR &= x^2 \sin^2\theta - y^2 \cos^2\theta \\ \text{and } PT^2 + PS^2 &= (a-x)^2 \sin^2\phi - (a-x)^2 y \sin 2\varphi \\ &\quad + y^2\cos^2\varphi + (b-x)^2\sin^2\gamma + (b-x)\, y \sin 2\gamma \\ &\quad + y^2 \cos^2\gamma \\ &= x^2(\sin^2\varphi + \sin^2\gamma) \\ &\quad + y^2(\cos^2\varphi + \cos^2\gamma) \\ &\quad + xy(\sin 2\varphi \div \sin 2\gamma) \\ &\quad - x\,(2a\sin^2\varphi + 2b\sin^2\gamma) \\ &\quad + y\,(b\sin 2\gamma - a\sin^2 2\varphi) \\ &\quad + a^2\sin^2\varphi + b\sin^2\gamma. \end{aligned}$$

Wherefore, since $PQ \times PR : PT^2 + PS^2 :: m : n$,

$$\begin{aligned} \frac{n}{m} \times (x^2 \sin^2 - y^2\cos^2\theta) &= x^2(\sin^2\varphi + \sin^2\gamma) \\ &\quad + y^2(\cos^2\varphi + \cos^2\gamma) \\ &\quad + xy\,(\sin 2\varphi - \sin 2\gamma) \\ &\quad - x\,(2a\sin^2\varphi + 2b\sin^2\gamma) \\ &\quad + y\,(b\sin 2\gamma - a\sin 2\phi) \\ &\quad + a^2\sin^2\varphi + b^2\sin^2\gamma; \end{aligned}$$

and, by transposition,

$$\begin{aligned} 0 = {} & x^2\left(\sin^2\varphi + \sin^2\gamma - \frac{n}{m}\sin^2\theta\right) \\ & + y^2\left(\cos^2\varphi + \cos^2\gamma + \frac{n}{m}\cos^2\theta\right) \\ & + xy(\sin 2\varphi - \sin 2\gamma) \\ & - x(2a\sin^2\phi + 2b\sin^2\gamma) \\ & + y(b\sin 2\gamma - a\sin 2\varphi) \\ & + a^2\sin^2\varphi + b^2\sin^2\gamma; \end{aligned}$$

an equation to a conic section, which is therefore the locus required.

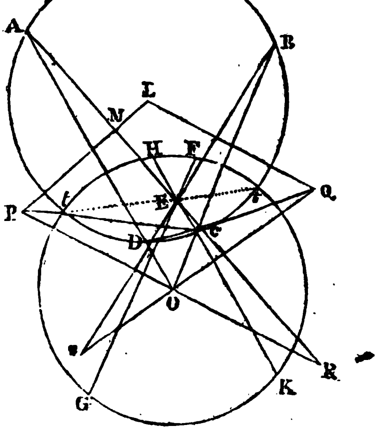
Answers were also given by Messrs. BAINES, DAVIES, *and* GODWARD.

XVII. QUESTION 487, *by* T. GALLOWAY, A.M. *R. M. College.*

Let A and B be two points without a circle of which O is the centre, and let BO × OC = square of radius and AO × OD = square of radius; and let BD and AC meet in E. Then, if any two chords FG and HK be drawn through E, making respectively equal angles with OA and OB, the four points A, F, C, G, as also the four points B, H, D, K, are situated in the circumference of a circle, and the diameter of the circle AFCG is to the diameter of the circle BHDK in a constant ratio. Required the demonstration.

FIRST SOLUTION, *by the Proposer.*

Since by hypothesis AO × OD = BO × OC, therefore the four points A, B, C, D are situated in the circumference of a circle. Let the circle be described, and let it intersect the given circle whose centre is O in the points t, t'. Then since by hypothesis AO × OD = square of radius = Ot^2, Ot and Ot' must be tangents to the circle ABCD in the points t and t'. It follows therefore from a well known property (equally true of any conic section as well as a circle, see Hamilton's *Sec. Con.*

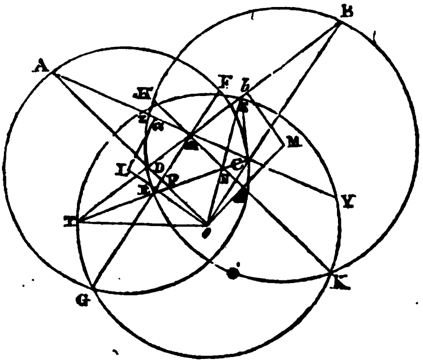


Lib. V. Prob. VIII.) that the three points t, E, t' are in the same straight line, or that the straight line tt' passes through E. Hence AC and tt' being chords of a circle intersecting each other in E, we have $AE \times EC = tE \times Et'$. For the same reason $FE \times EG = tE \times Et'$; consequently $AE \times EC = FE \times EG$; and therefore the four points A, F, C, G lie in the circumference of a circle. In the same manner it may be proved that B, H, D, K lie in the circumference of a circle, and it only remains to be demonstrated that the diameter of the circle passing through AFCG is to the diameter of the circle passing through BHDK in a given ratio.

Let L be the centre of the circle ABCD, and through L draw LM perpendicular to AC, and LN perpendicular to BD. Through O draw OP perpendicular to FG, and OQ perpendicular to HK, and let OP meet LM and AC (produced if necessary) in the points P and R, and OQ meet LN and BD in Q and S, and join CP, DQ. Because by hypothesis FG and HK make respectively equal angles with OA and OB, it follows that OP and OQ, which are perpendicular to FG and HK, also make equal angles with OA and OB. Hence the angle AOS is equal to BOR, and therefore BOS and AOR are also equal. But OAR and OBS being in the same segment of a circle DABC are equal, consequently, the two triangles AOR and BOS are similar, and, since $COR = DOS$, their bases AR and BS are similarly divided in C and D. We have therefore RC : CA :: SD : DB, or, since AC and BD are bisected by the perpendiculars LM and LN, RC : CM :: SD : DN, whence, by composition, RM : MC :: SN : ND. But by reason of the similar triangles AOR and BOS the angle ARC and BSO are equal, therefore the right angled triangles RMP and SNQ are similar, RM : MP :: SN : NQ. Combining this with the preceding analogy we shall have MC : MP :: ND : NQ, and consequently the triangles MCP and NDQ being right-angled at M and N are similar, whence CP : CM :: DQ : DN, or CP : CA :: DQ : DB. But CA : AO :: DB : BO; therefore CP : AO :: DQ : BO, or alternately, CP : DQ :: AO : BO. Now since in the circle AFCG the chords AC, FG are respectively bisected by the perpendiculars MP and OP, the point P is the centre of that circle, and CP its semi-diameter; and for a like reason Q is the centre of the circle AHDK, and DQ its semidiameter; hence the diameter of the circle passing through AFCG is to the diameter of that which passes through BHDK, in the ratio of AO to BO, which ratio is given.

Second Solution, *by Mr.* T. S. Davies, *Bath.*

The figure being drawn according to the directions of the proposition, and the intersections of AC, BD with the circles marked Y, Z, and S, T respectively, join OS, SC, OT, TC. Bisect AC, BD by the perpendiculars *b*M in *a*, *b*: and draw OR, OQ perpendicular to the chords FG, HK, cutting them in R, Q and meetng *ab*, *b*M in LM: and, lastly, join *ao*, CL, *bo*, DM.



1. Then, because, by hypothesis, BO . OC = AO . OD, each being equal to (radius)^2, the four points A, B, C, D are in a circle, and, therefore the triangles OAC, OBD are similar, as are likewise AED, BEC.

Again because CO . OB = OS^2, (OS being radius), we have

$$CO : OS :: OS : OB$$

the angle at O being common to the two triangles OBS, OSC, these triangles are also similar, and the angle OBS equal to OSC. In like manner it may be proved that the angle OTC is equal to OBS.

Hence since the angles OTC, OSC,* OAC are each equal to DBO, the points A, S, C, O, T are in the same circle: and therefore SE . ET = AE . EC. In the same way it may be shewn that ZE . EY = BE . ED = AE . EC, and, hence SE . ET = AE . EC = ZE . EY.

Moreover, because the chords ST, FG are in the circle and intersect in E, we have

$$FE . EG = SE . ET = AE . EC,$$

and, therefore, A, F, C, G are in one circle. In the same manner it may be proved that H, D, K, B are in one circle.†
Q. E. 1mo. *D.*

2. We come next to shew that the diameters of the circles have a given ratio.

---

* By the similar triangles AOC, OBD.

† *Remark.* This, it is clear, is independent of the particular inclination of the lines FG, HK to AO, BO.

Because of the similar triangles AOC, BOD, we have ACO = BDO, and, therefore,

$$CO : DO :: CA : DB :: Ca : Db,$$

the triangles aCO, bDO are similar, and the angles COa, CaO respectively equal to DOb, DbO. But the angles CaL, DbM are equal (constr.) and the angle AOC is common to the two triangles ACO, BDO; therefore the angles LaO, aOA are respectively equal to MbO, bOB. And because EPO (P being the intersection of FG, AO, and N that of HK, BO) is equal, by hypoth. to ENO and the angles at R and Q right angles, the angle ROP (LOA) = QON (BOM); and, therefore, the angles LaO, LOa are respectively equal to MbO, MOb, and

$$La : Mb :: Oa : Ob :: aC : bD.$$

Hence, the triangles LaC, MbD having the angles at a and b equal and the sides about them proportional are similar: and, therefore,

$$LC : MD :: Ca : Db, \text{ or } 2LC : 2MD :: CA : DB.$$

But because OL is drawn from the contre O of the circle FGHK perpendicular to the chord FG, it bisects it: and because OL bisects the chord FG in the circle AFCG at right angles it passes through the centre. Also aL bisects the chord AC of the latter circle at right angles, and therefore also passes through the centre. Hence L is the centre of the circle AFCG, and 2LC is its diameter. In the same manner we find that 2MD is the diameter of the circle BHDK. And it has been shewn that these have the ratio CA : DB; that is of two given lines.

Q.E.2°.D.

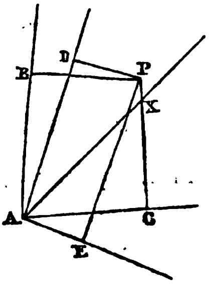
XVIII. QUESTION 488, *by Mr.* GALLOWAY.

Let AB, AC be two given straight lines, and $a : b$ a given ratio; two other straight lines AD, AE may be found, such, that if from any point whatever P in the plane of the given straight lines, there be drawn PB, PC, PD, PE respectively perpendicular to AB, AC, AD, AE, then $PD^2 + \frac{b}{a} PE^2$ is to $PA^2 + PB^2$ in a constant ratio.

FIRST SOLUTION, *by* L.

This question was wrong printed; it should have been PD^2

$+ \frac{b}{a}$ PE^2 : PC^2+PB^2 in a constant ratio, which is similar to the proposer's Lemma, page 105, last Number, when enunciated as a porism. The direct solution may be as follows: Let x and y be co-ordinates to the point P, axis AX bisecting the angle BAC. Put $\alpha = \angle$ BAX $=$ CAX, $\beta = \angle$ DAX, $\gamma = \angle$ EAX, and $m : n$ the ratio to be found.



Then
$$\text{PD} = x \sin\beta - y\cos\beta$$
$$\text{PE} = x \sin\gamma + y\cos\gamma$$
$$\text{PD}^2 = x^2 \sin^2\beta - 2xy\sin\beta\cos\beta + y^2\cos^2\beta$$
$$\text{PE}^2 = x^2 \sin^2\gamma + 2xy\sin\gamma\cos\gamma + y^2\cos^2\gamma.$$

Therefore
$$\text{PD}^2 + \frac{b}{a}\text{PE}^2 = x^2\left(\sin^2\beta + \frac{b}{a}\sin^2\gamma\right)$$
$$+ 2xy\left(\frac{b}{a}\sin\gamma\cos\gamma - \sin\beta\cos\beta\right)$$
$$+ y^2\left(\cos^2\beta + \frac{b}{a}\cos^2\gamma\right).$$

Again,
$$\text{PB} = x\sin\alpha - y\cos\alpha$$
$$\text{PC} = x\sin\alpha + y\cos\alpha$$

Therefore $\text{PB}^2 + \text{PC}^2 = 2(x^2\sin^2\alpha + y^2\cos^2\alpha)$

But, by the question, $\text{PD}^2 + \frac{b}{a}\text{PE}^2 = \frac{m}{n}(\text{PC}^2 + \text{PB}^2)$

therefore
$$x^2\left(\sin^2\beta + \frac{b}{a}\sin^2\gamma\right) + 2xy\left(\frac{b}{a}\sin\gamma\cos\gamma - \sin\beta\cos\beta\right) +$$
$$y^2\left(\cos^2\beta + \frac{b}{a}\cos^2\gamma\right) = \frac{2m}{n}(x^2\sin^2\alpha + y^2\cos^2\alpha);$$

or,
$$x^2\left(\sin^2\beta + \frac{b}{a}\sin^2\gamma - \frac{2m}{n}\sin^2\alpha\right)$$
$$+ 2xy\left(\frac{b}{a}\sin\gamma\cos\gamma - \sin\beta\cos\beta\right)$$
$$+ y^2\left(\cos^2\beta + \frac{b}{a}\cos^2\gamma - \frac{2m}{n}\cos^2\alpha\right) = 0$$

and making the coefficients of x^2, y^2 and $2xy$ severally equal to 0, we have the three equations of condition

$$\sin^2\beta + \frac{b}{a}\sin^2\gamma - \frac{2m}{n}\sin^2\alpha = 0 \ldots\ldots(1)$$

$$\cos^2\beta + \frac{b}{a}\cos^2\gamma - \frac{2m}{n}\cos^2\alpha = 0 \ldots\ldots(2)$$

$$\frac{b}{a}\sin\gamma\cos\gamma - \sin\beta\cos\beta = 0 \ \ldots\ldots(3).$$

From equation (3) we get $\frac{b}{a} = \frac{\sin\beta\cos\beta}{\sin\gamma\cos\gamma} = \frac{\sin 2\beta}{\sin 2\gamma}$,

and by adding (1), (2), $1 + \frac{b}{a} - \frac{2m}{n} = 0$

or, $\frac{a+b}{2a} = \frac{m}{n}$,

Again, subtracting (1), (2)

$$\cos^2\beta - \sin^2\beta + \frac{b}{a}(\sin^2\gamma - \cos^2\gamma) + \frac{2m}{n}(\cos^2\alpha - \sin^2\alpha) = 0$$

$$\text{or, } \cos 2\beta + \frac{b}{a}\cos 2\gamma = \frac{2m}{n}\cos 2\alpha,$$

therefore we have the two equations

$$a\cos 2\beta + b\cos 2\gamma = (a+b)\cos 2\alpha$$
$$a\sin 2\beta - b\sin 2\gamma = 0, \text{ to find } \beta \text{ and } \gamma.$$

From the 1st. $\cos^2 2\gamma = \left(\frac{(a+b)}{b}\cos 2\alpha - \frac{a}{b}\cos 2\beta\right)^2$;

from the 2nd. $\cos^2 2\gamma = b^2 - a^2\sin^2 2\beta$;

equating these values, and reducing, we get

$$\cos 2\beta = \frac{a - b + (a+b)\cos^2 2\alpha}{2a\cos 2\alpha}.$$

$$\text{and } \cos 2\gamma = \frac{b - a + (a+b)\cos^2 2\alpha}{2b\cos 2\alpha}.$$

Hence the position of the lines AD, AE and the ratio $\frac{m}{n}$ are now determined.

SECOND SOLUTION, *by Mr.* T. S. D'AVIES.

Join PA. Let AX bisect the angle CAB; denote each of these halves by α, the angle DAX and EAX by ϕ and χ, and let PAX $= \omega$, and PA $= v$: also for the present let $\frac{b}{a} = n$, and $p =$ *given* constant ratio.

Then the analytical expression of the conditions of the question is $v^2 \sin^2 (\phi - \omega) + n v^2 \sin (\chi + \omega) = p v^2 (\sin^2 (\alpha - \omega) + \sin^2 (\alpha + \omega))$, which must be independent of v and ω. Now it is already independent of v, since v^2 only appears as a factor common to all the terms. By expansion we obtain

$$\left.\begin{array}{l} \sin^2 \phi \cos^2 \omega - 2 \sin \phi \cos \phi \sin \omega \cos \omega + \cos^2 \phi \sin^2 \omega \\ + n \sin^2 \chi \cos^2 \omega - 2n \sin \chi \cos \chi \sin \omega \cos \omega + n \cos^2 \chi \sin^2 \omega \\ - 2p \sin^2 \alpha \cos^2 \omega \qquad\qquad - 2p \cos^2 \alpha \sin^2 \omega \end{array}\right\} = 0.$$

Hence the three equations for ϕ, χ, p are

$$\sin^2 \phi + n \sin^2 \chi = 2p \sin^2 \alpha \ldots\ldots\ldots (1)$$

$$\cos^2 \phi + n \cos^2 \chi = 2p \cos^2 \alpha \ldots\ldots\ldots (2)$$

$$\sin \phi \cos \phi = - n \sin \chi \cos \chi, \text{ or } \sin 2\phi = - n \sin 2\chi \ldots\ldots (3)$$

Subtract (1), (2) and we get $\cos 2\phi = 2p \cos^2 \alpha - n \cos 2\chi \ldots (4)$

Add squares of (3), (4) we have $\dfrac{n^2 - 1 - 4p^2 \cos^2 2\alpha}{4pn \cos 2\alpha} = \cos 2\chi \ldots (5)$

And in like manner $\dfrac{n^2 - 1 + 4p^2 \cos^2 2\alpha}{4p \cos 2\alpha} = \cos 2\phi. \ldots\ldots (6).$

Moreover, by adding (1), (2) we find $1 + n = 2p$, or restoring the value of n, we have $p = \dfrac{a + b}{2a}$; and the values of n and p also put in (5), (6) give a complete solution to the question: viz.

$$\cos 2\chi = \frac{(b - a) + (a + b) \cos^2 2\alpha}{2b \cos 2\alpha}.$$

$$\cos 2\phi = \frac{(a - b) + (a + b) \cos^2 2\alpha}{2a \cos 2\alpha}.$$

$$p = \frac{a + b}{2a}.$$

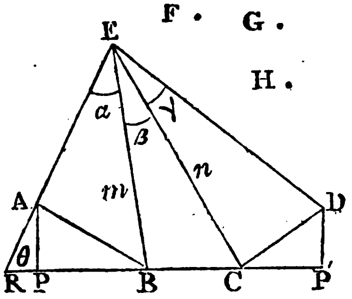
XIX. QUESTION 489, *by* PERIGRINATOR.

Let there be a system of four points, accessible (and either visible or invisible from one another). Let there be a system of four other points, inaccessible, but each of them visible from all the four points of the first system. The angles are observed under which the distances of all the four points of the second system are seen from each point of the first system: required the positions of these eight points in respect of one another.

FIRST SOLUTION, *by* L. PUISSANT.

Four points A, B, C, D, are inaccessible, but visible from four other points E, F, G, H; to find the respective distances of these eight points, supposing we have only observed from each of the points of the second system, the angles under which by the points of the first system are seen.

M. Lhuillier, of Geneva, in his Elements of Algebra and Geometry, p. 249, has satisfied himself with enunciating this problem, and referring for its solution to that given by Lambert. Let us put it into equations.



Prolong the straight lines AE, BC to R, and designate by θ the angle which they form: demit from the points A, D, upon the line BC (taken for unit of length) the perpendiculars AP, DP′. Then make in reference to this unit

$$\text{BP} = x,\quad \text{AP} = y,$$
$$\text{CP}' = x',\quad \text{D}'\text{P}' = y';$$

and represent by α, β, γ the observed angles AEB, BEC, CED, and by $m, n,$ the straight lines EB, EC.

In the first place, because the angle EBC $= \alpha + \theta$, and the angle ECP′ $= \alpha + \beta + \theta$, the triangle BEC gives

$$m \sin\beta = \sin(\alpha + \beta + \theta);$$ and

developing, we have

$$(1)\ldots\ldots m \sin\beta = \sin(\alpha + \beta)\cos\theta + \cos(\alpha + \beta)\sin\theta$$
$$= [\sin(\alpha + \beta) + \cos(\alpha + \beta)\tan\theta]\cos\theta$$

In the same triangle, we have again,

$$(2)\ldots\ldots \frac{m}{n} = \frac{\sin(\alpha + \beta + \theta)}{\sin(\alpha + \theta)} = \frac{\sin(\alpha + \beta) + \cos(\alpha + \beta)\tan\theta}{\sin\alpha + \cos\alpha\tan\theta}$$

The triangle REB gives, because of RP $= y\cot\theta$

$$m = \frac{\sin\theta\,(x + y\cot\theta)}{\sin\alpha}.$$

Substituting this value in (1), it becomes

$$\tan\theta \sin\beta\,(x + y\cot\theta) = \sin\beta\,(x\tan\delta + y)$$
$$= \sin\alpha\sin(\alpha + \beta) + \sin\alpha\cos(\alpha + \beta)\tan\theta,$$

from which we deduce

$$\tan\theta = \frac{\sin\alpha\sin(\alpha+\beta) - y\sin\beta}{x\sin\beta - \sin\alpha\cos(\alpha+\beta)}.$$

Putting this value in (2) it becomes, after having reduced to a common denominator and simplified;

$$\frac{m}{n} = \frac{x\sin(\alpha+\beta) - y\cos(\alpha+\beta)}{(1+x)\sin\alpha - y\cos\alpha}.$$

Such is the relation which exists between these four points E, A, B, C. There will exist a similar one between the four points E, D, C, B, viz.

$$\frac{n}{m} = \frac{x'\sin(\beta+\gamma) - y'\cos(\beta+\gamma)}{(1+x')\sin\gamma - y'\cos\gamma};$$

Thus, $$\frac{x\sin(\alpha+\beta) - y\cos(\alpha+\beta)}{(1+x)\sin\alpha - y\cos\alpha} = \frac{(1+x')\sin\gamma - y'\cos\gamma}{x'\sin(\beta+\gamma) - y'\cos(\beta+\gamma)}.$$

Now if we take away the denominators and convert it into a single member, we get

$$\left.\begin{array}{l} xx'[\sin(\alpha+\beta)\sin(\beta+\gamma) - \sin\alpha\sin\gamma] - yx'[\cos(\alpha+\beta)\sin(\beta+\gamma) - \cos\alpha\sin\gamma] \\ -yy'[\cos\alpha\cos\gamma - \cos(\alpha+\beta)\cos(\beta+\gamma)] - y'x[\sin(\alpha+\beta)\cos(\beta+\gamma) - \sin\alpha\cos\gamma] \\ +\ y'\sin\alpha\cos\gamma + y\cos\alpha\sin\gamma - (1+x+x')\sin\alpha\sin\gamma = 0 \end{array}\right\}(\text{A})$$

This equation is susceptible of a more simple form, for the coefficients of xx' and yy' are each equal to $\sin\beta\sin(\alpha+\beta+\gamma)$. In fact we have identically,

$$\begin{aligned} \sin\alpha\sin\gamma &= \sin(\alpha+\beta-\beta)\sin(\gamma+\beta-\beta) \\ &= [\sin(\alpha+\beta)\cos\beta - \cos(\alpha+\beta)\sin\beta] \\ &\quad [\sin(\gamma+\beta)\cos\beta - \cos(\gamma+\beta)\sin\beta] \end{aligned}$$

And if we effect the multiplication indicated and reduce, it will become

$$\sin\alpha\sin\gamma = \sin(\alpha+\beta)\sin(\gamma+\beta)\cos^2\beta - \sin\beta\cos\beta\sin(\alpha+2\beta-\gamma)$$
$$+\cos(\alpha+\beta)\cos(\gamma+\beta)\sin^2\beta;$$ and hence

$$\begin{aligned} \sin(\alpha+\beta)\sin(\beta+\gamma) - \sin\alpha\sin\gamma &= \sin(\alpha+\beta)\sin(\beta+\gamma)\sin^2\beta \\ &\quad -\cos(\alpha+\beta)\cos(\beta+\gamma)\sin^2\beta \\ &\quad +\sin\beta\cos\beta\sin(\alpha+2\beta+\gamma); \end{aligned}$$

but $\sin(\alpha+\beta)\sin(\beta+\gamma)\sin^2\beta - \cos(\alpha+\beta)\cos(\beta+\gamma)\sin^2\beta =$
$-\sin^2\beta\cos(\alpha+2\beta+\gamma)$; and, therefore,

$$\begin{aligned} &\sin(\alpha+\beta)\sin(\beta+\gamma) - \sin\alpha\sin\gamma \\ &= \sin\beta\cos\beta\sin(\alpha+2\beta+\gamma) - \sin^2\beta\cos(\alpha+2\beta+\gamma) \\ &= \sin\beta[\sin(\alpha+2\beta+\gamma)\cos\beta - \cos(\alpha+2\beta+\gamma)\sin\beta] \\ &= \sin\beta\sin(\alpha+\beta+\gamma). \end{aligned}$$

In the same way we may prove that

$$\cos\alpha\cos\gamma - \cos(\alpha+\beta)\cos(\beta+\gamma) = \sin\beta\sin(\alpha+\beta+\gamma),$$

and that the coefficients of yx' and $y'x$ are both equal to

$\sin\beta\cos(\alpha+\beta+\gamma)$. Hence the equation (A) takes the more simple form

$$(\text{B}) \left\{ \begin{array}{l} (xx'-yy')\sin\beta\sin(\alpha+\beta+\gamma)-(yx'+y'x)\sin\beta\cos(\alpha+\beta+\gamma) \\ +\,y'\sin\alpha\cos\gamma+y\cos\alpha\sin\gamma-(1+x+x')\sin\alpha\sin\gamma=0. \end{array} \right.$$

As this equation contains four unknowns, it is necessary, in order that the problem be determinate that there be four points such as E, F, G, H, from which we may see the four others A, B, C, D. Hence the only variables are α, β, γ. We deduce from this

$$\left. \begin{array}{l} 1+x+x'=y\cot\alpha+y'\cot\gamma \\ \qquad +(xx'-yy')\dfrac{\sin\beta\sin(\alpha+\beta+\gamma)}{\sin\alpha\sin\gamma} \\ \qquad -(yx'+y'x)\dfrac{\sin\beta\cos(\alpha+\beta+\gamma)}{\sin\alpha\sin\gamma} \end{array} \right\} \ldots\ldots(\text{C}),$$

and if, for brevity, we make

$$\text{A}=\cot\alpha,\ \text{B}=\cot\gamma,\ \text{C}=\frac{\sin\beta\sin(\alpha+\beta+\gamma)}{\sin\alpha\sin\gamma},\ \text{D}=\frac{\sin\beta\cos(\alpha+\beta+\gamma)}{\sin\alpha\sin\gamma};$$

$$z=xx'-yy',\ z'=yx'+y'x;$$

we shall have between the eight points under consideration, the four following equations

$$\left. \begin{array}{l} 1+x+x'=\text{A}y+\text{B}y'+\text{C}z-\text{D}z' \\ \qquad\quad =\text{A}'y+\text{B}'y'+\text{C}'z-\text{D}'z' \\ \qquad\quad =\text{A}''y+\text{B}''y'+\text{C}''z-\text{D}''z' \\ \qquad\quad =\text{A}'''y+\text{B}'''y'+\text{C}'''z-\text{D}'''z' \end{array} \right\} \ldots\ldots(\text{D})^*.$$

The method of eliminating to employ in this case is very natural. we have in the first place, by subtraction,

$$0=(\text{A}-\text{A}')y+(\text{B}-\text{B}')y'+(\text{C}-\text{C}')z-(\text{D}-\text{D}')z',$$
$$0=(\text{A}-\text{A}'')y+(\text{B}-\text{B}'')y'+(\text{C}-\text{C}'')z-(\text{D}-\text{D}'')z',$$
$$0=(\text{A}-\text{A}''')y+(\text{B}-\text{B}''')y'+(\text{C}-\text{C}''')z-(\text{D}-\text{D}''')z'; \text{ whence}$$

$$z=\frac{(\text{A}-\text{A}')y+(\text{B}-\text{B}')y'+(\text{C}-\text{C}')z}{\text{D}-\text{D}'}=\text{M}y+\text{N}y'+\text{P}z$$

$$z'=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots=\text{M}'y+\text{N}'y'+\text{P}'z$$

$$z''=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots=\text{M}''y+\text{N}''y'+\text{P}''z$$

and if we equate these values two and two we shall obtain

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[*] These equations are the same as those of M. Lefevre in his *Manual du Trigonometrie*, p. 90; but we have obtained them by a method purely trigonometrical.

$$0 = (M - M')y + (N - N')y' + (P - P')z$$

$$0 = (M - M'')y + (N - N'')y' + (P - P'')z;$$ and therefore

$$z = \frac{(M - M')y + (N - N')y'}{P - P'} = Qy + Ry'$$

$$z = \dots\dots\dots\dots = Q'y + R'y'.$$

Equating again these two values we shall find

$$y' = \frac{Q - Q'}{R' - R}y = Sy, \quad z = (Q + RS)y = Ty;$$ whence

$$z' = M'y + N'y' + P'z = (M' + N'S + P'T)y = Uy;$$

also because $z' = yx' + y'x$. we have

$$yx' + y'x = Uy = yx' + Sxy,$$

thus $x' + Sx = U$; and the first equation (D) will become

$$1 + x + x' = (A + BS + CT - DU)y = Vy.$$

Combining these two last equations, the result of the operation is

$$x = \frac{1 + U - Vy}{S - 1}, \quad x' = \frac{VSy - S - U}{S - 1}.$$

On the other hand, because $z = xx' - yy'$, we have

$$z = Ty = \left(\frac{1 + U - Vy}{S - 1}\right)\left(\frac{VSy - S - U}{S - 1}\right) - Sy^2;$$

thus y is definitely given by an equation of the second degree, of which the form is reducible to this:

$$y^2 + Fy - G.$$

This ordinate being known we shall have x, x' and y' which depend upon it; but it requires us to know which of the two values of y satisfy the equation: for if we determine the points A, B, C, D, E, after one of these values, and that the angles α, β, γ, have place at the point E between the four other points, the problem will be resolved. Otherwise it will be necessary to employ the other value of y.

That the solution should not be faulty it is necessary to take care that the angles α and γ be positive if they be one to the right and the other to the left of the angle β, and that they are negative if they have one part common with this last. In the case where the ordinates y, y' are negative, the points A, D will be below BC, and in the same state of x, x', these abscisses will have a situation opposite to that which we originally attributed to them. Finally, if the line BC be given in known measures, as for example, in metres, it will be necessary to multiply the values found for the other lines, by the number of metres contained in BC, as is obvious

Our solution supposes, moreover, that the points E, F, G, H, are all on the same side of BC; but if some be situated on one side and some on the other, it will be requisite to modify in a corresponding manner the formula (D) found above, to establish between them the necessary correlations. It is, however, to hinder ourselves with a problem, curious rather than useful, and which only requires a well exercised computor.

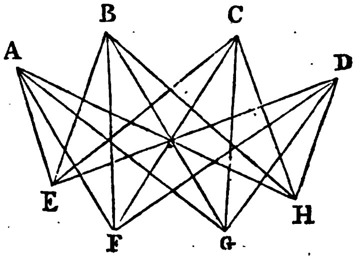
See Puissant, Traite de Topographie, 2nd. Ed. 1820, p. 55—60.

SECOND SOLUTION, *by* MEYER HIRSCH.

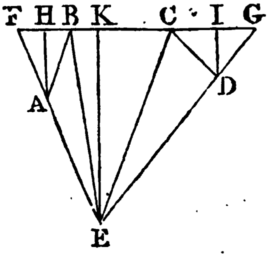
The four objects A, B, C, D are seen from the four stations E, F, G, H and there the angles

AEB $= \alpha$, BEC $= \beta$,
CED $= \gamma$, AFB $= \alpha'$,
BFC $= \beta'$, CFD $= \gamma'$,
AGB $= \alpha''$, BFC $= \beta''$,
CGD $= \gamma''$, AAB $= \alpha'''$,
BHC $= \beta'''$, CHD $= \gamma'''$,

are measured.



1. Join the points A, B, C, D by the lines AB, BC, CD; find first an equation for the point E. With this view, produce the lines BC, EA, ED, till they meet in F, G, and draw the perpendiculars AH, DI, EK. Since angles only are here given, consequently we cannot determine the actual magnitude of the lines belonging to the figure, but merely their proportion to one another. We can therefore assume one of these lines say BC, for unity, and put BC $= 1$. Now if besides the lines AB, CD, the angles ABC, BCD, or the adjacent angles ABF, DCG, are known; then the position of the four points A, B, C, D, and consequently also the situation of the points E, F, G, H, are determined. Assume therefore AB $= x$, CD $= y$, ABF $= \varphi$, DCG $= \psi$, and moreover, for shortness sake BAE $=$ A, CDE $=$ D, which last two angles vanish further on in the calculation. We then have AFB $=$ BAE $-$ ABF $=$ A $- \varphi$, CGD $=$ CDE $-$ DCG $=$ D $- \psi$, CBE $=$ AFB $=$ AFB $+$ AEB $=$ A $- \varphi + \alpha$, BCE $=$ CGD $+$ CED $=$ D $- \psi + \gamma$.



2. In the triangle ABE,

$$\text{BE} = \frac{\text{AB} \sin \text{BAE}}{\sin \text{AEB}} = \frac{x \sin \text{A}}{\sin \alpha}$$

and in the triangle CED,

$$\text{CE} = \frac{\text{CD} \sin \text{CDE}}{\sin \text{CED}} = \frac{y \sin \text{D}}{\sin \gamma};$$

consequently in the right angled triangle BEK,

$$\text{BK} = \text{BE} \cos \text{CBE} = \frac{x \sin \text{A} \cos (\text{A} - \varphi + \alpha)}{\sin \alpha},$$

and in the right angled triangle CEK,

$$\text{CK} = \text{CE} \cos \text{BCE} = \frac{y \sin \text{D} \cos (\text{D} - \psi + \gamma)}{\sin \gamma}.$$

Now since $\text{BK} + \text{CK} = \text{BC} = 1$: we then have the equation

$$\frac{x \sin \text{A} \cos (\text{A} - \varphi + \alpha)}{\sin \alpha} + \frac{y \sin \text{D} \cos (\text{D} - \psi + \gamma)}{\sin \gamma} = 1,$$

or,

$$x \sin \gamma \sin \text{A} \cos (\text{A} - \varphi + \alpha) + y \sin \alpha \sin \text{D} \cos (\text{D} - \psi + \gamma)$$
$$= \sin \alpha \sin \gamma.$$

3. But $\cos (\text{A} - \varphi + \alpha) = \cos [\text{A} - (\varphi - \alpha)] = \cos \text{A} \cos (\varphi - \alpha) + \sin \text{A} \sin (\varphi - \alpha)$, $\cos (\text{D} - \psi + \gamma) = \cos [\text{D} - (\psi - \gamma)] = \cos \text{D} \cos (\psi - \gamma) + \sin \text{D} \sin (\psi - \gamma)$. Substitute these values in the obtained equation, and at the same time put $1 - \cos^2 \text{A}$ for $\sin^2 \text{A}$, $1 - \cos^2 \text{D}$ for $\sin^2 \text{D}$; this gives,

$\sin \alpha \sin \gamma =$

$$x \sin \gamma \cos \text{A} [\sin \text{A} \cos (\varphi - \alpha) - \cos \text{A} \sin (\varphi - \alpha)]$$
$$+ x \sin \gamma \sin (\varphi - \alpha) + y \sin \alpha \cos \text{D} [\sin \text{D} \cos (\psi - \gamma) - \cos \text{D} \sin (\psi - \gamma)]$$
$$+ y \sin \alpha \sin (\psi - \gamma)$$
$$= x \sin \gamma \cos \text{A} \sin (\text{A} - \varphi + \alpha) + x \sin \gamma \sin (\varphi - \alpha) + y \sin \alpha \cos \text{D} \sin (\text{D} - \psi + \gamma) + y \sin \alpha \sin (\psi - \gamma)$$
$$= x \sin \gamma \cos \text{A} \sin \text{CBE} + x \sin \gamma \sin (\varphi - \alpha) +$$
$$y \sin \alpha \cos \text{D} \sin \text{BCE} + y \sin \alpha \sin (\psi - \gamma).$$

4. In the triangle BEC we have

$$\text{BC} = 1 = \frac{\text{BE} \sin \beta}{\sin \text{BCE}} = \frac{x \sin \text{A} \sin \beta}{\sin \alpha \sin \text{BCE}},$$

and ∴

$$\sin \text{BCE} = \frac{x \sin \text{A} \sin \beta}{\sin \alpha}.$$

In like manner

$$\text{BC} = 1 = \frac{\text{CE} \sin \beta}{\sin \text{CBE}} = \frac{y \sin \text{D} \sin \beta}{\sin \gamma \sin \text{CBE}},$$

and ∴

$$\sin \mathrm{CBE} = \frac{y \sin \mathrm{D} \sin \beta}{\sin \gamma}.$$

Substitute these values of sin BCE, sin CBE in the equation in 3; this gives $\sin \alpha \sin \gamma =$

$$xy \sin \beta \sin \mathrm{D} \cos \mathrm{A} + x \sin \gamma \sin (\varphi - \alpha) +$$
$$xy \sin \beta \sin \mathrm{A} \cos \mathrm{D} + y \sin \alpha \sin (\psi - \gamma)$$
$$= xy \sin \beta \sin (\mathrm{A} + \mathrm{D}) + x \sin \gamma \sin (\varphi - \alpha) + y \sin \alpha \sin (\psi - \gamma)$$

or, since $\mathrm{A} + \mathrm{D} = 540° - (\mathrm{AED} + \mathrm{ABC} + \mathrm{BCD}) = 180° - (\mathrm{E} - \varphi - \psi)$, when, for the sake of brevity, we put $\mathrm{AED} = \mathrm{E}$,

$$\sin \alpha \sin \gamma = xy \sin \beta \sin [\mathrm{E} - (\varphi + \psi)] + x \sin \gamma \sin (\varphi - \alpha)$$
$$+ y \sin \alpha \sin (\psi - \gamma).$$

5. In order to separate the unknown magnitudes from the known, solve this equation; this gives

$$\sin \alpha \sin \gamma = xy \sin \beta\, [\sin \mathrm{E} \cos (\varphi + \psi) - \cos \mathrm{E} \sin (\varphi + \psi)]$$
$$+ x \sin \gamma (\sin \varphi \cos \alpha - \cos \varphi \sin \alpha) + y \sin \alpha (\sin \psi \cos \gamma$$
$$- \cos \psi \sin \gamma)$$
$$= xy \sin \beta\, [\sin \mathrm{E}\, (\cos \varphi \cos \psi - \sin \varphi \sin \psi) -$$
$$\cos \mathrm{E}\, (\sin \varphi \cos \psi + \cos \varphi \sin \psi)] +$$
$$x \sin \gamma (\sin \varphi \cos \alpha - \cos \varphi \sin \alpha) +$$
$$y \sin \alpha (\sin \psi \cos \gamma - \cos \psi \sin \gamma).$$

6. If we put $x \sin \varphi = p$, $x \cos \varphi = q$, $y \sin \psi = r$, $y \cos \psi = s$; then this equation is transformed into the following one:

$$(qs - pr) \sin \beta \sin \mathrm{E} - (ps + qr) \sin \beta \cos \mathrm{E} + p \sin \gamma \cos \alpha$$
$$+ r \sin \alpha \cos \gamma - (q + s + 1) \sin \alpha \sin \gamma = 0,$$

or

$$(qs - pr) \frac{\sin \beta \sin \mathrm{E}}{\sin \alpha \sin \gamma} - (ps + qr) \frac{\sin \beta \cos \mathrm{E}}{\sin \alpha \sin \gamma}$$
$$+ p \cot \alpha + r \cot \gamma - (q + s + 1) = 0,$$

where $\mathrm{E} = \alpha + \beta + \gamma$.

Likewise each of the other stations F, G, H gives a similar equation. We therefore have four equations in all, whence the value of $qs - pr$, $ps + qr$, p, r, $q + s + 1$, and hence again the values of p, q, r, s, may be determined: If these last are found, then we also have the values of x, y, φ, ψ.

When the situation of the four points A, B, C, D, are determined by means of the angles and lines φ, ψ, x, y, when calculated, it is easy to determine the position of the points E, F, G, H.

See Hirsch's Geometry, 1805. *Translated by the Rev. J. A. Ross, A.M. and Edited by J. M. F. Wright, A. B.* 1827.

XX. PRIZE QUESTION 490, *by* ——.

Let a portion of the surface of an oblate spheroid be projected on the surface of a given sphere in such a manner that its differential elements and their projections may be similar; it is required 1st. to assign a relation between any point on the spheroid and its corresponding point on the sphere in terms of their respective latitudes and longitudes; and 2nd. to determine at a given latitude the ratio of the corresponding linear elements of the two surfaces, supposing that at another given latitude it approaches the nearest possible to a ratio of equality.

SOLUTION.

Some time after this question had been proposed in the Repository, the translation of a Memoir of the celebrated Gauss, from which it was taken, appeared in the Philosophical Magazine, (August and September 1828). As no satisfactory solution has come to hand we will here give the translation we had previously made from the German. The Memoir was originally published in Professor Schumacher's *Astronomische Abhandlungen*, and was composed in answer to a Prize Question proposed by the Royal Society of Sciences at Copenhagen:

1. The nature of a curve surface is defined by an equation between the three co-ordinates x, y, z, of any point on the surface; and any one of the three variable quantities in this equation may be expressed in a function of the two others. It is still more general, however, to consider x, y, z as functions of two new variable quantities t, u; whereby, generally speaking at least, known values of t and u will always be obtained for every given point upon the proposed surface, and conversely.

2. In defining a second surface, we suppose the letters X, Y, Z, T, U, to have respectively similar significations to those which have been given to x, y, z, t, u, in defining the first.

3. To project one surface upon another means to establish a law, according to which a determinate point upon the second surface shall correspond to every point on the first. This will be accomplished when T and U are also expressed in known functions of the two variables t, u. When the projection must satisfy certain conditions, these functions can no longer be arbitrary: X, Y, Z, being in this case also functions of t, u, these functions, in the equations which determine the two surfaces, must satisfy every condition required by the projection.

4. The problem requires that the projection be similar to the

projected surface in its least elements. We have first then, to express this condition analytically.

By differentiating the functions which express x, y, z, X, Y, Z, in terms of t and u, we obtain the following equations:—

$$\begin{aligned} dx &= adt + a'du \\ dy &= bdt + b'du \\ dz &= cdt + c'du \\ dX &= Adt + A'du \\ dY &= Bdt + B'du \\ dZ &= Cdt + C'du \end{aligned}$$

The condition specified requires, first, that all the infinitely small elements of the lines which pass through any point on the first surface, and are contained in that surface, shall be proportional to the corresponding linear elements in the second surface; and secondly, that the angle contained between any two such lines in the first surface shall be equal to that contained between the two corresponding lines of the second surface.

A linear-element of the first surface is expressed by

$$\sqrt{\{(aa+bb+cc)dt^2+2(aa'+bb'+cc')dtdu+(a'a'+b'b'+c'c')du^2\}}$$

and the corresponding linear-element of the second by

$$\sqrt{\{(AA+BB+CC)dt^2+2(AA'+BB'+CC')dtdu+(A'A'+B'B'+C'C')du^2\}}.$$

Now since these elements must have a certain given relation to each other, independently of t and u, it is manifest that each of the three quantities

$$aa+bb+cc, \quad aa'+bb'+cc', \quad a'a'+b'b'+c'c',$$

must be respectively proportional to each of the three following

$$AA+BB+CC, \quad AA'+BB'+CC', \quad A'A'+B'B'+C'C'.$$

If the extremities of a second element upon the first surface are denoted by

$$t, u \text{ and } t+\delta t, u+\delta u,$$

then the cosine of the angle which this second element makes with the first will be expressed by*

$$\frac{(adt+a'du)(a\delta t+a'\delta u)+(bdt+b'du)(l\delta t+b'\delta u)+(cdt+c'du)(\delta t+c'\delta u)}{\sqrt{\{(adt+a'du)^2+(bdt+b'du)^2+(cdt+c'du)^2\}\{(a\delta t+a'\delta u)^2+(l\delta t+b'\delta u)^2+(c\delta t+c'\delta u)^2\}}};$$

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* If V denote the angle which two straight lines in space make with each other, X, Y, Z, the angles which the first straight line makes with the co-ordinates, and X', Y', Z' those which the second makes with the same co-ordinates, then, as is well known,

$$\cos V = \cos X \cos X' + \cos Y \cos Y' + \cos Z \cos Z'.$$

and to denote the cosine of the angle between the two corresponding elements of the second surface it is only necessary to change a, b, c, a', b', c', into A, B, C, A', B', C', respectively. The two resulting expressions will be manifestly similar when the above mentioned proportion has place, and the two conditions are therefore both included in the first; which indeed is rendered self-evident by regarding the superficial elements as very small plane surfaces.

The analytical expression of the condition of our problem is consequently that

$$\frac{AA + BB + CC}{aa + bb + cc} = \frac{AA' + BB' + CC'}{aa' + bb' + cc'} = \frac{A'A' + B'B' + C'C'}{a'a' + b'b' + c'c'};$$

and if we denote the value of these equal quantities by m^2, then m^2 must be a finite function of t and u. The function m therefore expresses the proportion in which the values of the linear elements of the first surface increase or diminish when projected on the second surface (according as m is greater or less than 1). Generally speaking this proportion will be variable from one element to another; in the particular case in which m is constant, the finite parts of the first surface will be perfectly similar to those of the second; and when $m = 1$ the two surfaces will be perfectly equal, and coincide in every part when one is applied to the other.

5. If we now assume

$$(aa+bb+cc)\, dt^2 + 2(aa'+b'b+c'c)\, dtdu + (a'a'+b'b'+c'c')\, du^2 = w,$$

we remark that the equation $w = 0$ admits of two integrals, because when the trinomial is resolved into two factors, each of

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Applying this formula to the case in question, we have

$$\cos X = \frac{dx}{\sqrt{(dx^2 + dy^2 + dz^2)}}, \quad \cos Y = \frac{dy}{\sqrt{(dx^2 + dy^2 + dx^2)}}$$

$$\cos Z = \frac{dz}{\sqrt{(dx^2 + dy^2 + dz^2)}}; \text{ and also } \cos X' = \frac{\delta x}{\sqrt{(\delta x^2 + \delta y^2 + \delta z^2)}}$$

$$\cos Y' = \frac{\delta y}{\sqrt{(\delta x^2 + \delta y^2 + \delta y^2)}}, \quad \cos Z' = \frac{\delta z}{\sqrt{(x^2 + \delta y^2 + \delta z^2)}},$$

consequently

$$\cos V = \frac{dx\delta x + dy\delta y + dz\delta z}{\sqrt{(dx^2 + dy^2 + dz^2)} \, . \, \sqrt{(\delta x^2 + \delta y^2 + \delta z^2)}}.$$

Now since δx, δy, δz correspond to δt, δu in the same manner as dx, dy dz correspond to dt, du, we shall have

$$\delta x = a\delta t + a'\delta u, \text{ \&c.}$$

and by substituting these values the expression which has been given above is obtained. *Tr.*

which will be linear in respect of dt and du, either the one or the other of the factors must be $= 0$. One of these factors will give the equation

$$0 = (aa + bb + cc)\,dt$$
$$+ \left\{aa' + bb' + cc' + i\sqrt{\left((aa + bb + cc)(a'a' + b'b' + c'c') - (aa' + bb' + cc')^2\right)}\right\}du$$

(in which for the sake of brevity we have put i for $\sqrt{-1}$, and it is easy to see that the irrational part of the equation must be imaginary): the other factor will furnish an equation entirely similar, only $-i$ must be written instead of i. The integral of the first factor will be of this form

$$p + iq = \text{Const.}$$

where p and q are real functions of t and u. The integral of the other factor will be likewise

$$p - iq = \text{const.}$$

and the nature of the thing requires that

$$(dp + idq)\,(dp - idq) \text{ or } dp^2 + dq^2$$

shall be a factor of w, or that

$$w = n\,(dp^2 + dq^2)$$

where n is a finite function of t and u.

We will now designate by Ω the trinomial into which the expression

$$dX^2 + dY^2 + dZ^2$$

is changed by substituting for dX, dY, and dZ their values in terms of U, T, dU, dT, and assume at once that, as before, the two integrals of the equation $\Omega = 0$ are these

$$P + iQ = \text{Const.}$$
$$P - iQ = \text{Const.}$$
$$\text{and that } \Omega = N\,(dP^2 + dQ^2),$$

where P, Q, N denote real functions of T and U.

These integrations (leaving aside the general difficulties which may occur in integrating the functions) may all be effected before the resolution of the problem.

If we now substitute for T and U such functions of t and u as satisfy the conditions of the problem, the quantity Ω will become m^2w, and we shall have

$$\frac{(dP + idQ)\,.\,(dP - idQ)}{(dp + idp)\,.\,(dp - idq)} = \frac{m^2n}{N}:$$

And the numerator of the first member of this equation will evidently be divisible by the denominator when either

$dP + idQ$ is divisible by $dp + idq$, and $dP - idQ$ by $dp - idq$,
or $dP + idQ$ by $dp - idq$, and $dP - idQ$ by $dp + idq$.

In the first case $dP + idQ$ will vanish when $dp + idq = 0$, or $P + iQ$ will be constant when $p + iq$ is constant; that is to say, $P + iQ$ must be an entire function of $p + iq$, and also $P - iQ$ of $p - iq$. In the second case $P + iQ$ will be a function of $p - iq$, and $P - iQ$ of $p + iq$. It is also easy to see that these consequences are reciprocally true, that is, if we make $P + iQ$, and $P - iQ$ functions of $p + iq$ and $p - iq$ (either respectively or conversely) the finite division of Ω by w may always be effected, and the proportion which the conditions of the problem require satisfied.

It is besides easy to perceive that if we make

$$P + iQ = f(p + iq), \quad P - iQ = f'(p - iq)$$

the nature of the function f' will correspond with that of f. For if among the constant quantities which this last may involve, there are none found but such as are real, then the function f' must be entirely identical with f in order that real values of P and Q may be obtained from every such value of p and q. In the contrary case the functions f and f' will only differ in this respect, that $-i$ must be written throughout instead of i.

We have therefore immediately

$$P = \tfrac{1}{2}f(p + iq) + \tfrac{1}{2}f'(p - iq)$$
$$iQ = \tfrac{1}{2}f(p + iq) - \tfrac{1}{2}f'(p - iq),$$

or, which amounts to the same thing, in order that the function f may be entirely arbitrary, (including at pleasure constant imaginary quantities) P must be made equal to the real, and iQ (or in the second solution $-iQ$) to the imaginary part of $f(p + iq)$, and then, by eliminating T and U, P and Q will be expressed in functions of t and u. Hence the problem is generally and completely resolved.

6. When $p' + iq'$ is a determinate function of $p + iq$, assumed at pleasure (provided p', q' are real functions of p, q) it is easy to see that

$$p' + iq' = \text{Const. and } p' - iq' = \text{Const.}$$

represent the integrals of the differential equation $w = 0$; in fact each will, respectively, be perfectly synonymous with the expressions

$$p + iq = \text{Const. and } p - iq = \text{Const.}$$

given above. So also in the differential equation $\Omega = 0$,

$$P' + iQ' = \text{Const and } P' - iQ' = \text{Const.}$$

will be respectively synonymous with

$$P + iQ = \text{Const. } P - iQ = \text{Const.}$$

when $P' + iQ'$ is any determinate function of $P + iQ$ (P', Q' being real functions of P, Q). Hence it follows that in the general solution of the problem which has been given in the preceding articles, the quantities p and q may be changed into p' and q', and also P and Q to P$'$ and Q$'$. Since the generality of the solution is not affected by this change, either the one form or the other may be assumed as shall be most convenient in the applications.

7. Denoting by φ and φ' respectively the functions which arise from the differentiation of the arbitrary functions f and f', so that $d\,.\,fv = \varphi x\,.\,dv$, $d\,.\,f'v = \varphi'v\,.\,dv$, our general solution will become

$$\frac{d\mathrm{P} + id\mathrm{Q}}{dp + idq} = \varphi(p + iq),\ \frac{d\mathrm{P} - id\mathrm{Q}}{dp - idq} = \varphi'(p - iq);$$

and

$$\frac{mmn}{\mathrm{N}} = \varphi(p + iq)\ .\ \varphi'(p - iq).$$

And the ratio of the enlargement of the elements will hence be given by the formula

$$m = \sqrt{\left\{\frac{dp^2 + dq^2}{w}\ .\ \frac{\Omega}{d\mathrm{P}^2 + d\mathrm{Q}^2}\ .\ \varphi(p + iq)\ .\ \varphi'(p - iq)\right\}}.$$

8. We will now apply this general solution to the case proposed in the question.

If a and b denote the semi-conjugate axes of the ellipsoid of revolution, about the shorter axis $2b$, we may make

$$x = a \cos t \sin u$$
$$y = a \sin t \sin u$$
$$z = b \cos u.$$

In this case we have

$$w = a^2 \sin^2 u dt^2 + (a^2 \cos^2 u + b^2 \sin^2 u)du;$$

and putting for the sake of brevity $\sqrt{\left(1 - \frac{b^2}{a^2}\right)} = \varepsilon$, the differential equation $w = 0$ gives

$$0 = dt \mp idu \sqrt{\{\cot^2 u + 1 - \varepsilon\varepsilon\}}.$$

Let $\sqrt{1 - \varepsilon\varepsilon}\ .\ \tan u = \tan w$. In the application to the terrestrial spheroid 90° — w will denote the geographical latitudes, and t the longitudes; and the above equation becomes

$$0 = dt \mp idw\ .\ \frac{1 - \varepsilon^2}{(1 - \varepsilon^2 \cos^2 w) \sin w},$$

the integral of which is

$$\text{Const.} = t \pm i \log \left\{\cot \tfrac{1}{2} w \left(\frac{1 - \varepsilon \cos w}{1 + \varepsilon \cos w}\right)^{\frac{1}{2}\varepsilon}\right\}.$$

Next, if A denote the radius of the given sphere, we may put in like manner

$$\mathrm{X} = \mathrm{A} \cos \mathrm{T} \sin \mathrm{U}$$
$$\mathrm{Y} = \mathrm{A} \sin \mathrm{T} \sin \mathrm{U}$$
$$\mathrm{Z} = \mathrm{A} \cos \mathrm{U}.$$

On applying the general solution obtained in Art. 5, to this case, it will be found that, when f is an arbitrary function, T must be put equal to the real, and $i \log \operatorname{cotang} \frac{1}{2}\mathrm{U}$ equal to the imaginary part of

$$f\left(t + i \log \left\{\operatorname{cotang} \tfrac{1}{2}w \,.\, \left(\frac{1 - \varepsilon \cos w}{1 + \varepsilon \cos w}\right)^{\frac{1}{2}\varepsilon}\right\}\right).$$

The simplest solution will be obtained by making $fv = v$, and then we shall have

$$\mathrm{T} = t,\ \operatorname{tang} \tfrac{1}{2}\mathrm{U} = \operatorname{tang} \tfrac{1}{2}w \,.\, \left(\frac{1 + \varepsilon \cos w}{1 - \varepsilon \cos w}\right)^{\frac{1}{2}\varepsilon}.$$

This offers a very useful application in problems of the higher geodesy, but of its advantages we can at present, give only a single and brief indication.

Suppose that on the surfaces of the spheroid and sphere we assume a number of points corresponding to each other, that is, having the same longitude on both surfaces, and whose respective latitudes 90° — U, 90° — w, are connected by the equation just obtained, these points may be considered as belonging to a series of small triangles (which will always be the case when they are obtained by actual measurement) whose sides on the surface of the spheroid are the shortest lines that can be drawn between the angular points, and as belonging also to a series of triangles on the surface of the sphere, the angles of which are exactly equal to the corresponding angles on the surface of the spheroid, and whose sides differ so little from arcs of great circles, that in most cases, where the utmost rigour is not necessary, they may be considered as coinciding with those arcs; and besides, when great exactitude is required, the difference from arcs of great circles may be readily computed, with all the necessary precision, by very simple formulæ. Hence, after transferring in the requisite manner the sides of a triangle to the surface of the sphere, the whole system may be computed by means of the angles, in the same manner as if it belonged to the sphere alone, at least with the modication just pointed out. Values of T and U may be obtained for every point of the system, and from these last the values of w are found by means of an auxiliary table extremely easy to construct.

Where the sides of a triangle extend over only a very moderate portion of the earth's surface, the required object may be attained with still greater accuracy, by rendering the solution more general, and instead of taking $fv = v$, by taking $fv = v +$ Const. It is manifest that no advantage will be gained by this change in the form of the function, as long as a real value is assigned to the constant, because T and t will then only differ by a constant quantity, and the sole change will be in the points from which the origin of the longitudes is reckoned. The case however will be entirely different when an imaginary value is assigned to the constant. Taking for example the constant equal to $i \log k$, we shall have

$$\mathrm{T} = t,\ \text{tang}\ \tfrac{1}{2}\mathrm{U} = k\ \text{tang}\ \tfrac{1}{2}w \left(\frac{1 + \varepsilon \cos w}{1 - \varepsilon \cos w}\right)^{\frac{1}{2}\varepsilon}.$$

In this case before we can discover what value of k it will be most convenient to adopt, we must first of all assign the ratio of the enlargement.

Here, as was indicated in Articles 5 and 6, we shall have

$$n = aa \sin^2 u$$

$$\mathrm{N} = \mathrm{AA} \sin^2 \mathrm{U}$$

$$\varphi v = 1.$$

Also
$$m = \frac{\mathrm{A} \sin \mathrm{U}}{a \sin u} = \frac{\mathrm{A} \sin \mathrm{U}}{a \sin w} \cdot \sqrt{(1 - \varepsilon\varepsilon \cos^2 w)}$$

$$= \frac{\mathrm{A}}{a} \cdot \frac{k(1 - \varepsilon^2 \cos^2 w)^{\frac{1}{2}+\frac{1}{2}\varepsilon}}{\cos^2 \tfrac{1}{2}w\, (1 - \varepsilon \cos w)^{\varepsilon} + k^2 \sin^2 \tfrac{1}{2}w\, (1 + \varepsilon \cos w)^{\varepsilon}},$$

A proportion which also depends entirely upon the latitude. The least possible deviation from perfect similarity will be obtained, when k is taken in such a manner, that the values of m at the extreme latitudes are equal, for then at the middle latitudes, the values of m will approach very nearly to their maximum or minimum. If we represent the extreme values of w by w° and w' we shall obtain

$$k = \sqrt{\frac{\dfrac{\cos^2 \frac{1}{2}w^{\circ}\,(1 - \varepsilon \cos w^{\circ})^{\varepsilon}}{(1 - \varepsilon^2 \cos^2 w^{\circ})^{\frac{1}{2}+\frac{1}{2}\varepsilon}} - \dfrac{\cos^2 \frac{1}{2}w'\,(1 - \varepsilon \cos w')^{\varepsilon}}{(1 - \varepsilon^2 \cos^2 w')^{\frac{1}{2}+\frac{1}{2}\varepsilon}}}{\dfrac{\sin^2 \frac{1}{2}w'\,(1 + \varepsilon \cos w')^{\varepsilon}}{(1 - \varepsilon^2 \cos^2 w')^{\frac{1}{2}+\frac{1}{2}\varepsilon}} - \dfrac{\sin^2 \frac{1}{2}w^{\circ}\,(1 + \varepsilon \cos w^{\circ})^{\varepsilon}}{(1 - \varepsilon^2 \cos^2 w^{\circ})^{\frac{1}{2}+\frac{1}{2}\varepsilon}}}}.$$

In order to ascertain at what latitude the value of m is a maximum or minimum, we have

$$\frac{dm}{m} = \text{cotang } \mathrm{U} \,.\, d\mathrm{U} - \text{cotang } w \,.\, dw + \frac{\varepsilon^2 \cos w \sin w \,.\, dw}{1 - \varepsilon^2 \cos^2 w};$$

$$\frac{d\mathrm{U}}{\sin \mathrm{U}} = \frac{dw}{\sin w} - \frac{\varepsilon^2 \sin w \,.\, dw}{1 - \varepsilon^2 \cos w} = \frac{(1 - \varepsilon^2)\, dw}{(1 - \varepsilon^2 \cos^2 w) \sin w};$$

and hence

$$\frac{dm}{m} = \frac{(1 - \varepsilon^2)\, dw}{\sin w\,(1 - \varepsilon^2 \cos^2 w)} \,.\, (\cos \mathrm{U} - \cos w).$$

It is manifest from this that the value of m will be a maximum or minimum when $\mathrm{U} = w$; let us denote in this case the value of w by W, and then

$$k = \left(\frac{1 - \varepsilon \cos \mathrm{W}}{1 + \varepsilon \cos \mathrm{W}}\right)^{\frac{\varepsilon}{2}} \text{ or } \cos \mathrm{W} = \frac{1 - k^{\frac{2}{\varepsilon}}}{\varepsilon\left(1 + k^{\frac{2}{\varepsilon}}\right)},$$

from which we can obtain the value of W when that of k has been calculated from the above formulæ. In practice, indeed, a perfect equality in the values of m can hardly be attained at the extreme latitudes, and it will be sufficient to choose the middle latitudes nearly for 90° — W, and thence deduce k. The general connexion between U and W, is obtained from the equation

$$\text{tang } \tfrac{1}{2}\mathrm{U} = \text{tang } \tfrac{1}{2}\mathrm{W} \left\{\frac{(1 - \varepsilon \cos \mathrm{W})(1 + \varepsilon \cos \mathrm{W})}{(1 + \varepsilon \cos \mathrm{W})(1 - \varepsilon \cos \mathrm{W})}\right\}^{\frac{\varepsilon}{2}}.$$

For actual numerical calculation it is preferable to employ series to which we may give different forms, but at present we must omit their developement.

Hence it is also easy to see when $w < \mathrm{W}$, $\mathrm{U} > w$, that $\cos \mathrm{U} - \cos w$ and therefore $\frac{dm}{dw}$ is negative; and also when $w > \mathrm{W}$, $\mathrm{U} < w$, that $\frac{dm}{dw}$ will be positive, it is therefore evident that for $w = \mathrm{U} = \mathrm{W}$ the value of m will always be a minimum, and besides equal to $\frac{\mathrm{A}}{a}\sqrt{(1 - \varepsilon^2 \cos^2 \mathrm{W})}$.

If we also choose the radius of the circle $\mathrm{A} = \frac{a}{\sqrt{(1 - \varepsilon \cos^2 \mathrm{W})}}$, the elements of projection of the surface of the ellipsoid at the latitude 90° − w, will be not only similar to the original surface, but equal, and at any other latitude greater.

The logarithm by which m is expressed may be transformed with advantage into a series proceeding according to the powers of cos u — cos w, of which the first terms, (and these are sufficient in the applications,) are the following:

$$\text{hyp log } m = \log \left\{ \frac{A}{a} \sqrt{(1 - \varepsilon^2 \cos^2 w)} \right\}$$
$$+ \frac{\varepsilon^2}{2(1 - \varepsilon^2)} \cdot (\cos u - \cos w)^2$$
$$- \frac{2\varepsilon^4 \cos w}{3(1 - \varepsilon^2)^2} \cdot (\cos u - \cos w)^3 \ldots\ldots$$

Suppose, for example, the surface of the kingdom of Denmark (confined between the latitudes of 53° and 58°) were projected in this manner upon the surface of a sphere, and that we make $w = 34° 30'$, then allowing the ellipticity of the earth to be $\frac{1}{303}$, the projection, at its limits, would only be increased by a $\frac{1}{530000}$ part, according to this formula.

END OF THE FIRST PART OF THE FIFTH VOLUME.

THE
MATHEMATICAL REPOSITORY.

VOL. V. PART II.

ORIGINAL ESSAYS.

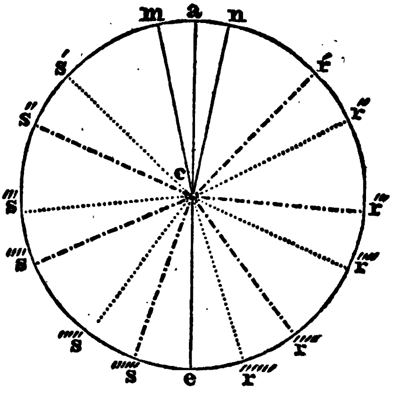
ARTICLE I.

On the Composition of Forces.

By Mr. THOMAS KNIGHT, *Papcastle.*

Problem.

If two equal forces (F) act on a point *c*, in the directions *cm*, *cn*, it is required to find what force (R), acting in the direction *ca*, which bisects the angle *mcn*, will produce an effect equivalent to that arising from the joint action of the two others.



Call the angle *acn* θ. The data we have for the solution of the Problem are, that, when $\theta = 0$, $R = 2F$; and that, *if* $2n+1$ *equal forces are inclined towards one another, in angles each of* $\frac{2\pi}{2n+1}$, *any one of them will balance the joint action of all the others.* It is, moreover, easy enough to shew that we may assume, generally, $R = F.\varphi(\theta)$*; Resolve the force, acting along *cn*, into $2n$ other equal forces, acting in the directions cr'', cr'''', &c. cs'

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* Gregory's Mechanics, 2nd Ed. Art. 37, Cor. 2nd.—Francœur, 3rd Ed. Art. 15.

cs''', &c. and making the angles ncr'', $r''cr''''$, &c. each $= \frac{2\pi}{2n+1}$. In like manner, resolve the force, acting along cm, into $2n$ other equal forces acting in the directions cs'', cs'''', &c. cr', cr''', &c. and making the same angles as the former ones.

Let $R^{''\ldots 2n}$ be the force, in the direction ce, resulting from those acting in the directions $c\,r^{''\ldots 2n}$, $c\,s^{''\ldots 2n}$*; $R^{''\ldots\overline{2n-1}}$ the force, in the same direction, resulting from those acting in the directions $c\,r^{''\ldots\overline{2n-1}}$, $c\,s^{''\ldots\overline{2n-1}}$; and so on. Let R' be the force, in the direction ca, resulting from those acting in the directions cr', cs'; R'' the force in the same direction resulting from those acting in the directions cr'', cs'', &c., then

$$R = R^{''\ldots 2n} + R^{''\ldots\overline{2n-1}} + \&c. \ldots\ldots - R' - R'' - \&c. \quad \ldots\ldots (1)$$

But because $acr' = \frac{2\pi}{2n+1} - \theta$, $acr'' = \frac{2\pi}{2n+1} + \theta$, &c.,

$$ec\,r^{''\ldots 2n} = \frac{\pi}{2n+1} - \theta,\quad ec\,r^{''\ldots\overline{2n-1}} = \frac{\pi}{2n+1} + \theta, \ \&c.$$

we have

$$R^{''\ldots 2n} = F\cdot\varphi\left(\frac{\pi}{2n+1} - \theta\right),\quad R^{''\ldots\overline{2n-1}} = F\cdot\varphi\left(\frac{\pi}{2n+1} + \theta\right), \&c.$$

$$R' = F\cdot\varphi\left(\frac{2\pi}{2n+1} - \theta\right),\quad R' = F\cdot\varphi\left(\frac{2\pi}{2n+1} + \theta\right), \ \&c.$$

which values put in equation (1) give

$$\varphi(\theta) = \varphi\left(\frac{\pi}{2n+1} - \theta\right) + \varphi\left(\frac{\pi}{2n+1} + \theta\right) + \varphi\left(\frac{3\pi}{2n+1} - \theta\right) + \&c. \quad (2)$$
$$- \varphi\left(\frac{2\pi}{2n+1} - \theta\right) - \varphi\left(\frac{2\pi}{2n+1} + \theta\right) - \varphi\left(\frac{4\pi}{2n+1} - \theta\right) - \&c.$$

The number of terms, in the right-hand member, being $2n$. From this equation we derive $\varphi(\theta) = A \cos.\theta$ (Euleri ; Introd. in Analys. Infin. Tom 1, p. 207), where A is an arbitrary constant to be determined from the nature of the Problem.

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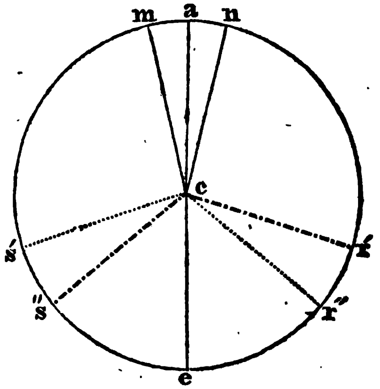
* In the figure $2n$ is 6.

Put, then, in the equation $R = F . \phi(\theta)$, $A \cos. \theta$, for its equal $\phi(\theta)$, and it becomes $R = F \times A \times \cos. \theta$. If we now make $\theta = 0$, and consequently $R = 2F$, the last equation becomes $2F = A . F$, whence $A = 2$, and finally $R = 2F . \cos. \theta$. *Q. E. I.*



The above investigation would become much simpler, and better adapted for an elementary treatise on mechanics, if $n = 1$, or if cm were resolved only, into the two forces, $c r'$, $c s''$ forming angles of 120°; and $c n$, in like manner, into $c r''$, $c s'$, forming the same angles. In this case equation (1) becomes

$$R = R' + R''; \quad \text{and (2) is simply}$$

$$\phi(\theta) = \phi(60^\circ + \theta) + \phi(60^\circ - \theta).$$

ARTICLE II.

Three Indeterminate Problems.

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Problem I.

To find three right-angled triangles, such, that their perimeters may be equal to each other, and their areas in arithmetical progression.

Solution.

Let the sides of the first triangle be denoted by $2amn$, $a \times (m^2 - n^2)$ and $a \times (m^2 + n^2)$; the sides of the second by $2bpq$, $b \times (p^2 - q^2)$ and $b \times (p^2 + q^2)$; and the sides of the third by $2crs$, $c \times (r^2 - s^2)$ and $c \times (r^2 + s^2)$: then, by the question, the perimeters of the triangles being all equal, there will be had, $a \times (m^2 + mn) = b \times (p^2 + pq) = c \times (r^2 + rs)$. It is well-known, that the area of any plane triangle is equal to that of a rectangle under half the perimeter, and the radius of its inscribed circle; and the perimeters of the three triangles being equal, their areas will be as the radii of their inscribed circles; and when these radii are in arithmetical progression, their areas will be so likewise.

Now the radius of the inscribed circle of any right-angled triangle is equal to half the difference between the hypothenuse and the sum of the two legs; wherefore $an \times (m - n)$, $bq \times (p - q)$, and $cs \times (r - s)$ are the respective radii of the inscribed circles of the three preceding triangles; and when these radii are in arithmetical progression, supposing the second the mean, we shall have, $an \times (m - n) + cs \times (r - s) = 2bq \times (p - q)$; and from this expression, exterminating b and c, by means of the equation $a \times (m^2 + mn) = b \times (p^2 + pq) = c \times (r^2 + rs)$, it will become $an \times (m - n) + a \times$

$$\frac{(m^2 + mn)}{r^2 + rs} \times s \times (r - s) = 2a \times \frac{(m^2 + mn)}{p^2 + pq} \times q \times (p - q);$$

and hence

$$\frac{m^2 + mn}{n \times (m - n)} = \frac{(r^2 + rs) \times (p^2 + pq)}{(r^2 + rs) \times 2q \times (p - q) - (rs - s^2) \times (p^2 + pq)} = e,$$

putting $e = \dfrac{m^2 + mn}{n \times (m - n)}$; and the equation

$$\frac{(r^2 + rs) \times (p^2 + pq)}{(r^2 + rs) \times 2q \times (p - q) - (rs - s^2) \times (p^2 + pq)} = e,$$

by reduction, becomes

$$p^2 + pq \times \frac{(1 - 2e)r^2 + rs \times (1 - e) - es^2}{r^2 + rs \times (1 + e) - es^2} = -\frac{2eq^2 \times (r^2 + rs)}{r^2 + rs \times (1 + e) - es^2}.$$

Completing the square on the first side of the equation,

$$p^2 + pq \times \frac{(1 - 2e)r^2 + rs \times (1 - e) - es^2}{r^2 + rs \times (1 + e) - es^2}$$

$$+ q^2 \times \frac{\{(1 - 2e)r^2 + rs \times (1 - e) - es^2\}^2}{4 \times \{r^2 + rs \times (1 + e) - es^2\}^2} =$$

$$q^2 \times \frac{\left\{\begin{array}{c}\{(1 - 2e)r^2 + rs \times (1 - e) - es^2\}^2 \\ -8e \times (r^2 + rs) \times \{r^2 + rs \times (1 + e) - es^2\}\end{array}\right\}}{4\{r^2 + rs\,(1 + e) - es^2\}^2}$$

= a square; and consequently

$$\left.\begin{array}{l}\left\{(1-2e)\,r^2+rs\,(1-e)-es^2\right\}^2\\ -\,8e\times(r^2+rs)\times\left\{r^2+rs\times(1+e)-es^2\right\}\end{array}\right\}$$

$= r^4 \times (1 - 12e + 4e^2) + 2r^3 s \times (1 - 11e - 2e^2)$
$+ r^2 s^2 \times (1 - 12e + 5e^2) - 2ers^3 \times (1 - 5e) + e^2 s^4 =$
a square. Assume $r^2 \times (1 + 10e) + rs \times (1 - 5e) - es^2$ for the root; that is, put

$$\begin{array}{c} r^4 \times (1 - 12e + 4e^2) + 2r^3 s \times (1 - 11e - 2e^2) \\ + r^2 s^2 \times (1 - 12e + 5e^2) - 2ers^3 \times (1 - 5e) + e^2 s^4 \\ = \left\{r^2 (1 + 10e) + rs \times (1 - 5e) - es^2\right\}^2 = \\ r^4 \times (1 + 20e + 100e^2) + 2r^3 s \times (1 + 5e - 50e^2) \\ + r^2 s^2 \times (1 - 12e + 5e^2) - 2ers^3 \times (1 - 5e) + e^2 s^4 \end{array}$$

whence

$$\begin{array}{c} r^4 \times (1 - 12e + 4e^2) + 2r^3 s \times (1 - 11e - 2e^2) \\ = r^4 \times (1 + 20e + 100e^2) + 2r^3 s (1 + 5e - 50e^2); \end{array}$$

from whence, after proper reduction $\frac{r}{s} = \frac{3e - 1}{3e + 1}$; wherefore we may take $r = 3e - 1$, and $s = 3e + 1$; but r being here less than s, will obviously make one side of the triangle to which it belongs negative, it will therefore be necessary to obtain a value for s that shall be less than r.

We have hitherto proceeded generally with the solution, and the problem might, from what has been deduced, be done generally for any value of $e = \frac{m^2 + mn}{n \times (m - n)}$; but some of the resulting expressions would be rather complex, yet still quite manageable. However, for the sake of abridging the labour of calculation, we shall, for the remaining part of the solution, proceed in a manner not quite so general: for which purpose, take $m = 2$, and $n = 1$; then $e = \frac{m^2 + mn}{n \times (m - n)} = 6$; $r = 3e - 1 = 17$, and $s = 3e + 1 = 19$; and in order to obtain a value for s that shall be less than r, put $r = 17$, and

$s = 19 - v$; then writing these values in the before found expression, viz.

$$p^2 + pq \times \frac{(1-2e)r^2 + rs \times (1-e) - es^2}{r^2 + rs \times (1+e) - es} = - \frac{2eq^2 \times (r^2 + rs)}{r^2 + rs \times (1+e) - es^2},$$

it becomes

$$p^2 + pq \times \frac{(6v^2 - 313v + 6960)}{6v^2 - 109v - 384} = - \frac{12q^2 \times (17v - 612)}{6v^2 - 103v - 384}.$$

Completing the square, $p^2 + \frac{pq \times (6v^2 - 313v + 6960)}{(6v^2 - 109v - 384}$

$$+ q^2 \times \frac{(6v^2 - 313v + 6960)^2}{4 \times (6v^2 - 109v - 384)^2} = \frac{q^2 \times (6v^2 - 313v + 6960)^2}{4 \times (6v^2 - 109v - 384)^2}$$

$$- \frac{48q^2 \times (17v - 612) \times (6v^2 - 109v - 384)}{4 \times (6v^2 - 109v - 384)^2}$$

$$= q^2 \times \frac{(36v^4 - 8652v^3 + 446689v^2 - 7245600v + 37161216)}{4 \times (6v^2 - 109v - 384)^2} =$$

a square; and therefore $36v^4 - 8652v^3 + 446689v^2 - 7245600v + 37161216$ must be a square. Assume $6v^2 - 721v + 6096$ for its root, that is, put $36v^4 - 8652v^3 + 446689v^2 - 7245600v + 37161216 = (6v^2 - 721v + 6096)^2 = 36v^4 - 8652v^3 + 592993v^2 - 8790432v + 37161216$; whence $v = \frac{1544832}{146304} = \frac{1341}{127}$; and therefore $s = 19 - v = 19 - \frac{1341}{127} = \frac{1072}{127}$, and $r = 17 = \frac{2159}{127}$; wherefore we may take $r = 2159$, and $s = 1072$.

Recollecting now that we took $6v^2 - 721v + 6096$ for the root of $36v^4 - 8652v^3 + 446689v^2 - 7245600v + 37161216$, and taking the square root of each side of the expression

$$p^2 + pq \times \frac{(6v^2 - 313v + 6960)}{6v^2 - 109v - 384} + q^2 \times \frac{(6v^2 - 313v + 6960)^2}{4 \times (6v^2 - 109v - 384)^2}$$

$$= q^2 \times \frac{(36v^2 - 8652v^3 + 446689v^2 - 7245600v + 37161216)}{4 \times (6v^2 - 109v - 384)^2},$$

we shall have

$$p + q \times \frac{(6v^2-313v+6960)}{2\times(6v^2-109v-384)} = q \times \frac{(6v^2-721v+6096)}{2\times(6v^2-109v-384)};$$

and hence $\frac{p}{q} = \frac{204v + 432}{384 + 109v - 6v^2} = \frac{41710356}{13967314} = \frac{4572}{1531}$; by writing for v its value found above, and reducing the fraction to its lowest terms. Therefore we may take $p = 4572$ and $q = 1531$; and we found $r = 2159$, and $s = 1072$; also we took $m = 2$ and $n = 1$; whence, and from the equality of the perimeters, viz. $a \times (m^2 + mn) = b \times (p^2 + pq) = c \times (r^2 + rs)$, that is, $6a = 27902916b = 6975729c$; whence $a = 4650486b$, and $c = 4b$. Hence $2mna = 4a = 18601944b$, $(m^2 - n^2) \times a = 3a = 13951458b$, and $(m^2+n^2) \times a = 5a = 23252430b$, which are the two legs and hypothenuse of the first triangle; and $2pqb = 13999464b$, $(p^2 - q^2) \times b = 18559223b$, and $(p^2 + q^2) \times b = 23247145b$, which are the two legs and hypothenuse of the second, or mean triangle. Also, $2rsc = 18515584b$, $(r^2 - s^2) \times c = 14048388b$, and $(r^2 + s^2) \times c = 23241860b$, are the two legs and hypothenuse of the third triangle. Or, omitting the common factor b, 18601944, 13951458, and 23252430, are the sides of the first triangle; 13999464, 18559223, and 23247145, are the sides of the mean triangle; and 18515584, 14048388, and 23241860, are the sides of the other triangle. For it will appear, that all these triangles have the same perimeter, viz. 55805832; and the radii of their inscribed circles are 4650486, 4655771, and 4661056, which are in arithmetical progression.

We shall now proceed to give a more general solution to the question; and for this purpose put $r = 3e - 1$ and $s = 3e + 1 - v$; and these being written in the expression

$$p^2 + pq \times \frac{(1-2e)r^2 + rs(1-e) - es^2}{r^2 + rs(1+e) - es^2} = - \frac{2eq^2(r^2+rs)}{r^2 + rs(1+e) - es^2},$$

before deduced, it becomes

$$p^2 + pq \times \frac{ev^2 - v\,(9e^2 - 2e + 1) + 4e\,(9e^2 - 6e + 2)}{ev^2 - v\,(3e^2 + 1) - 4e\,(3e - 2)}$$

$$= - \frac{2eq^2\left\{v\,(3e-1) - 6e\,(3e-1)\right\}}{ev^2 - v\,(3e^2+1) - 4e\,(3e-2)};$$ completing the square

$$(a)\quad p^2 + pq \times \frac{ev^2 - v\,(9e^2 - 2e + 1) + 4e\,(9e^2 - 6e + 2)}{ev^2 - v\,(3e^2 + 1) - 4e\,(3e - 2)}$$

$$+\frac{q^2\left\{ev^2-v\,(9e^2-2e+1)+4e\,(9e^2-6e+2)\right\}^2}{4\left\{ev^2-v\,(3e^2+1)-4e\,(3e-2)\right\}^2}$$

$$\pm\frac{q^2\left\{ev^2-v\,(9e^2-2e+1)+4e\,(9e^2-6e-2)\right\}^2}{4\left\{ev^2-v\,(3e^2+1)-4e\,(3e-2)\right\}^2}$$

$$-\frac{2eq^2\left\{v\,(3e-1)-6e\,(3e-1)\right\}}{ev^2-v\,(3e^2+1)-4e\,(3e-2)}$$

$$=\frac{q^2}{4\left\{ev^2-v\,(3e^2+1)-4e\,(3e-2)\right\}^2}$$

$$\times\left\{\begin{array}{l}e^2v^4-2ev^3(21e^2-6e+1)+v^2(369e^4-156e^3+62e^2-12e+1)\\-8ev(135e^4-126e^3+93e^2-24e+2)+16e^2(9e^2-12e+2)^2\end{array}\right\}$$

= a square, and therefore

$$ev^4-2ev^3(21e^2-6e+1)+v^2(369e^4-156e^3+62e^2-12e+1)$$
$$-8ev^3(135e^4-126e^3+93e^2-24e+2)+16e^2(9e^2-12e+2)^2,$$

must be a square.

Assume $ev^2 - v\,(21e^2-6e+1)+4e\,(9e^2-12e+2)$ for the root of the last expression, that is, put

$$e^2v^4-2ev^3(21e^2-6e+1)+v^2(269e^4-156e^3+62e^2-12e+1)$$
$$-8ev^3(135e^4-126e^3+93e^2-24e+2)+16e^2(9e^2-12e+2)^2$$
$$=\left\{ev^2-v\,(21e^2-6e+1)+4e\,(9e^2-12e+2)\right\}^2$$
$$=e^2v^4-2ev^3(21e^2-6e+1)+v^2(513e^4-348e^3+94e^2-12e+1)$$
$$-8ev\,(189e^4-306e^3+123e^2-24e+2)+16e^2(9e^2-12e+2)^2,$$

whence, by transposition,

$$v^2\,(144e^4-192e^3+32e^2)=8ev\,(54e^4-180e^3+30e^2),$$

which gives $v=\dfrac{3e\,(9e^2-30e+5)}{9e^2-12e+2}$, and hence

$r=3e-1=(3e-1)\times\dfrac{9e^2-12e+2}{9e^2-12e+2}=\dfrac{27e^3-45e^2+18e-2}{9e^2-12e+2}$, and

$s=3e+1-v=3e+1-\dfrac{3e(9e^2-30e+5)}{9e^2-12e+2}=\dfrac{63e^2-21e+2}{9e^2-12e+2}$; and therefore we may take $r = 27e^3 - 45e^2 + 18e - 2$, and $s = 63e^2 - 21e + 2$; or r and s may be expounded by any integer numbers, in the ratio of $27e^3 - 45e^2 + 18e - 2$ to $63e^2 - 21e + 2$, that is, we may put $\frac{r}{s} = \frac{27e^3 - 45e^2 + 18e - 2}{63e^2 - 21e + 2}$.

Again, taking the square root of each side of the expression (a), and recollecting what was taken for the root of the irrational part,

$$p + q \times \frac{ev^2 - v(9e^2 - 2e + 1) + 4e(9e^2 - 6e + 2)}{2\left\{ev^2 - v(3e^2 + 1) - 4e(3e - 2)\right\}}$$

$$= q \times \frac{ev^2 - v(21e^2 - 6e + 1) + 4e(9e^2 - 12e + 2)}{2\left\{ev^2 - v(3e^2 + 1) - 4e(3e - 2)\right\}},$$

and by transposition, &c. $\frac{p}{q} = \frac{v(6e^2 - 2e) + 12e^2}{4e(3e - 2) + v(3e^2 + 1) - ev^2}$, and in this expression for v, writing its value found above,

$$\frac{p}{q} = \frac{v(6e^2 - 2e) + 12e^2}{4e(3e - 2) + v(3e^2 + 1) - ev^2}$$

$$= \frac{\frac{3e(9e^2 - 30e + 5)(6e^2 - 2e)}{9e^2 - 12e^2 + 2} + 12e^2}{4e(3e - 2) + \frac{3e(3e^2 + 1)(9e^2 - 30e + 5)}{9e^2 - 12e + 2} - \frac{9e^3(9e^2 - 30e + 5)^2}{(9e^2 - 12e + 2)^2}}$$

$$= \frac{6e(9e^2 - 12e + 2)(27e^3 - 81e^2 + 21e - 1)}{\left\{\begin{array}{c}4(3e-2)(9e^2-12e+2)^2 + 3(3e^2+1)(9e^2-12e+2)(9e^2-30e+5)\\ -9e^2(9e^2 - 30e^2 + 5)^2\end{array}\right\}}$$

and the denominator of this fraction being properly expanded by multiplication, &c. will be found to be divisible by $27e^3 - 81e^2 + 21e - 1$, which is a factor of the numerator, by which means it finally reduces to $\frac{3e(9e^2 - 12 + 2)}{45e^2 - 15e + 1}$; that is $\frac{p}{q} = \frac{3e(9e^2 - 12e + 2)}{45e^2 - 15e + 1}$, or p and q may be expounded by any two numbers in the ratio of $3e(9e^2 - 12e + 2)$ to $45e^2 - 15e + 1$; and we have put $e = \frac{m(m + n)}{n(m - n)}$, where m and n may be taken at pleasure.

Example. Take $m = 2$ and $n = 1$; then $e = \frac{m(m+n)}{n(m-n)} = 6$; and from what has been deduced

$$\frac{r}{s} = \frac{27e^3 - 45e^2 + 18e - 2}{63e^2 - 21e + 2} = \frac{4318}{2144} = \frac{2159}{1072}; \text{ therefore}$$

we may take $r = 2159$, and $s = 1072$.

Also $\frac{p}{q} = \frac{3e(9e^2 - 12e + 2)}{45e^2 - 15e + 1} = \frac{4572}{1531}$; therefore we may take $p = 4572$ and $q = 1531$.

The preceding values of the quantities r, s; p, q, obtained from the conclusions deduced in the general solution, being precisely the same as those obtained in the particular solution, prove the accuracy of the various calculations.

Problem II.

To find any number of numbers, *ad libitum*, such, that if their sum be either added to, or subtracted from the square of each particular number, the several results shall all be rational squares.

Solution.

Let a, b, h; a', b', h'; a'', b'', h'', &c. denote the two sides and hypothenuse (all rational) of any number of right-angled triangles, having the same area;—that is,

$$ab = a'b' = a''b'' = a'''b''', \text{ \&c.}$$

then we shall obviously have

$$\begin{aligned}
&a^2 + b^2 \pm 2ab = h^2 \pm 2ab = \text{a square},\\
&a'^2 + b'^2 \pm 2a'b' = h'^2 \pm 2a'b' = h'^2 \pm 2ab = \text{a square},\\
&a''^2 + b''^2 \pm 2a''b'' = h''^2 \pm 2a''b'' = h''^2 \pm 2ab = \text{a square},\\
&a'''^2 + b'''^2 \pm 2a'''b''' = h'''^2 \pm 2a'''b''' = h'''^2 \pm 2ab = \text{a square};\\
&\qquad\text{\&c.} \qquad\qquad \text{\&c.} \qquad\qquad \text{\&c.}
\end{aligned}$$

and the same properties must necessarily still prevail, if each of the quantities, a, b, h; a', b', h', &c. is multiplied by any common factor, as g, that is,

$$\begin{aligned}
&g^2 h^2 \pm 2g^2 ab\\
&g^2 h'^2 \pm 2g^2 ab\\
&g^2 h''^2 \pm 2g^2 ab\\
&g^2 h'''^2 \pm 2g^2 ab\\
&\text{\&c.} \quad \text{\&c.} \quad \text{\&c.}
\end{aligned}$$

must all be rational squares.

A method of finding the sides of any number of right-angled triangles, having the same area, in rational numbers, may be seen at page 60, art. 13, part 11, vol. 3, of the Repository; and the hypothenuses of such right-angled triangles may be taken for the quantities, h, h', h'', h''', &c. and four times the areas of each of the triangles will manifestly be equal to $2ab$.

Now in order to apply what has been premised to the solution of the question, let gh, gh', gh'', gh''', &c. denote the required numbers, and take $2abg^2$, for their sum, that is, put $gh + gh' + gh'' + gh'''$, &c. $= 2abg^2$, whence $g = \dfrac{h+h'+h''+h''' \text{ \&c.}}{2ab}$;

and hence, the required numbers will be

$$gh = \frac{h(h+h'+h''+h''' \text{ \&c.})}{2ab}, gh' = \frac{h'(h+h'+h''+h'''+ \text{ \&c.})}{2ab}$$

$$gh'' = \frac{h''(h+h'+h''+h''' \text{ \&c.})}{2ab}, gh''' = \frac{h'''(h+h'+h''+h''' \text{ \&c.})}{2ab} \text{ \&c.}$$

At page 62, of the above mentioned art. and vol., we have 58, 74 and 113 for the hypothenuses of three right-angled triangles, four times the area of each of which or $2ab = 3360$.

Wherefore we may put $h = 58$, $h' = 74$ and $h'' = 113$; and hence

$$\frac{h(h+h'+h'')}{2ab} = \frac{203}{48} = \frac{406}{96}, \frac{h'(h+h'+h'')}{2ab} = \frac{259}{48} = \frac{518}{96}, \text{ and } \frac{h''(h+h'+h'')}{2ab} = \frac{791}{96},$$ which are three numbers that will answer.

Ex. 2. In the same article, and upon the same page, we have also 218, 233 and 394 for the hypothenuses of three right-angled triangles, four times the area of each of which or $2ab = 43680$; from whence, in the same manner, we shall find $\frac{2834}{672}$, $\frac{3029}{672}$, and $\frac{5122}{672}$ for three other numbers that will answer; that is, if their sum be either added to or subtracted from, the square of each particular number, the several results will all be rational squares.

Ex. 3. To find four rational numbers, such, that if their sum be either added to, or, subtracted from the square of each particular number, the several results shall all be rational squares.

At page 64, of the vol. and art. before quoted, we have 58, 74, 113 and $\frac{1412881}{1189}$ for the hypothenuses of four right-angled triangles, four times the area of each of which is $2ab = 3360$, wherefore we may put $h = 58, h' = 74, h'' = 113$, and $h''' = \frac{1412881}{1189}$, whence $h + h' + h'' + h''' = \frac{1704186}{1189}$; and hence

$$\frac{h(h + h' + h'' + h''')}{2ab} = \frac{58}{3360} \times \frac{1704186}{1189} = \frac{98842788}{3995040}$$

$$\frac{h'(h + h' + h'' + h''')}{2ab} = \frac{74}{3360} \times \frac{1704186}{1189} = \frac{126109764}{3995040}$$

$$\frac{h''(h + h' + h'' + h''')}{2ab} = \frac{113}{3360} \times \frac{1704186}{1189} = \frac{192573018}{3995040}$$

$$\frac{h'''(h + h' + h'' + h''')}{2ab} = \frac{1412881}{3360 \times 1189} \times \frac{1704186}{1189} = \frac{240781201966}{4750102560}$$

which four numbers will answer the question.

We may next take $h^{\text{iv}} = \frac{2579761}{39997}$, as appears at page 64 of the above mentioned art. and proceeding as before, we shall obtain five numbers that will answer, but it is manifest from what has been done that they will be very large fractional numbers.

Problem III.

To find any number of fractions, ad libitum, such, that if the sum of their cubes, be either added to, or subtracted from the square of each particular fraction, the several results shall all be rational squares.

Solution.

Let gh, gh', gh'', gh''', &c. denote the required fractions; the quantities h, h', h'', h''', &c. being the same as in the preceding problem. Put $2abg^2$ for the sum of their cubes; that is, put

$$g^3h^3 + g^3h'^3 + g^3h''^3 + g^3h'''^3 + \text{&c.} = 2abg^2;$$

whence

$$g = \frac{2ab}{h^3 + h'^3 + h''^3 + h'''^3 + \&c.};$$

and therefore

$$gh = \frac{2abh}{h^3+h'^3+h''^3+h'''^3\&c.},\ gh' = \frac{2abh'}{h^3 + h'^3 + h''^3 + h'''^3\ \&c.},$$

$$gh'' = \frac{2abh''}{h^3+h'^3+h''^3+h'''^3\&c.},\ gh''' = \frac{2abh'''}{h+h'^3+h''^3+h'''^3+\&c.};$$

&c. &c.

Example. Let it be required to find three rational fractions, such, that if the sum of their cubes be either added to, or subtracted from the square of each particular fraction, the several results shall all be rational squares.

Take $h = 58$, $h' = 74$, $h'' = 113$, and $2ab = 3360$; and we shall have

$$g = \frac{2ab}{h^3 + h'^3 + h''^3} = \frac{3360}{2043233},$$

$$gh = \frac{58 \times 3360}{2043233} = \frac{194880}{2043233},\ gh' = \frac{74 \times 3360}{2043233} = \frac{248640}{2043233},$$

and $gh'' = \dfrac{113 \times 3360}{2043233} = \dfrac{379680}{2043233}$ which are three fractions that will answer the proposed question; that is, if the sum of their cubes be either added to, or subtracted from the square of each particular fraction, the several results shall be rational squares.

If it were required to find four fractions of the kind, we shall have $h''' = \dfrac{1412881}{1189}$; it is therefore evident that the required fractions would be very large, or consist of a great number of figures.

The manner of extending the question to any assigned number of fractions that will answer, will be obvious enough from what has been done in the preceding examples.

ARTICLE III.

Mathematical Scraps.

By Mr. THOMAS WHITE, *Dumfries.*

On the transformation of equations and of slowly converging infinite series into others that shall have their coefficients arbitrarily assumable as given or known, by which the *direct* and *general resolution* of the one and the *summation* of the other are accomplished. (Occurred 23 November 1817).

(*a*) Suppose ... $x = 1 + \frac{z}{\alpha} + \frac{z^2}{\beta} + \frac{z^3}{\gamma} + \frac{z^4}{\delta} + \ldots\ldots\ldots$; $\alpha, \beta, \gamma, \delta \ldots\ldots$ being transient indeterminates. Then

$$x^2 = 1 + \frac{2z}{\alpha} + \left(\frac{1}{\alpha^2} + \frac{2}{\beta}\right)z^2 + \left(\frac{2}{\alpha\beta} + \frac{2}{\gamma}\right)z^3 + \left(\frac{2}{\alpha\gamma} + \frac{1}{\beta^2} + \frac{2}{\delta}\right)z^4 + \ldots\ldots$$

$$x^3 = 1 + \frac{3z}{\alpha} + \left(\frac{3}{\alpha^2} + \frac{3}{\beta}\right)z^2 + \left(\frac{6}{\alpha\beta} + \frac{3}{\gamma} + \frac{1}{\alpha^3}\right)z^3 + \left(\frac{6}{\alpha\gamma} + \frac{3}{\beta^2} + \frac{3}{\delta} + \frac{3}{\alpha^2\beta}\right)z^4 + \ldots\ldots$$

$$x^4 = 1 + \frac{4z}{\alpha} + \left(\frac{6}{\alpha^2} + \frac{4}{\beta}\right)z^2 + \left(\frac{12}{\alpha\beta} + \frac{4}{\gamma} + \frac{4}{\alpha^3}\right)z^3 + \left(\frac{12}{\alpha\gamma} + \frac{6}{\beta^2} + \frac{4}{\delta} + \frac{12}{\alpha^2\beta} + \frac{1}{\alpha^4}\right)z^4 + \ldots\ldots$$

$$x^5 = 1 + \frac{5z}{\alpha} + \left(\frac{10}{\alpha^2} + \frac{5}{\beta}\right)z^2 + \left(\frac{20}{\alpha\beta} + \frac{5}{\gamma} + \frac{10}{\alpha^3}\right)z^3 + \left(\frac{20}{\alpha\gamma} + \frac{10}{\beta^2} + \frac{5}{\delta} + \frac{30}{\alpha^2\beta} + \frac{5}{\alpha^4}\right)z^4 + .$$

&c. &c.

Now, let $0 = -a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \ldots\ldots$(A), be the equation to be transformed; substitute for x, we have

$$\begin{aligned}
0 = \; & -a+b+c+d+e+f\ldots\ldots \\
& + \left\{(b+2c+3d+4e+5f\ldots\ldots)\cdot\frac{1}{\alpha}\right\}\cdot z \\
& + \left\{\begin{array}{c}(b+2c+3d+4e+5f\ldots\ldots)\cdot\frac{1}{\beta} \\ (c+3d+6e+10f\ldots\ldots)\cdot\frac{1}{\alpha^2}\end{array}\right\}\cdot z^2 \\
& + \left\{\begin{array}{c}(b+2c+3d+4e+5f\ldots\ldots)\cdot\frac{1}{\gamma} \\ (c+3d+6e+10f\ldots\ldots)\cdot\frac{2}{\alpha\beta} \\ (d+4e+10f\ldots\ldots)\cdot\frac{1}{\alpha^3}\end{array}\right\}\cdot z^3
\end{aligned}$$

$$+\left\{\begin{array}{l}(b+2c+3d+4e+5f\ldots\ldots).\ \frac{1}{\delta}\\ \qquad(c+3d+6e+10f\ldots\ldots).\left(\frac{2}{\alpha\gamma}+\frac{1}{\beta^2}\right)\\ \qquad\quad(d+4e+10f\ldots\ldots).\frac{3}{\alpha^2\beta}\\ \qquad\qquad(e+5f\ldots\ldots).\frac{1}{\alpha^4}\end{array}\right\}.\ z^4$$

$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now, when $x=1$, the given equation (A) gives $-a+b+c+d+e+f+\ldots\ldots=\text{C}$, suppose; and the other coefficients are most easily found thus, viz.

$\overset{\cdot}{(\text{A})}$ gives, when $x=1$, ... $b+2c+3d+4e+5f+\ \ldots\ldots=\text{C}_1$

$\overset{\cdot\cdot}{(\text{A})}$ (casting out divisors) ...... $c+3d+6e+10f+\ \ldots\ldots=\text{C}_2$

$\overset{\therefore}{(\text{A})}$ .... $d+4e+10f+\ \ldots\ldots=\text{C}_3$

$\overset{::}{(\text{A})}$ .... $e+5f+\ \ldots\ldots=\text{C}_4$

$\overset{\overset{\cdot}{v}}{(\text{A})}$ .... $f+\ \ldots\ldots=\text{C}_5$

&c. &c.

Hence, by substitution, (A) becomes

$$0=\text{C}+\frac{\text{C}_1}{\alpha}z+\left[\frac{\text{C}_1}{\beta}+\frac{\text{C}_2}{\alpha^2}\right]z^2+\left[\frac{\text{C}_1}{\gamma}+\frac{2\text{C}_2}{\alpha\beta}+\frac{\text{C}_3}{\alpha^3}\right]z^3$$

$$+\left[\frac{\text{C}_1}{\delta}+\text{C}_2\left(\frac{2}{\alpha\gamma}+\frac{1}{\beta^2}\right)+\frac{3\text{C}_3}{\alpha^2\beta}+\frac{\text{C}_4}{\alpha^4}\right]z^4+\ldots\ldots\text{(B)}$$

For brevity, this may be called "The Formula of Transformation." If, in applying this formula to any particular equation, C be $=0$, one root of such equation is unity; and the equation must be depressed before the process commences: And if, in any case, C_1 be $=0$ (see n° IV. below) the roots of the proposed equation must be either increased or diminished, previous to the operation, to avoid this circumstance. The reasons will soon appear.

Now, (A) .... $a=bx+cx^2+dx^3+\ldots\ldots$; then, for particular equations, $\text{C}, \text{C}_1, \text{C}_2\ \ldots\ldots$ are found as below ......

Equations.	Roots.	C	C_1	C_2	C_3	C_4
I. $6 = 10x + x^2 - x^3$....	-3, and those of $x^2 + 2 = 0$	4	9	-2	-1	0
II. $10 = 7x + 4x^2 - x^3$..	1; -2 and 5; to be depressed	0	12	1	-1	0
III. $32 = 8x + 0 + 4x^3 - x^4$	2; 4 & the roots of $x^2 + 2x + 4 = 0$..	-21	16	6	0	-1
IV. $10 = 4x - 5x^2 + 2x^3$	$\frac{5}{2}$, and the roots of $x^2 + 2 = 0$	-9	0	1	2	0
V $180 = 9x + 29x^2 - x^3 - x^4$	3; 4; -3 and -5	-144	60	20	-5	-1
VI. $24 = 16x - 8x^2 + 2x^3$	$\frac{3}{2}$, and the roots of $x^2 + 1 = 0$	-9	16	3	2	0
VII. $5 = 12x - 5x^2 - x^3 + x^4$	roots of $x^2 + 2x - 2 = 0$ and $x^2 - 3x + 3 = 0$	1	5	-2	3	1
VIII. $1 = 4x - 5x^2 + 4x^3 - x^4$	roots of $x^2 - x + 1 = 0$ & $x^2 - 3x + 1 = 0$..	1	2	1	0	-1
&c.	&c.					

(*b*) Let us now assume the equation

$$o = c + (c + \varphi)z + \frac{z^2}{\beta_1} + \frac{z^3}{\gamma_1} + \frac{z^4}{\delta_1} + \ldots\ldots\ldots\ldots \quad (B_1)$$

as equivalent, term by term, to (B); and having, *within certain limits*, φ; and, *absolutely*, $\beta_1, \gamma_1, \delta_1$ .... *assumable* as *given*, c and φ having the same sign. Comparing (B) and (B_1), we have $\frac{c_1}{\alpha} = c + \varphi$; and $\frac{1}{\alpha} = \frac{c + \varphi}{c_1}$ which *must* be a fraction; that is $c + \varphi$ *must* be $< c_1$; or $\varphi < c_1 - c$. Again if (B) were to be inverted, or z to be found in terms of c, c_1 .... and α, β, γ ...., the series for z would proceed by the powers of $c \div \frac{c_1}{\alpha}$, or its equal $\frac{c}{c + \varphi}$ which, to secure convergency, must be a fraction; and therefore, $c + \varphi$ *must* be $> c$; or $\varphi >$ than o. Consequently, c_1 *must not* be $= c$: Should this happen, the roots of the equation to be solved, must be either increased or diminished; and it has been observed that, neither c_1 nor c must be = o. It results, then, that φ may be assumed $>$ than o, but $<$ than $c_1 - c$, and having the same sign as c, either + or —.

We have found $\frac{1}{\alpha} = \frac{c + \varphi}{c_1}$; and the 3d terms of (B) and (B_1) give

$$\frac{1}{\beta} = \frac{1}{\beta_1 c_1} - \frac{c_2}{\alpha^2 c_1} = (\text{substituting for } \tfrac{1}{\alpha})\frac{1}{\beta_1 c_1} - \frac{c_2}{c_1}\left(\frac{c + \varphi}{c_1}\right)^2;$$

the 4th term gives

$$\frac{1}{\gamma} = \frac{1}{\gamma_1 c_1} - \frac{2c_2}{\alpha\beta c_1} - \frac{c_3}{\alpha^3 c_1} = \frac{1}{c_1 \gamma_1} - \frac{2c_2}{\beta_1 c_1^2} \cdot \frac{c + \varphi}{c_1} + 2\left(\frac{c_2}{c_1}\right)^2 \cdot \left(\frac{c + \varphi}{c_1}\right)^3;$$

and so on for $\frac{1}{\delta}$; thus, the coefficients, of the assumed value of x, are known and are obviously convergent. Hence, we are thus enabled to assign such a seriesal assumption for x, in any equation, as shall transform it into another having *arbitrarily assumable* or *given* coefficients. Regarding the limits of φ .... it may be observed that $\frac{\pm(c+\varphi)}{\pm c_2}$ is $= \frac{c+\varphi}{c_2}$; and, as above $\varphi \angle c_2 - c$; also $\pm \frac{c+\varphi}{c_2} = \frac{+(c+\varphi)}{-c_2} = \frac{-(c+\varphi)}{+c_1}$; and $c + \varphi \angle - c_2$; or $\varphi \angle$ than $-(c_1 + c)$; but, the limits are regulated by the numerical values of $(c_1 - c)$ and $(c_1 + c)$ without attention to the sign which precedes these expressions.

If we revert (B_2) or $-c = \pm(c+\varphi)z + \frac{z^2}{\beta_2} + \frac{z^3}{\gamma_2} + \ldots.$, which is equivalent to (B), we have

$$z = \frac{c}{\mp(c+\varphi)} - \frac{1}{\pm(c-\varphi)\cdot\beta_2}\cdot\left(\frac{c}{\pm(c+\varphi)}\right)^2$$

$$+\left[\frac{2}{(\pm(c+\varphi))^2\cdot\beta_1^2} - \frac{1}{\pm(c+\varphi)\cdot\gamma_1}\right]\cdot\left(\frac{c}{(c+\varphi)}\right)^3$$

$$+\left[\frac{5}{(c+\varphi)\cdot\beta_1}\left(\frac{1}{\pm(c+\varphi)\cdot\gamma_2} - \frac{1}{(\pm c+\varphi)^2\cdot\beta_2^2}\right) - \frac{1}{\pm(c+\varphi)\cdot\delta_2}\right]$$

$$\times\left(\frac{c}{\mp(c+\varphi)}\right)^4 + \ldots. = \mp\frac{c}{c+\varphi} \mp \frac{1}{(c+\varphi)\cdot\beta_2}$$

$$\times\left(\frac{c}{c+\varphi}\right)^2 \mp \left[\frac{2}{(c+\varphi)^2\cdot\beta_2^2} \mp \frac{1}{(c+\varphi)\cdot\gamma_2}\right]\cdot\left(\frac{c}{c+\varphi}\right)^3$$

$$+\left[\frac{5}{(c+\varphi)\cdot\beta_2}\cdot\left(\frac{1}{(c+\varphi)\cdot\gamma_2} \mp \frac{1}{(c+\varphi)^2\cdot\beta_2^2}\right) \quad \frac{1}{(c+\varphi)\cdot\delta_2}\right]$$

$\times\left(\frac{}{c+\varphi}\right)^4 + \ldots\ldots$; the upper signs for $+(c+\varphi)$; the under for $-(c+\varphi)$. Here, as has been observed, $\varphi > 0$: Consequently, if this be substituted for z; and, the values of $\alpha, \beta, \gamma, \ldots\ldots$, already found, in $x = 1 + \frac{z}{\alpha} + \frac{z^2}{\beta} + \ldots.$, a root r, of the given equation, becomes directly known, every term being obviously convergent.

(c) Again; for a slowly converging infinite series .... Suppose the sum s, of such a series, be required, when x is given, *i. e.* let $s = \frac{1}{a} + \frac{x}{b} + \frac{x^2}{c} + \frac{x^3}{d} + \frac{x^4}{e} + \ldots\ldots$; then assume, as before, $x = 1 + \frac{z}{\alpha} + \frac{z^2}{\beta} + \ldots\ldots$, the formula of transformation (B) will arise in which, for five terms, $C = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \ldots$; $C_1 = \frac{1}{b} + \frac{2}{c} + \frac{3}{d} + \frac{4}{e} + \ldots$; $C_2 = \frac{1}{c} + \frac{3}{d} + \frac{6}{e} + \ldots$; $C_3 = \frac{1}{d} + \frac{4}{e} + \ldots\ldots$; $C^4 = \frac{1}{e} + \ldots$; &c. &c.

Now, let us assume $S = C + (C + \varphi)z + \frac{z^2}{\beta_1} + \frac{z^3}{\gamma_1} + \ldots (B_1)$ as equivalent, term by term, to (B); and having *within certain* limits, φ; and *absolutely* β_1, γ_1, δ_1, ... assumable as given; C and φ having the same sign: .... the second terms give $\frac{1}{\alpha} = \frac{C + \varphi}{C_1}$ which is the coefficient of the second term of $x = 1 + \frac{z}{\alpha} + \ldots$; and, this seriesal equation x is given, because it is, on this supposition, that S is to be found. Again the 3rd terms give

$\frac{C_1}{\beta} + \frac{C}{\alpha^2} = \frac{1}{\beta_1}$; hence, $\frac{1}{\beta} = \frac{1}{\beta_1 C_1} - \frac{C_2}{C_1 \alpha^2} = \frac{1}{\beta_1 C_1} - \frac{C_2}{C_1}\left(\frac{C+\varphi}{C_1}\right)^2$;

in like manner, $\frac{1}{\gamma} = \frac{1}{\gamma_1 C_1} - \frac{2C_2}{\beta_1 C_1^2} \cdot \frac{C+\varphi}{C_1} + 2\left(\frac{C_2}{C_1}\right)^2 \cdot \left(\frac{C+\varphi}{C_1}\right)^3$;

and thus for $\frac{1}{\delta}$, &c. It remains to discover such circumstances as will enable us to assign a value for φ: Now, since x is given, and $\frac{1}{\alpha}$; $\frac{1}{\beta}$, ... are found, with the exception of φ, if we were to invert the seriesal equation $x = 1 + \frac{z}{\alpha} + \frac{z^2}{\beta} + \ldots$, the series arising for the value of z would proceed by the powers of $(x - 1) \div \frac{1}{\alpha}$ which is $= \frac{C_1(x - 1)}{C + \varphi}$; and which, to secure convergency, *must* be a *proper* fraction, *i. e.* $\frac{C_1 x}{C + \varphi} - \frac{C_1}{C + \varphi}$ *must* be *proper* fractions. And, here are three cases to be considered; viz. 1st. x may be a whole number > unity; 2nd x may be a fraction; 3rd x may be = unity.

In the first case, $c + \varphi$ must exceed $x\ c_1$; and therefore, $\varphi > c_1\ x - c$: In the 2nd case $c + \varphi$ must exceed c_1; and $\varphi > c_1 - c$; and, in the 3rd case, when $x = 1$, we have

$$o = \frac{z}{\alpha} + \frac{z^2}{\beta} + \frac{z^3}{\gamma} + \ldots\ldots; \text{ or } o = 1 + \frac{\alpha}{\beta}z + \frac{\alpha}{\gamma}z^2 + \frac{\alpha}{\delta}z^3 + \ldots$$

Here the series found for z, by inversion, will proceed by the powers of $(-1) \div \frac{\alpha}{\beta} = -\frac{\beta}{\alpha} = -\frac{\beta_1 c_1^3}{c_1^2 - \beta_1 c_2 (c+\varphi)^2} \times \frac{c+\varphi}{c_1}$

$= \frac{\beta_1 c_1^2 (c+\varphi)}{-c_1^2 + \beta_1 c_2 (c+\varphi)^2}$ which must be a *proper* fraction; and therefore $\beta_1 c_2 (c + \varphi)^2 - c_1^2 > \beta_1 c_1^2 (c + \varphi)$, that is ....

$$\beta_1 c_2 (c + \varphi)^2 - \beta_1 c_1^2 (c + \varphi) > c_1^2; \text{ consequently}$$

$$(c+\varphi)^2 - c_1 \cdot \frac{c_1}{c_2}(c+\varphi) + c_1^2\left(\frac{c_1}{2c_2}\right)^2 > c_1^2 \cdot \left(\frac{c_1}{2c_2}\right)^2 + c_1^2 \cdot \frac{4c_2}{\beta_1 \cdot 4c_2^2};$$

and, hence $(c + \varphi) > \frac{c_1}{2c_2} \cdot \left\{c_1 \pm \sqrt{\left(c_1^2 + \frac{4c_2}{\beta_1}\right)}\right\}$ and,

$\varphi > \frac{c_1}{2c_2}\left\{(c_1 \pm \sqrt{\left(c_1^2 + \frac{4c_2}{\beta_1}\right)}\right\} - c_1$. Remember that $(c + \varphi)$ may be preceded by either + or —, as it may happen. This limit enables us to assume φ so as to secure convergency; and β, γ_1, δ_1 .... in (B_1) which is equivalent to (B) may be any numbers we please.

Since, in the first two cases, $x - 1 = \frac{c+\varphi}{c_1}z + \frac{z^2}{\beta} + \frac{z^3}{\gamma} + \ldots$, we have, by reversion, ...

$$z = \frac{(x-1) \cdot c_1}{c+\varphi} - \frac{c_1}{\beta \cdot (c+\varphi)} \cdot \left(\frac{(x-1) \cdot c_1}{c+\varphi}\right)^2$$

$$+ \left[2 \cdot \left(\frac{c_1}{\beta \cdot (c+\varphi)}\right)^2 - \frac{c_1}{\gamma \cdot (c+\varphi)}\right]\left(\frac{(x-1) \cdot c_1}{c+\varphi}\right)^3$$

$$+ \left[5 \cdot \frac{c_1}{\beta \cdot (c+\varphi)}\left(\frac{c_1}{\gamma \cdot (c+\varphi)} - \frac{c_1^2}{\beta^2 \cdot (c+\varphi)^2}\right) - \frac{c_1}{\delta \cdot (c+\varphi)}\right]$$

$\times \left(\frac{(x-1) \cdot c_1}{c+\varphi}\right)^4 + \ldots\ldots$; and in the 3rd case,

$-1 = \frac{\alpha}{\beta}z + \frac{\alpha}{\gamma}z^2 + \frac{\alpha}{\delta}z^3 + \ldots\ldots$; by inversion, gives

$$z = -\frac{\beta}{\alpha} - \frac{\beta}{\gamma}\left(\frac{\beta}{\alpha}\right)^2 - \left[2 \cdot \frac{\beta^2}{\gamma^3} - \frac{\beta}{\delta}\right]\left(\frac{\beta}{\alpha}\right)^3$$

$$+ \left[5 \cdot \frac{\beta}{\gamma}\left(\frac{\beta}{\delta} - \frac{\beta^2}{\gamma^3}\right) - \frac{\beta}{\varepsilon}\right]\left(\frac{\beta}{\alpha}\right)^4 + \ldots\ldots:$$ Here $\frac{\beta}{\alpha}$ is a proper fraction, by what is done above; and, $\alpha, \beta, \gamma \ldots.$ are known in terms of φ, β_1, γ_1 .... and of C, C_1, C_2 .... as has been shewn.

By substituting either of the above values of z (to suit the case) in (B_1) we shall have the sum S of the proposed series by means of the series (B_1) of *arbitrary* or *chosen* convergency.

(*d*). We are now directly conducted to a result which, from hopelessness of ever obtaining it, has never been hitherto attempted in the doctrine of infinite series: viz. to exhibit the sum of *any proposed* converging infinite series in terms of the known *sum* of a *given* infinite series. Suppose that, we any how know that, the function f (z) produces, by expansion, or is the sum of the converging infinite series $A + \alpha_1 z + \frac{z^2}{\beta_1} + \frac{z^3}{\gamma_1} + \frac{z^4}{\delta_1} + \ldots$ ....; that is, let us know

$f(z)$ to be $= A + \alpha_1 z + \frac{z^2}{\beta_1} + \ldots.$; then

$f(z) - (A + \alpha_1 z) = \frac{z^2}{\beta_1} + \frac{z^3}{\gamma_1} + \ldots.$; and, therefore,

$$f(z) - (A + \alpha_1 z) + C + (C + \varphi)z = C + (C + \varphi)z + \frac{z^2}{\beta_1} + \frac{z^3}{\gamma_1} + \frac{z^4}{\delta_1} + \ldots.$$

(B_1) which is equivalent to (B), or to any proposed series $\frac{1}{a} + \frac{x}{b} + \frac{x^2}{c} + \ldots.$ of which the sum S is required.

Now, if we substitute the value of z, found above, in $f(z) - (A + \varphi z) + C + (C + \varphi)\, z$, which is the sum of the chosen or arbitrary series, we shall have S, expressed in terms of f (z), the required sum of the proposed series.

ARTICLE IV.

HORÆ ARITHMETICÆ.

To the Editor of the Mathematical Repository.

SIR,

The title prefixed to the short series of essays which I have the pleasure of offering for your acceptance, sufficiently indicates their nature. Among the observations which shall be made, some, it is hoped, of an original character, will be found to facilitate the management, or illustrate the theory, of some interesting points, both of the common and diophantine algebra. The principle, that no numerical solution can be regarded as perfect, which is not reduced to an exercise of simple arithmetic, has been steadily kept in view.

I am, SIR,

Your's, very truly,

Bath, July, 1820. W. G. HORNER.

NO. I. ON NUMERICAL EQUATIONS.

A new process, which "may be regarded as a universal instrument of calculation, extending to the composition as well as analysis of functions of every kind," has been recently published in the Philosophical Transactions.* The investigation is there conducted on the strictly universal principle; and the application to the solution of numerical equations illustrated by a praxis sufficiently copious and minute to convey an impressive idea of its facility, simplicity, and comprehensiveness. Preceptive details were not deemed necessary in addition to a very intelligible general formula, and ample practical elucidations. They will be supplied in the present essay, in which I propose to present, in as succinct and perspicuous a form as possible, those parts of the theory and practice of the new method, which peculiarly regard *finite equations;* accompanied with such improvements as a more intimate consideration of the subject has suggested.

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* *Note.* Subsequent and totally unexpected circumstances require me to assert explicitly my undivided claim to the invention, and developement of every particular, wherein the new process differs from those which have been long familiar to all well-informed mathematicians. I intend to resume this subject in the sequel.

PROB. III. When the roots of the transformed, are to be less than those of the given, equation, by any assigned quantity r.

Rule. Under the second coefficient write r times the first, n times repeated. Under the third, r times the successive sums of the second and the several quantities beneath it, omitting the last. Under the fourth, r times the like successive sums of the third and the numbers beneath it, omitting the last. Proceed thus through all the coefficients: under the last of which, or the absolute term, one addend only (of course) will appear.

Add up each column separately; and the respective amounts will be the corresponding coefficients of the transformed equation.—The first remains unaltered.

Note 1. In the practical application of each of these rules, *the first addend in every set may be found and used before the remaining addends are placed in column.* This plan is not only almost indispensable to the convenient use of Prob. III., but has the important advantage of conducting us in the speediest manner possible, to the new absolute term; which is often the only coefficient of the new equation, that is requisite to be ascertained.

2. It is quite obvious, that *each addend,* in any set constructed on the principles of these rules, *is equal to r times the corresponding addend in the preceding set, together with the preceding addend in its own set.*

3. If these remarks be kept in view, the whole process, when r is a simple digit, may be carried on figure by figure mentally; the results alone appearing in column, and no bye-operations being requisite.

Example. Transform the equation $x^5 - 8x^4 + 11x^3 - 39x^2 + 31x - 101 = 0$ to another whose roots shall be less by 4.

1	− 8	+ 11	− 39	+ 31	− 101	(4
	4	− 16	− 20	− 236	− 820	
1	− 4	− 5	− 59	− 205	− 921	
	4	0	− 20	− 316		
	4	16	44			
	4	32				
	4					
1	+ 12	+ 43	− 35	− 521	− 921	Coefficients of the required equation.

PROB. IV. When the roots of the transformed, are to be less, or greater, than the reciprocals of those of the given equation, by 1 or r.

RULE. Suppose the order of the coefficients to be reversed, and then work as in the preceding problems. The results, reversed, and with all their signs changed if requisite, will be the coefficients of the new equation.

Example. Transform the equation $x^4 - 12x^2 + 12x - 3 = 0$, to another, whose roots shall be equal to the reciprocals of the given one, diminished by 1.

$$\begin{array}{llllll}
1) & 1 & +\ 0 & -\ 12 & +\ 12 & -\ 3 \\
 & -\ 3 & -\ 3 & 9 & -\ 3 & \\
 & & 3 & 6 & -\ 3 & \\
 & & & 3 & -\ 3 & \\
 & & & & -\ 3 & \\
\hline
 & -\ 2 & +\ 0 & +\ 6 & +\ 0 & -\ 3
\end{array}$$

Consequently the required equation is $3y^4 - 6y^2 + 2 = 0$.

Criteria of Possibility.

According to the theorem of DESCARTES, no equation can have more real affirmative roots than it has changes of sign, nor more real negative roots than permanences of sign. Incongruous indications resulting from the application of this test, are proofs of the existence of imaginary roots in the equation under examination, and of a certain number of such roots. The two criteria which I am going to state, are the most useful formulae that I am acqainted with, to aid in conducting this enquiry. The first is a particular case of a general theorem due to DE GUA; and the second is an improvement on BUDAN's *criterium**.

1. Every instance in which zero appears, in an equation and its transformees, as a coefficient, between two coefficients affected with like signs, indicates a distinct pair of imaginary roots.

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* It is a striking proof of the want of clearness and precision in the conception and praxis of this part of Budan's system, that of 14 transformed reciprocals (or as he calls them, collateral transformees) employed in his 5th chapter, only *two* were at all necessary; and only 3 of the 10 in chap. 6.

2. If, in passing from one equation to another whose roots are less by unity, a certain number of changes of sign are lost; take the reciprocal of the former equation, and diminish its roots also by 1, (by Problem IV). Then, if the changes of sign, which appear in this transformee, are not as many as were lost in the other, every unit in the difference indicates a distinct imaginary root.

Example 1. Let the equation $x^4 - 4x^3 + 8x^2 - 16x + 20 = 0$, be proposed. By two successive applications of Prob. I. we have

$$\begin{array}{rrrrrrrrrl}
1 & - & 4 & + & 8 & - & 16 & + & 20 & (1 \\
 & & 1 & - & 3 & + & 5 & - & 11 & \\
 & & 1 & - & 2 & + & 3 & & & \\
 & & 1 & - & 1 & & & & & \\
 & & 1 & & & & & & & \\
\hline
1 & \pm & 0 & + & 2 & - & 8 & + & 9 & (1 \\
 & & 1 & + & 1 & + & 3 & - & 5 & \\
 & & 1 & + & 2 & + & 5 & & & \\
 & & 1 & + & 3 & & & & & \\
 & & 1 & & & & & & & \\
\hline
1 & + & 4 & + & 8 & \pm & 0 & + & 4 &
\end{array}$$

and, observing that zero has twice appeared between affirmative coefficients, we know that four, that is all, of the roots are imaginary.

Example 2. Try $x^5 - 8x^4 + 11x^3 - 39x^2 + 31x - 101$, the formula used in Prob. III.

$$\begin{array}{rrrrrrrrrrrl}
1 & - & 8 & + & 11 & - & 39 & + & 31 & - & 101 & (1 \\
 & & 1 & - & 7 & + & 4 & - & 35 & - & 4 & \\
 & & 1 & - & 6 & - & 2 & - & 37 & & & \\
 & & 1 & - & 5 & - & 7 & & & & & \\
 & & 1 & - & 4 & & & & & & & \\
 & & 1 & & & & & & & & & \\
\hline
1 & - & 3 & - & 11 & - & 44 & - & 41 & - & 105 &
\end{array}$$

Here *four* changes of signs have vanished. Diminish the root of the reciprocal equation by 1.

$$\begin{array}{lrrrrrrrrrrr}
1) & 1 & - & 8 & + & 11 & - & 39 & + & 31 & - & 101 \\
 & & - & 106 & - & 98 & - & 109 & - & 70 & - & 101
\end{array}$$

but, without completing the transformation, it is already evident that the new coefficients will all be affected with like signs, or will have *no* change, ∴ $4 - 0 = 4$ roots are imaginary.

No. II.

The two criteria of possibility, with which my last paper concluded, were selected on account of their being most generally applicable, but were not intended to exclude the use of any others which may happen to be strikingly suitable to any particular equation, which arises in the course of transformation. In fact, it is no small recommendation of the new method, that it multiplies the chances of introducing these subsidiary aids with advantage.

The first criterion itself can only occur incidentally, but its application is capable of an extension beyond what is at once apparent. To cite a single example; when the m^{th} coefficient changes its sign in passing from one set to another, while those which immediately precede and follow it are, and continue to be, identical, the existence of zero between like signs somewhere in the interval may be suspected. Should all the previous signs in these sets be also alike, the probability is increased; and, in this case especially, may usually be put to a decisive proof, by an operation plainly suggested by the first set of formulae for *transformation.*

The utility of the first or *figurate formulae* for transformation, which is manifest in this instance, will occasionally be illustrated by the detection of a close *inferior* limit; and, in short, whenever it is desirable to investigate the state of any particular coefficient or coefficients, separately from the rest.

On the subject of the second criterion, it will be understood, that if only *one* change of sign is lost in passing from any transformation to the next, the diminished reciprocal need not be taken, as this circumstance proves the existence and situation of one real root. And, in short, since the alternations of signs are even or odd, according as the signs of the extreme terms agree or differ, and, in the latter case, there is certainly one real root, the place of such a root will be shewn every time the absolute term changes its sign.

Initial Solution.

The first step toward the complete solution of a numerical equation is, to obtain what are usually, tho' not with sufficient precision, called the nearest integral values* of the real roots.

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* When correct writers employ such expressions, their practice shews that they have a reference to that transient relation between the successive portions

And in order to this, the nearest superior limit of the greatest root both affirmative and negative, must be investigated. The best methods of doing which, are founded on a rule given by **Newton** in the latter portion of his *Universal Arithmetic.* Perhaps therefore the most natural transition to the new method will be made by translating Newton's rule and example into its algorithm.

Prob. V. To find a limit greater than the greatest root of an equation.

For the limit of affirmative roots: By successive applications of Prob. I. find the coefficients of the equations whose roots are less by 1, 2, 3, &c. than those of the given equation. Continue the process until a set of coefficients arise, which are all affirmative. The index of the number of transformations employed, is the limit required.

For the limit of negative roots: Change the alternate signs, and proceed and estimate as before.

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of the root, according to which (as I have said in art. 25 of my Essay) in passing from one digit to the next, we tacitly regard the former as units, and the latter as a decimal fraction. Conformably to this principle, the equation is always understood to be so reduced in the first instance, that the absolute term does not contain more integral places than are indicated by the dimension of the equation. To effect which, the neatest mode perhaps is, after distributing the integral part of the absolute term into *periods* as in common evolution, to point off just as many *digits* from the integral part of the 2nd coefficient, as many periods of 2 digits each, from that of the 3rd, as many of 3 digits each from the 4th, and so on; ciphers being prefixed for that effect, if necessary.

If Mr. Nicholson had paid attention to this circumstance, he would perhaps have omitted the 3rd and 4th props. of his Essay on Involution and Evolution. Certainly he would have suppressed one among the strange misrepresentations which he has made of my method of Continuous Approximation. So far from having stated, as he affirms, "that there is no advantage in applying it to the integral part of a root," I have distinctly shewn, in the only article (21) to which he can allude, not only "its sufficiency for effecting the development of the real root independently of any previous knowledge of R," but the very manner of arriving, by means of it, "at R, an unambiguous limiting value of x," which value in all ordinary cases is the first digit of the root, as it is most obviously in the example to which he refers, solved on p. 44 of his Essay.

This attack is the more unwarrantable, because the terms *integer* and *integral*, or any equivalent to them, do not occur in any sentence I wrote, nor any allusion to the distinction between integral and fractional decimals, except in the expression quoted in the beginning of this note, and the incidental mention of "decimal places" in some of the examples; and because the solution so pointingly held up by Mr. N. is *in every particular*, excepting a want of precision in his way of determining the first digit, a direct exemplification of my rules.

Example 1. Required the superior limits of the positive and negative roots of

$$x^5 - 2x^4 - 10x^3 + 30x^2 + 63x - 120 = 0$$

For the positive roots.	For the negative roots.
1—2—10+30+63—120 (1	1+2—10—30+63+120 (1
1— 1—11+19+ 82	1 3— 7—37+ 26
1 0—11+ 8	1 4— 3—40
1 1—10	1 5+ 2
1 2	1 6
1	1
1+3— 8— 2+90— 38 (1	1 +7+ 8—38—14+146 (1
1 4— 4— 6+ 84	1+ 8+16—22— 36
1 5 1— 5	1+ 9+25+ 3
1 6 7	1+10+35
1 7	1+11
1	1
1+8+14+ 2+79+ 46	1+12+46+38-33+110 (1
	1+13+59+97 &c.

The positive limit is thus attained at two steps, and the negative at three; consequently $x < 2, > -3$.

Moreover, by a merely mental application of the first criterion to the last set of coefficients in the first operation, and of the second to the last set in the other, it is obvious that two positive roots and two negative are impossible. The equation therefore has *only one real* root, whose first digit is 2.

By the figurate formulae, the coefficient — 10 in the first operation would become at the next step $-10 - 4\,(2) + 10\,(1)$, obviously negative; the same may be said of — 30 in the 2nd operation, which would become $-30 - 3\,(10) + 6\,(2) + 10\,(1)$.

Observing this, we might have omitted the first step in each operation, by substituting 2 for 1, and working by Prob. III. thus

$$\begin{array}{lcl}
1-2-10+30+63-120\ (2 & \qquad & 1+\ 2-10-30+63+120\ (2 \\
\quad 2\quad 0-20+20+166 & & \quad 2+\ 8-\ 4-68-\ 10 \\
\hline
1-0+10+10+83+\ 46 & & 1+\ 4-\ 2-34-\ 5+110 \\
\quad 2+\ 4-12-\ 4 & & \quad 2+12+20-28 \\
\quad 2+\ 8+\ 4 & & \quad 2+16+52 \\
\quad 2+12 & & \quad 2+20 \\
\quad 2 & & \quad 2 \\
\hline
1+8+14+\ 2+79+\ 46 & & 1+12+46+38-33+110 \\
\hline\hline
\end{array}$$

But on this plan, there is a manifest risk of overstepping the places or criteria of intermediate roots.

Example 2. Find near limits to the negative root of $x^3-7x+7=0$. [LAGRANGE *Res. des Eq. Num. p.* 122.]

$$\begin{array}{cccccccc}
1 & \pm & 0 & - & 7 & - & 7 \\
 & & 3 & & 1 & - & 6 \\
 & & & & 2 & & \\
\hline
1 & + & 3 & - & 4 & - & 13 \\
 & & 3 & & 4 & & 0 \\
 & & & & 5 & & \\
\hline
1 & + & 6 & + & 5 & - & 13 \\
 & & 3 & & 7 & & 12 \\
 & & & & 8 & & \\
\hline
1 & + & 9 & + & 20 & - & 1 \\
\hline\hline
\end{array}$$

The next transformation would obviously produce coefficients entirely affirmative; and therefore 3 may be taken for an inferior, and 4 for a superior, limit. But the simple inspection of the last set of coefficients, indicates the root of the corresponding equation to be $> \frac{1}{21} < \frac{1}{20}$. Consequently the root of the proposed equation is $> 3\frac{1}{21} < 3\frac{1}{20}$, abstracting the sign. The limits found by La Grange, certainly with much greater trouble, are 3 and $3\frac{1}{10}$.

Prob. VI. To obtain the complete initial solution of a numerical equation.

1. Find the superior limits of the positive and negative roots by the last problem.

2. Apply, at the proper places, the criteria of possibility, mentioned in the conclusion of the last and beginning of the present No.

3. If the aggregate of real and imaginary roots, respectively indicated by the criteria, is equal to the index of the equation, *the solution is already effected;* as was the case with *Example* 1, Prob. V. For, all the real roots are distinct, and also correct as far as they are determined; that is, in this instance, to one digit.

4. If otherwise, return to the first equation, whose diminished reciprocal indicated a probability of a certain number of roots commencing with the same digit; and multiply the successive coefficients by 1, 10, 100, &c.

5. Find the superior limit of the *positive* roots of this new equation; and proceed according to the foregoing precepts, until the nature of those roots has been ascertained, and such of them as are real have been traced as far as to the first digit in which they differ.

6. Perform a similar course of operations, if necessary, upon another of the equations referred to in the 4th precept; and so on, until the requisite number of roots has been classified, and all the real roots traced to distinctness.

Example 1. Characterise and distinguish the roots of $x^3 - 7x + 7 = 0$.

From *Example* 2, Prob. V. it is obvious that there is but one negative root; whose distinct initial value was found to be -3.

For the positive roots.

$$\begin{array}{rrrrrrrl}
1 & & 0 & - & 7 & + & 7 & (1 \\
 & & 3 & & 1 & - & 6 & \\
 & & & & 2 & & & \\
\hline
1 & + & 3 & - & 4 & + & 1 & (1 \\
 & & 3 & & 4 & \text{\&c.} & & \\
\hline
\end{array}$$

all positive.

∴ if real, there are two, each having 1 for its first digit.

∴ Apply the 2nd criterion.

$$\begin{array}{lrrrrrrr}
1) & 1 & + & 3 & - & 4 & + & 1 \\
 & 0 & - & 3 & & 3 & & \\
 & & - & 2 & & & & \\
\hline
 & 1 & - & 2 & - & 1 & + & 1 \\
\hline
\end{array}$$

Here are *two* alternations of sign, as before; ∴ two roots *may* be real. See the next column.

Investigation of these two roots.

$$\begin{array}{rrrrrrrl}
1 & + & 30 & - & 400 & + & 1000 & (1 \\
 & & 3 & & 31 & - & 369 & \\
 & & & & 32 & & & \\
\hline
1 & + & 33 & - & 337 & + & 631 & (1 \\
 & & 3 & & 34 & - & 303 & \\
 & & & & 35 & & & \\
\hline
1 & + & 36 & - & 268 & + & 328 & (1 \\
 & & 3 & & 37 & - & 231 & \\
 & & & & 38 & & & \\
\hline
1 & + & 39 & - & 193 & + & 97 & (1 \\
 & & 3 & & 40 & - & 153 & \\
 & & & & 41 & & & \\
\hline
1 & + & 42 & - & 112 & - & 56 & (2 \\
 & & 2 & & 88 & - & 48 & \\
\hline
1 & + & 44 & - & 24 & - & 104 & \\
 & & 2 & + & 92 & & & \\
 & & 2 & & & & & \\
\hline
1 & + & 48 & + & 68 & - & 104 & (1 \\
 & & 3 & + & 49 & + & 117 & \\
\end{array}$$

&c. all positive.

Hence the roots in question are *real*, and their respective second digits are 3 and 6. Consequently 1 . 3, 1 . 6, and — 3, are approximations to the roots of the proposed equation.

REMARKS. 1. In the course of proceeding by the foregoing problem, if a *rational* root should happen to be accurately attained, the absolute term in the resulting equation would be 0.

2. If two or more successive terms, including the last, should become 0 together; this would shew, that the quantity attained was the accurate value of just as many *equal roots*.

3. The equations whose last terms have thus vanished, are by that very circumstance *depressed* just so many degrees, and may be treated as such in the subsequent process: — which is another remarkable advantage of the new calculus.

4. If in pursuing the directions of precepts 4 and 5, there should appear an increasing tendency, in the last terms of the equations, to simultaneous disappearance; as this argues a probability of two or more equal roots, it may sometimes be advisable to apply the usual test, namely to seek the greatest common measure of the last two *figurate formulae* concerned in the transformation. For if that measure exceed 1, it will be of the form $(r-a)^m$, a and m being given quantities, and r will have $m+1$ values, each $= a$, and the given equation in r may be depressed $m+1$ degrees by being divided by $(r-a)^{m+1}$. But even in this well-known operation, the principle of working with the *coefficients only*, will be found beneficial.

No. III.

Abbreviations of the Initial Solution.

Considering the extreme degree of simplicity to which the general formula of transformation was reduced, it cannot be expected that a substitute, easier than itself, should be attainable otherwise than casually; or that the convenience gained by contracting the work, if the operations can still be mentally connected, will not be purchased by some other sacrifice. What appears really useful on this head, is subjoined.

(A). *General Compendia.* The variations in sign and magnitude which the absolute term undergoes, supply, in most cases, satisfactory indications of the place of the superior, and even of the intermediate limits. When, therefore, the transformations are likely to be numerous, it becomes an object to procure the series of absolute terms separately. For this purpose, LAGRANGE, after substituting 0, 1, 2, 3, &c. in the common way, until the number of trials exceeds by unity the index of the equation, finds, by subtraction as usual, the several orders of differences of the results, and thence continues the set at pleasure by addition.

In an Addendum to my Essay, I have advanced on this idea, by shewing how the several orders of differences due to all the coefficients, may be simultaneously deduced from the given equation itself. The substance of that amendment is contained in the following problem.

PROB. VII. To deduce, &c.

1. Proceed according to Prob. I.; except, that in taking the sums of the columns, the number at the head of each column is to be omitted*. The several amounts will then be the 1st diffs. of the coefficients under which they respectively stand.

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* In the example, these numbers are separated from the rest by a line drawn below them.

2. Perform a similar operation, but adapted to an equation one degree lower, on this series of differences; and the results will be the 2nd differences of the coefficients, under which they stand; and so on. The work will terminate of itself when the set of differences is completed.

Example. Find the differences, due to the coefficients of $x^3 - 63x + 189 = 0$. [LAGRANGE. *Res. des Eq. N. p.* 16].

	1	0	− 63	+ 189
		1	1	− 62
		1	2	
		1		
1st diffs.		3	3	− 62
			3	6
			3	
2nd diffs....			6	6
3rd diffs....				6

Still, the subject appeared capable of further improvement. The separate courses of addition, for constructing the separate coefficients, were an encumbrance. And since the coefficients, taken in retrogression from the absolute term, are functions of the same class as the respective orders of differences belonging to that term, it was natural to infer, that the two sets of quantities were mutually transformable.

The result of pursuing this train of thought, is given in the two following propositions, which appear to me to simplify this point to the utmost degree of which it is capable.

PROB. VIII. The inferior coefficients in any equation being regarded as a set of partial differences belonging to the absolute term, to find other sets, to any extent, similarly connected with the succeeding absolute terms; supposing the roots of the given equation to be diminished by a succession of units.

1. Under the given set of coefficients, draw horizontal lines at convenient intervals. Over each coefficient, write the index of that power of the unknown quantity to which the coefficient appertains. Write the same indices, but in the natural order, 1, 2, 3, &c. against the successive horizontal spaces; the greatest index being understood before all the spaces that may still remain.

2. The first number to be entered in each interval is, the given first coefficient. The next and each succeeding term is to be found by multiplying the number last entered, by the smaller of the two indices, which stand horizontally and vertically in line with it, and adding the product to the number directly over the place you are going to fill.

3. This process, applied to the intervals in regular succession, will produce series having the required property: the last terms being the absolute terms in question with their proper signs.

Prob. IX. From the set of partial differences appertaining to any absolute term, to find the remaining coefficients of the equation to which that term belongs.

1. Draw horizontal lines as before; and against the successive intervals write the set of multiplier-indices which were employed in constructing the said set of differences, omitting the first of those indices.

2. In each interval enter, first, the given first coefficient; secondly, multiply the number so entered by the index which stands against it, and add the product to the number directly over the place you are going to fill; thirdly, having entered this amount, multiply it by the next inferior index, and add the product to the number directly over the next space to be filled; and so of the rest.

3. Applying these precepts in regular order to the filling of the intervals, the number of terms in the successive series produced, will be found to diminish continually by one. And the last terms in these series, taken in reverse order, will be the coefficients required.

Example. Shew the distinct initial values, or the impossibility, of the several roots of $x^5-8x^4+11x^3-39x^2+31x-101=0$.

By Prob. VIII.

	5	4	3	2	1	
	1	− 8	+ 11	− 39	+ 31	− 101
1	1	− 7	+ 4	− 35	− 4	− 105
2	1	− 5	− 6	− 47	− 98	− 203
3	1	− 2	− 12	− 83	− 264	− 467
4	1	+ 2	− 4	− 95	− 454	− 921
5	1	+ 7	+ 24	− 23	− 500	− 1421
	1	+ 12	+ 72	+ 193	− 114	− 1535
	1	+ 17	+ 140	+ 613	+ 1112	− 423

At the next step all the terms would be positive; consequently 8 is the superior limit of the roots. And it is manifestly the immediate limit of a single *real* root, whose distinct initial value is therefore 7.

The coefficients due to the further evolution of that root, are thus found by prob. IX.

	1	+ 17	+ 140	+ 613	+ 1112	− 423
4	1	+ 21	+ 203	+ 1019	+ 2131	
3	1	+ 24	+ 251	+ 1270		
2	1	+ 26	+ 277			
1	1	+ 27				

∴ 1 + 27 + 277 + 1270 + 2131 − 423 are the coefficients:

Again, as the absolute term has neither changed its sign nor diminished, until we approach the superior limit, it is probable that the other roots, if possible, are all less than 1; particularly as several alternations of sign appear to have been lost at the first step. Returning therefore to that step, we find by Prob. IX.

$$\begin{array}{ll}
 & 1 - 7 + 4 - 35 - 4 - 105 \\ \hline
1 & 1 - 6 - 2 - 37 - 41 \\ \hline
1 & 1 - 5 - 7 - 44 \\ \hline
1 & 1 - 4 - 11 \\ \hline
1 & 1 - 3 \\ \hline\hline
\end{array}$$

∴ coefficients $1 - 3 - 11 - 44 - 41 - 105$.

These coefficients exhibit only *one* change of sign, that is, *four less* than the preceding, or given, equation. The diminished reciprocal gives

$$\begin{array}{ll}
1) & 1 - 8 + 11 - 39 + 31 - 101 \\
 & - 106 - 98 - 109 - 70 - 101 \\ \hline
 & \&\text{c. all negative.}
\end{array}$$

Consequently, by the second criterion, the *four roots* in question are *imaginary*.

(B). Of *Partial Compendia* of this class, one only appears worthy of recollection, both on account of its facility, and of the frequent occurrence of cubic equations; to which it is adapted. It is intimated by Budan, and consists in the following observations: In a series of transformees of this order:

The *first* term is *constant*.

The *second* increases by a *constant increment*, equal to three times the first term.

The *third* is equal to the aggregate of the *former third* term, and of the *former and new second* terms.

The *fourth*, as usual, is equal to the aggregate of the *entire preceding set*.

Hence the steps in the initial solution of a cubic equation, may be mentally connected; *e.g.* Example 2. Prob. V. would be complete under this form,

$$\begin{array}{llll}
1 & 0 & - 7 & - 7 \\ \hline
1 & 3 & - 4 & - 13 \\ \hline
1 & 6 & 5 & - 13 \\ \hline
1 & 9 & 20 & - 1 \\ \hline\hline
\end{array}$$

By the same artifice, two lines may be spared from transformations of the higher orders.

(C). Other scales of notation would develope some particular roots, more aptly than the decimal scale; but the place for this artifice can seldom be anticipated. Some advantage, however, may occasionally be gained by introducing the *binary* principle, previous to the decimal, in the practice of the fourth and fifth precepts of Prob. VI. Thus, multiply by 1, 2, 4, 8, &c. instead of 1, 10, 100, &c. and transform the result by Prob. I: then, if no changes of sign are lost, we may infer that the first five transformations prescribed by precept 6 would not lead to a root of the equation; and the binary transformee may be at once changed into the *fifth* decimal one by multiplying its terms by 1, 5, 5^2, &c. *i. e.* by 1, $\frac{10}{2}$, $\frac{100}{4}$, $\frac{1000}{8}$, &c.

(D). My readers will have anticipated an observation, which should have been subjoined to Prob. I: If the last addend in any column be added to the last in the next, the amount will be the same as the sum of the former of the two columns.

ARTICLE V.

Letter from Mr. W. G. Horner.

Bath, 20 *January*, 1821.

To the Editor of the Mathematical Repository,

Dear Sir,

I am really sorry to occupy your valuable pages with a statement of particulars perhaps merely personal; but if you will indulge me with the opportunity of speaking a few words in self-defence, I promise to be as brief as possible.

Three months after my Method of Continuous Approximation had been read before the Royal Society, you wrote me that you had just received from Mr. Evans of Woolwich, a printed prospectus for publishing by subscription a new, &c. method of solving equations, &c. by Theophilus Holdred, &c. *This was the first time I heard of that Gentleman's name or of any discovery made by him.*

I heard nothing more of Mr. Holdred until last summer, when some person unknown to me, but apparently familiar with Mr. Nicholson, was so kind as to send me a copy of the latter Gentleman's Essay on Involution and Evolution, then recently published. Reference being therein made to a Treatise on Algebra by Mr. N. I procured that work also.

This Treatise on Algebra contained the *first public statement of Mr. Holdred's Method*; and it was published, as Mr. Nicholson himself states, subsequently to the reading of my Essay before the Royal Society. Supposing therefore, that the two methods (as has been insinuated) were essentially the same, the question of *priority* of date is clearly decided in my favour, With regard to superiority in the mode of operating (still supposing the fundamental identity of the methods); Mr. Nicholson, who has indisputably presented Mr. Holdred's mode in the most perfect form possible, and might be supposed to retain a partiality for the improvements he had himself introduced, distinctly says, "the advantage certainly remains with the nonfigurate" (my Theorem III); "and this will appear more decidedly the case as the dimension of the equation increases."

But to reverse the hypothesis: supposing that these gentlemen's methods had not been thus inferior in date and facility; still, if two methods of effecting a given purpose, one of which is partial and the other general and complete, be published nearly at the same time by two authors, who knew nothing at all of each others studies or performances, can there be any doubt with whom the claim of superiority rests? Now, it does not appear, from any thing advanced by Messrs. N. and H. that either of them had any view beyond the solution of a finite equation, whose roots are, to a certain extent, previously known*. On the contrary, in addition to this, my Essay conveys an easy and popular method of getting unambiguous limiting values, or, as I now prefer to call them, distinct initial values of all the real roots of a finite equation; and conducts the whole developement on a principle manifestly unrestricted, and applying as directly to irrational and transcendental as to finite expressions. Nor, in reverting, on the other hand to the minutiæ of calculation, ought it to be forgotten, that by my Compendious Operation, the subject of arithmetical facility was carried to its acme.

Perhaps, Sir, I might here safely close the argument: but, wishing to remove every, the slightest, shade of suspicion that has been attempted to be fixed on my claims, I request your forbearance a little longer.

Mr. Nicholson says, that about twelve months before the publication of his Essay, I purchased Budan's work at Mr. Dickson's shop.

The facts are these, some time after my Method was completely developed, and I was considering of the most effectual way of ushering it into notice, an expression in the introduction to BARLOW's *Mathematical Tables* caught my attention. It excited, in fact, an apprehension that Budan had anticipated me; and, in consequence, I wrote with much anxiety to Mr. Barlow, requesting the loan of his copy of the work. That gentleman had parted with it, and all my enquiries, for a time, were fruitless. At last I was fortunate enough to meet with it at Mr. Dickson's. This was in the first or second week of *July*, 1818.

The use I made of that author is exactly stated in my Essay, and was very slight. In fact, perceiving that he had entirely missed the principle of continuity, I candidly confess that I skimmed over the general matter of his Treatise too hastily to perceive the value of some of his remarks, and, in consequence gave, in a note on my Essay, an opinion, which, on subsequent perusal, I find reason to believe not perfectly just. A more accurate account will be given in the Horæ Arithmeticæ.

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* What Mr. N. has said, on the subject of limits, is directly in elucidation of an observation quoted from me.

The date and circumstances of this affair are a sufficient reply to Mr. Holdred's remarks in Phil. Mag. for November last, about an article connected with his method, which appeared in the number for October, 1818. My method was finished long before that date; I never saw, never till this day heard of that article; and, once for all, I AM NOT INDEBTED TO ANY PERSON THAT EVER LIVED, FOR A HINT BEYOND WHAT I HAVE FAIRLY AND BROADLY STATED IN THE PHIL. TRANSACTIONS.

Among the advantages secured to my little Tract by its appearance in that respectable record, that of very general perusal cannot be included. Having, besides, as little of the book-maker in my disposition as an author can well possess, I feel myself in a sense at the mercy of those whose names are more familiar to the mass of readers. At the same time I am conscious that to an invention avowedly presented to public use, the public have a right. For which reason, if Mr. Nicholson had restricted himself to a fair account of my discovery, or even annexed a more elementary developement of some useful points, than had accorded with my more enlarged plan, I should have been under obligation to him. But his frequent and strange mistakes and misrepresentations greatly alter the case.

I am obliged to you, Sir, for a sight of Mr. Holdred's late publication, which, under the notion of its being long since published, my bookseller had repeatedly ordered in vain; and am exceedingly surprised to see annexed to a work which makes its appearance so long after my method has been published and republished, a supplement detailing, as the latest discovery of the author, MY NONFIGURATE METHOD!!!

Mr. Nicholson blames Mr. Holdred for not acknowledging this circumstance, nor mentioning my name in his whole Treatise. Mr. Holdred replies by saying, "I should have made no mention of Mr. N. if I had not thought myself ill-used by him."*—The surest way to obtain justice of Mr. Holdred, it seems, is to use him ill!

But to proceed. If this Gentleman has read my original paper—as he ought to have done, before he pronounced so confidently that he had been "anticipated by no one"* he must be aware that my discovery includes *three* distinct theorems, each rising above the former in simplicity, but all connected by a common and original principle; viz. that the constant multiplier employed in completing each transformation is that portion of the root by which the *previous* reduction of the absolute term was effected. That which Messrs. H. and N. employ is due to the *subsequent* reduction. And for want of adverting to this essential distinction, they speak of my Figurate Method as if it were identical with theirs, or only differed from it in having the figurate numbers "run back one order." Whereas the methods themselves are totally distinct, and mine will be found higher up in the direct track of improvement, and considerably easier.

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* Phil. Mag. *Nov.* 1820.

In the first of my three theorems (art 9), the multiplicands are reduced *figurate functions* of the once-corrected coefficients.

In the second (art. 14) they are the *successive sums* of each coefficient and its series of corrective addends.

In the third (art. 16, &c.) they are the *addends* themselves.—In practice I have adhered to this least plan as being by far the most elegant;—demanding the least possible space, and the smallest multiplicands.

Mr. H. on the contrary, having taken a step backward from this station, and stumbled on my second theorem, styles it, under the notion of its being an invention of his own, "an improvement, beyond which, he believes, nature could not go!"

The real advantage attending it, is the facility with which it may be expressed in words, which renders it an almost necessary introduction to the third: otherwise it is in practice both roomy and laborious. The first paragraph of Prob. III. in the Horæ Arithmeticæ is a *bona-fide* interpretation of the method in question; and Mr. Holdred's practice, it will be seen, is a literal *pons asinorum* compliance with that precept.

Taken in one point of view, Mr. Holdred's supplement is a suffrage in my favour: it proves that he feels tacitly the conviction which Mr. Nicholson has avowed, that my method is "*decidedly*" superior to his; and the implied probability that *mine alone will be adopted* by mathematicians. All I ask of them in recompence for the facilities consigned to their use in the Nonfigurate Method is, to bear in mind that I alone am the author of it.

I am, Sir,

Your's, very respectfully,

W. G. HORNER.

P.S. On looking back for the causes of my hasty perusal and long neglect of Budan, I find that I was engaged in preparing for the press and publishing a most unmathematical effusion—a Poem; and that in addition to the ordinary classical avocations of my own seminary, I was visited for several months by an unusual number of private pupils. Need I add, that anticipating no rivalry, all anxiety on that head ceased, from the moment I ascertained that Budan had missed the mark which I had hit; and that more than three months intervened between my finished transcript leaving my possession and being acknowledged by the Royal Society.

ARTICLE VI.

HORÆ ARITHMETICÆ.

No. IV.

Extremely simple Theory of Transformation. Part First.

A clear and early apprehension of an easy mode of transforming a given equation to another having the roots diminished by a given quantity, is indispensable to the ready practice of any approximative, that is any numerical, solution whatever. Nor is it less necessary as a means of estimating the degree of perfection, of which those methods are severally capable, and their relative value as practical rules.

In fact, no other consideration is wanted to enforce its importance than this, that the defects of the old approximating rules are all to be traced to that laborious mode of transformation, which induced calculators to allow a general uncertainty in the result, rather than employ the only proper and natural check, which is obtained by transforming the complete set of coefficients.

For these reasons, although I have already given two distinct investigations of my mode of transforming, I hope to be pardoned for introducing a third, deduced from principles quite intelligible to the youngest algebraist*.

Discarding the intervention of figurate numbers, which are more to the purpose in a corollary, being merely incidental to the examination of *detached* coefficients, we have only to reflect:

First, That if any expression in terms of x, be divided n times successively, by $x-r$; the remainders, in the order they arise, being l, k, $h \ldots . c$, b; and a the quantity left undivided; then will this transformed expression in $x-r$, viz.

$$a(x-r)^n + b(x-r)^{n-1} \ldots . + h(x-r)^2 + k(x-r) + l$$

be exactly equivalent to the original expression in x. For, this is only saying in other terms, that if $\frac{L}{x-r} = K + \frac{l}{x-r}$, $\frac{K}{x-r} = H + \frac{k}{x-r}$ ..... $\frac{C}{x-r} = B + \frac{c}{x-r}$, and $\frac{B}{x-r} = a + \frac{b}{x-r}$, then is $L = K(x-r) + l = H(x-r)^2 + k(x-r) + l =$ .... $=$ finally to $a(x-r)^n$ + &c. as above. *Q. E. D.*

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* Though Columbus's feat with the egg *(absit invidia!)* was not more directly in every body's track than the propositions that follow, they have never, I believe, been taken for *practical* principles by any one before the present writer. Setting aside the use made of them in this Essay to establish conclusions which are not *now* new; the first fruit of their application as instruments of investigation is another original and very easy method of transformation; a portion of which was given in my last number, without the investigation, for the sake of exciting curiosity. The rest must be reserved for a separate paper.

Secondly. That in dividing by $x - r$ in the usual manner, the coefficients in the quotient form themselves from those in the dividend thus: the 1st in the quotient is = the 1st in the dividend; the 2nd is = the 2nd + r times the 1st; the 3rd = the 3rd + r times this sum, &c. Or, in other words, the mth in the quotient is = the mth in the dividend + the $(m-1)$th in the quotient. As is evident from observation.

Thirdly. That the quotient contains one term fewer than the dividend.

Now in the practice of the New Method, the coefficients are all we have occasion for; and these, agreeably to the indication of the foregoing principles, will be found by this

ELEMENTARY RULE. 1. To the first coefficient add the correction 0; and to each succeeding coefficient add r times the amount of the preceding coefficient and its correction. Proceed thus with all the coefficients in order, until the absolute term is corrected.

2. Apply a similar process to these results, and then to those which are deduced from them; and so on, as often as it can be done; stopping each process one term short of the preceding. The last term in each row of results will be one of the new coefficients, standing vertically under the given one for which it is to be substituted.

This rule differs in enunciation only from the text of Prob. III. in these Essays; and, like it, is a strict verbal interpretation of the Theorem in Art. 14, of my original Essay, (Phil. Trans. for 1819). As in the passages referred to, we pass on to a more commodious and elegant mode of operating, by observing, that as the addends to the dividend-coefficients (see the *second principle above*) are constant multiples of preceding quotient coefficients, the addends flow from each other in the same manner as the quotient-coefficients, and independently of them after the first process. Hence this

IMPROVED RULE. 1. Perform the first process, or that by which the absolute term is corrected, as directed in the former rule.

2. Apply a similar process to the series of *corrections*, and then to those which are deduced from them; and so on, as often as it can be done; stopping each process one term short of the preceding.

3. Add up each column of additional corrections, together with the once-corrected coefficient under which it stands; and the sum will be the new coefficients, standing vertically under those given ones for which they are to be substituted.

Very moderate expertness in arithmetic will enable the analyst to form and add the *principal* corrections (in the Elementary Rule) or the *secondary* ones (in the Improved Rule) under one mental operation. The latter rule in this form, has been the most frequently employed of any of the new methods; nor am I aware that more than one person has objected to it. But how weak any such objections must be, even in reference to the only incident which seems to sanction the adoption of the elementary rule, *viz.* a frequent intermixture of + and — corrections in the columns, will be manifest from the following considerations:

1st. The principal corrections to the 2nd coefficient are all equal, and amount, in one sum, to $(n - 1)ar$: Those to the 3rd increase by a constant difference, ar^2; as do the secondary corrections to the 4th, *viz.* by ar^3: So that the corrections, for equations under the fifth order, may be said to be formed *by addition only*.

2nd. As to the rest, not to mention the obvious resource of drawing the pen across each secondary correction as soon as it is written, to prevent its being included in the sum of the columns; and the various ways in which the aggregate of a set of + and – terms may be taken without writing the arithmetical equivalents*, the calculator has the choice of at least two very elegant artifices for avoiding every thing like complex mental operations: viz.

(A). Retain the secondary corrections: Omit the last principal correction in each set. Find each new coefficient by adding up *all* the numbers which stand in column *below the old one*. Or,

(B). Perform precepts 1 and 2 of the Improved Rule, either with mental or written secondary corrections: To the last principal correction in each column add $\frac{1}{r}$ of the last in the next superior set: The sum will be the new coefficient belonging to the former of the two columns.

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* The transition from Gunter's well-known arithmetical complements to these equivalents is so slight, that more than one person possibly has made it; but *Francœur*, I believe, was the first, and he has elucidated the subject most amply. Leslie also has touched upon it with his usual ingenuity in the Philosophy of Arithmetic.

The practice (A), applied to the equations of the inferior dimensions, is particularly neat: I shall adhere to it, in regard to quadratics and cubics at least, in the remainder of these papers. The other (B) is a useful check upon the results of precept 3 of the Improved Rule.

That this sketch may serve, as completely as possible, the purpose intimated in the outset, of promoting the perfection of approximation in general, it only remains for me to remind the the reader that surds and transcendentals are readily convertible into the ordinary form of a finite equation, by means of as many terms of Taylor's Theorem as are requisite for the degree of accuracy intended. The process, for surds at least, can be made plain even to tyroes, by an intelligent teacher; and is easier for that effect than expanding by the Binomial Theorem.

March, 1827. Perspicuity, as well as the reader's advantage, may be promoted by a brief explanation of the mode of DIVISION adopted in these transformations; which differs materially in point of arrangement from that which is familiar to algebraists, and is intrinsically recommendable for its compactness, adaptation to continuity, and ready introduction to recurring series, &c. The principle is simply this; that Division has the same affinity to Multiplication, which Subtraction has to Addition. Every one sees this in theory; adopt the idea therefore in practice, and the two rules may be thus symbolized:

Multiplication.		*Division.*	
	Q by $(1-x)$		P by $(1-x)$
To	Q	To	P
Add	$-Qx$	Add	$+Qx$
Product	P	Quotient	Q

The *change of signs* in the terms characterized by x, and the *gradual developement* of those expressed by Q, are all that distinguishes the latter rule from the former. An example will make the practice clear. The following will be interesting for the intimation it conveys of the familiar light in which a trivial practical improvement may place an old speculative fact.

Divide $3x^3 - bx$ *by* $x^3 - bx - c$, *or* $3 - bx^{-2}$ *by* $1 - bx^{-2} - cx^{-3}$.

$$\text{Here}\quad P = {}^{(1)}3 + {}^{(1)}0x^{-1} - {}^{(1)}bx^{-2}$$

$$\begin{array}{l}
+bx^{-2}Q = \ldots\ldots\ldots\ldots\ldots\ldots {}^{(3)}3bx^{-2} + {}^{(3)}0x^{-3} + {}^{(5)}2b^2x^{-4} + {}^{(5)}3bcx^{-5} + \&c.\\
+cx^{-3}Q = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots {}^{(3)}3cx^{-3} + {}^{(5)}0x^{-4} + {}^{(5)}2bcx^{-5} + {}^{(5)}3c^2x^{-6} + \&c.\\
\hline
Q = {}^{(2)}3 + {}^{(2)}0x^{-1} + {}^{(4)}2bx^{-2} + {}^{(4)}3cx^{-3} + {}^{(6)}2b^2x^{-4} + {}^{(6)}5bcx^{-5} + \&c.
\end{array}$$

The bracketted numerals shew the order of developement of the work: the calculator has no occasion for them. In fact, since convenience requires that the set of coefficients should be completed at the outset by inserting zeroes where requisite, he may dispense with the use of x and its powers during the operation; restoring them, if requisite, in the final result, but *operating with the coefficients alone.* This remark, slight as its pretensions appear, considerably extends the boundaries of common arithmetic

This may serve as a general elucidation. In our transformations, the divisor being always binomial, the process assumes its most simple form; of which this is the type.

Divide $A_0x^n + B_0x^{n-1} + \ldots\ldots L_0x^2 + M_0x + N_0$ *by* $x - r$, *or* $A_0x^{n-1} + B_0x^{n-2} + \&c.$ *by* $1 - r^{-1}$.

$$\begin{array}{lccccccc}
\text{To} & A_0 & B_0 & C_0 & \ldots\ldots\ldots & L_0 & M_0 & N_0\\
\text{Add} & 0 & A_1r & B_1r & \ldots\ldots\ldots & K_1r & L_1r & M_1 \;\&c.\\
\hline
 & A_1 & B_1 & C_1 & \ldots\ldots\ldots & L_1 & M_1 & N_1 \;\&c.
\end{array}$$

Whence the quotient is $A_1x^{n-1} + B_1x^{n-2} + \ldots\ldots\ldots L_1x + M_1 + \dfrac{N_1}{x - r}$,

Progress of Solution.—Three Stages.—Rules for the Second.

Since the progressive solution of an equation consists in the alternate operations of determining a certain portion of the root and preparing by proper transformation to determine a further portion, it is worthy remark how happily the principle of the foregoing sketch harmonizes with, and completes, the idea thrown out in Art. 29, of my Essay, respecting the continuous evolution of the root: where the term Evolution is employed in its familiar arithmetical acceptation, and is defined to be *Division by a variable divisor.* Such, it now appears, is, in the most strict and simple algebraic point of view, the character of the preparatory and connecting class of operations, as well as the inventive class.

With regard to the latter, the fact is manifest, and elucidates the practical part of our enquiry as much as the theoretical in the former. For, if we turn to the first precept in the Rules above, it will appear that if the quantity by which we are proceeding to diminish the root is a root, it will, when its sign is changed, be accurately equal to the quotient of the absolute term by the preceding coefficient once corrected; and that if it is an initial approximation to the root, it will be identical with the first digits of that quotient. This test which it is sometimes necessary to apply, may properly be termed trying or *proving the approximation.* And on account of their subserviency to this use, we may have occasion to designate the absolute term with its sign changed, by the term *dividend* or *resolvend,* and the once corrected previous coefficient by that of *proof divisor.*

Again, the youngest student will readily perceive that when one or two of the first digits of the root have been truly determined, by whatsoever means, the subsequent corrections, partaking of the progressive diminution in decimal value of the root figures, of which they are nearly constant multiples, will become so far reduced as not to affect the first digit or digits of the divisor. When this happens, it is evident that the penultimate coefficient may be employed with certainty as a *search-divisor* to get the approximation true to an ascertainable extent; and that, from this point, not only will the same advantage attach to all the subsequent transformees, but *the proof-divisor of one may be made the search-divisor of the next,* and therefore the whole of the remaining work may be done by continuous evolution.

But between this final stage of progress and the initial one already explained, and in which the direct use of division is precluded by the ambiguity of the conditions, there occurs a second stage, when we perceive that one quotient, and only one, is contained within the limits of a known place in the decimal scale, but the penultimate coefficient is not an accurate search-divisor.

In this case, the cautious advance by units, which could seldom be dispensed with during the first stage, may in general be at least partially superseded by more decided expedients which practice will suggest, such as the following.

[A] Employ the last coefficient but one, or the sum of the absolute term and the greatest term of a contrary sign, or the aggregate of all the terms except the absolute one, as a search-divisor: Take the quotient as far as one digit, for an approximation; and if the process of proving it leads to a different quotient, either this last will be the correct approximation, or it will be found between the two.

This rule applies almost exclusively to fractional approximations; the following, principally to integral ones, and more especially when the root in question is the greatest. It includes the case of pure powers.

[B] When some *one coefficient* is so *large* that the correction traced from it will constitute the chief portion of the proof-divisor. Employ it as a search-divisor: Take that root of the quotient whose index is the distance of the divisor from the absolute term, for an approximation; and thereby correct the same or a more advanced coefficient for a proof-divisor: Either the proper root, estimated as above, of the new quotient will be the correct approximation, or it lies between the two.

N. B. When the first coefficient predominates, the degree of reliance due to it as a divisor is matter of intuition rather than trial.

[C]. When we want the integral part of the affirmative root of an equation having but one change of signs and a relatively small absolute term, as in the final stage of the solution by continued fractions, the common doctrine of the superior limit will lead us to operate with two predominant terms having contrary signs, in a manner similar to that last mentioned[*].

Bath, June 11, 1821. W. G. HORNER.

Correction. For paragraph (B) in No. III, read:

(B) The following *Partial Compendia* appear worthy of recollection:

The *first* term is *constant.* The *second* increases by a *constant increment* = the first term × its index. The *last* is equal to the aggregate of the *entire preceding set.* In cubics, the *third* is equal to the aggregate of the *former third term,* and of the *former and new second terms.*

¶ Hence, agreeable to a suggestion of Budan's, the steps, &c.

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[*] As far as continued fractions are concerned, this tentating method is superseded by an exact one, which I have investigated in a late number of the Quarterly Journal.—*March,* 1827.

No. V.

Third Stage of Solution.

Having copiously explained this portion of my discovery in another place, there is the less necessity for me to repeat the detail. A portion of its precepts has indeed been necessarily anticipated in treating of the previous stages, and most of those which remain will be implied in the contents of the present paper; in which, agreeably to an idea thrown out in the last, I propose to shew by rules and examples the efficacy of the new principles in *improving old methods* of solution. The improvements may be classed under the heads of *entireness*, in regard that all the coefficients co-operate; *conciseness;*—*completeness*, in regard of the transformations, which require no bye-work; *continuity*, since every transformation takes up the work where the preceding one left it;—unprecedented *facility;*—and almost intuitive *certainty*.

Sketches will also be given of two modern methods, which have a kind of historical connexion with my own, and, as far as equations in their final stage are concerned, participate with it the characteristics of entireness and continuity.

1. *The Ordinary* (Newton's or Raphson's) *Approximation, improved.*

The principle of this method is, that *the last coefficient but one is an adequate search-divisor.* Consequently, (1). If the proposed equation does not accord with this description, it must be reduced to such a form, by applying the precepts of one or both the foregoing stages. By this means, at the same time, a number nearly equal to the root will be found; and that by an easier process than the *trials* prescribed by the old rules.

(2). To ascertain the extent to which the next approximation may be confidently carried, consider what single digit is the nearest in value to the quotient of the last term (with its sign changed) by the search-divisor; and by means of this digit-approximation and as many of the coefficients as are requisite for the trial, by a process resembling the first under the Elementary Rule in No. IV. determine the first digits of the *proof-divisor.*

(3). After this scrutiny, which in most cases may be applied mentally, take the aforesaid quotient, to a number of digits equal to those in which the search and proof divisors agree;* and it will, to that extent, be a root of the reduced equation.

(4). Diminish the roots of the said equation by this approximate root, and to the resulting equation apply the two last recited precepts; and so on, until the root has been approximated to the degree of accuracy desired.

EXAMPLE. Given $x^3 + x^2 + x - 90 = 0$, to find x. (Bonnycastle, Vol. I. p. 172.)

Commencing as at the 1st Stage.

$$\begin{array}{llll}
1+1 & +1 & -90 & (1 \\ \hline
1+4 & +6 & -87 & (1 \\ \hline
1+7 & +17 & -76 & (1 \\ \hline
1+10 & +34 & -57 & (1 \\ \hline
1+13 & +57 & -6 & \left(\frac{6}{58} = .10\right. \\
 & .1 \quad 1.31 & & 5.831 \\ \hline
 & 13.1 \quad 58.31 & & \\
 & .1 \quad .01 & & \\ \hline
\end{array}$$

$$1 + 13.3 + 59.63 - 0.169 \left(\frac{.169}{59.66} = .002832 \right. \therefore \text{The root is } 4.102832.$$

3

Ditto as at the 2nd Stage.

$$\begin{array}{llll}
1 + & 1 + & 1 - 90 & (4 < \sqrt[3]{90} \\
 & 4 & 20 \quad 84 & \\ \hline
 & 5 & 21 & \\
 & 4 & 16 & \\ \hline
1 + & 13 + & 57 - 6 & \\
\end{array}$$

&c. as before.

Remark. 1. In these operations, the number to the right of the curve is the approximation by which the roots are next to be diminished. 2. The diminution by 1 is effected by compendium (B) of No. III.; and the others, by that marked (A) in the former part of No. IV. 3. The effect of precept 2 is seen in the figure under the last set of coefficients in the work: at the preceding step it was applied by inspection. 4. Not a figure is wanting to complete the work, except the division of 6 by 57 or 58, and of 169 by 59.63 or 59.66.

2. *The Method of Double Position, improved.*

The principle of this method, as appears from the equivalent formula in my paper at the conclusion of Vol. IV. of this Repository, is, that *the aggregate of all the coefficients save the last is an adequate search-divisor.* This being understood, the rule may be said to be the same as in the preceding article. But ob-

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[*] Or, divide the last term by the *proof-divisor*, to one digit more than the two divisors agree.

serve, that, *previously* to taking the aggregate, the decimal point in each coefficient should be removed as many periods* to the right, as can be done without causing the last term to become more than twice or thrice as great as the last but one; and *after* taking the aggregate, the points should be restored to their places, and the aggregate pointed as the last term but one. The reason for this procedure is easily understood.

Example. Supposing that the 5th line in the foregoing example weie proposed, viz.

$1 + \quad 13 + 57 - 6.$ The work might stand thus

$58.31,)\ 1 + 13.0, + 57.00, - 6.000, (\ .1029 -$

In this example a closer approximation is made than by the preceding method; but this will seldom be the case unless the quotient is little more than .1 or < 1.

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3. *Halley's and other Methods, improved*†.

(A). The principle of Halley's or Delagny's rational method, is, that if the penultimate coefficient be of any value as a search-divisor, it will be improved by a correction deduced from the preceding term. For the sake of simplification the division is most conveniently expressed in a fractional formula; the denominator, or search divisor, being the sum of the last coefficient but one and a fraction whose denominator is the said coefficient, and its numerator the product of the last and last but two. The quotient may be taken to *twice* the extent indicated in the first article: in other respects the rule is the same.

Example. In the case assumed in the preceding article, the approximation is expressed by $\dfrac{6}{57 + \dfrac{6 \times 13}{57}} = \dfrac{2}{19 + \frac{26}{57}} = \dfrac{2}{19\frac{5}{11}}$ very nearly $= \dfrac{22}{214} = \dfrac{11}{107} = .1028 +$

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* A period consists of 1 digit in the 2nd coefficient, 2 in the 3rd, &c. agreeably to the note under No. II.

† This 3rd class belongs to the formula $p + \dfrac{1}{a + \dfrac{n-1}{p + F}}$ which occurs incidentally in a Paper of mine in No. 42, of the Journal of Science; by making $p = 4$, $a = \dfrac{57}{6}$, $n = 3$, and $F = \dfrac{2 \times 57}{13} - 4$, or $\dfrac{2 \times 13}{3} - 4$ respectively. More accurate values of F would of course give still more accurate approximations, if it were worth while to seek them. *May*, 1827.

Remark. In ordinary cases, as Newton's approximation exceeds the truth, so Halley's will fall short of it, because the fractional addend in Halley's divisor is a multiple of the Newtonian approximation. For this reason, the principal *minor* fraction converging to the value of that addend is generally a preferable addend; as in the example above, where $19\frac{5}{11}$ is not only a more convenient, but a more correct divisor than $19\frac{26}{57}$; and the approximation is in consequence very close. I am not aware that this remark has been made by any other person.

(B). In solving the equation of a pure power, Halley's or Hutton's correction of the search-divisor would be the fraction whose numerator is the product of the last term by the *index* of the second, and its denominator twice the root already determined.

Mr. Barlow has extended this method to the solution of all equations in the third stage, which are transformees of a given equation whose root is > 1.

Example. *The same as before.* Here, because $1 + 13 + 57 - 6$ is derived from a previous equation by diminishing the roots by 4, which therefore is the whole root hitherto determined, we have for a further approximation $\dfrac{6}{57 + \frac{2 \times 6}{2 \times 4}} = \dfrac{2}{19\frac{1}{2}} = \dfrac{4}{39} =$ $.1026 +$, which is nearer than Newton's.

(C). The connection of our transformees suggests another mode of extending the same principle. Instead of supposing the equation in the third stage to be derived from the equation arbitrarily given, let us assume it to be derived from one, in which the second term was 0, as this is the case which makes the nearest approach to the condition of an equation of pure power. Now the quantity by which the roots of the assumed original equation have been diminished is always $\frac{1}{n}$ of the 2nd term of the actual equation. Consequently,

On this hypothesis, the correction of the search-divisor will be the fraction, whose numerator is $\frac{1}{2}(n \times n - 1)$ times the last term, and its denominator the 2nd term.

Example. *As before.* The approximation is $\dfrac{6}{57 + \frac{3 \times 6}{13}} =$ $\dfrac{2}{19\frac{6}{13}} = \dfrac{26}{253} = .1028 -$.

The variation from Newton's or Halley's method, which would be required in case of the absence of the penultimate coefficient, must be sufficiently obvious from the observations made regarding the second stage of progress. The calculator will appreciate the clear and connected view of his subject, which the entireness of our transformations furnishes him with, and by which at every step he is enabled to select that mode of approximation which most happily unites ease with certainty.

4. *Budan's (supplementary) Method.*

Besides his *Méthode Génèrale*, which, for the investigation of roots exceeding two or three decimals*, he significantly consigns over to the day labourers (*manouvriers*) of science, as a tool which they may handle *mechanically*, this author, in a Note (V.) on his 6th Chapter, indulges clever mathematicians with the use of an instrument which is somewhat less cumbersome.

His principle is that of the ordinary inferior limit; or, according to our mode of expression, his search-divisor is the sum of the last term and the greatest coefficient of a contrary sign.

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* Of the utility of this method as an instrument in aid of *initial* approximation it cannot be necessary for me to repeat my opinion. I commended it, even under the erroneous impression that Budan's transformations did not exclude the use of figurative factors; an error to which, I must say, a want of clearness in the introductory portion of his work, and contracted operations throughout the remainder, greatly contributed. And from the time when I discovered the proximity of our principles of initial solution (which was not until I sat down to commence this course of Essays) my sentiments have been sufficiently marked by the use I have made of his labours, accompanied *in every instance* with due acknowledgement.

Some of my readers may possibly require to be reminded that this author's *General* Method (so called) is substantially, both in principle and practice, a *a particular case of mine:* a fact, which fully explains how it could happen that I passed from my method to his, without being immediately aware of the coincidence (I refer, of course, to *initial* approximation only); while, on the other hand, his principles never led him on to the general ground which I assumed for my basis; nor, probably, would they have led me to it, had I been acquainted with them before the developement of my own, which, however, I most assuredly was not.

Nothing is more usual than unconscious anticipation of portions of a general theory. How long has Square Root been known, and yet none reflected that the same principle extended to Quadratic Equations, and opened the way to the solution of all equations. And again, the very method of extracting the Cube Root which my theory inculcates, is to be found in Ingram's edition of Melrose's Arithmetic, a little work well known, no doubt, to gentlemen in the North, though I never happened to meet with it till this year; yet to whom did that rule ever suggest the solution of a cubic equation? Most probably, not even to its author.

Now this principle, even if united with my mode of transformation would give slow approximations, compared even with the Newtonian; for example, in the problem which has been solved in this paper, the first decimal approximation would be .0952 or .10—, and the second .002826 or .00283—. But, strange as it may appear, there is nothing in Budan's work, that I am aware of after habitual perusal, which indicates so much as the possibility of diminishing the roots of an equation by more than *a unit a time*. This is the well known basis of his Méthode Générale; and the comparative rapidity of the supplementary operation is accomplished by the (certainly very dexterous) manner, in which he contrives to throw his approximations into the form of a series of connected reciprocals, like $x + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd}$, &c. by which artifice the diminution of roots is reduced to a single unit for each term of the series, and the denominators are somewhat more manageable than if unconnected. But, after all, the reductions and counter-reductions which they require, render the operation sufficiently intricate, as will be understood from the following epitome of the contents of four pages of the aforesaid note.

The EXAMPLE is taken from Newton, $x^3 - 2x = 5$.

$$\begin{array}{lrrrrl}
 & 1 & 0 & -2 & -5 & (1 + 1 = 2.\text{ first term.} \\
\hline
 & 1 & 3 & 1 & -6 & (1 \\
\hline
 & 1 & 6 & 10 & -1 & \left(\frac{1}{10+1} = \frac{1}{11},\text{ 2nd term.}\right. \\
\text{Multiply by} & 1 & 11 & 11^2 & 11^3 & \\
\hline
 & 1 & 66 & 1210 & -1331 & (1 \\
\hline
 & 1 & 69 & 1345 & -54 & \\
\text{Multiply by} & 11^3 & 11^2 & 11 & 1 & \\
\hline
 & 1331 & 8349 & 14795 & -54 &
\end{array}$$

Hence $\frac{54}{14795+54} = \frac{1}{275+} = \frac{1}{11.25}$ nearly, is the 3rd term.

$$\begin{array}{lrrrrl}
\text{Multiply} & 1 & 69 & 1345 & -54 & \\
\text{By} & 1 & 25 & 25^2 & 25^3 & \\
\hline
 & 1 & 1725 & 840625 & -843750 & (1 \\
\hline
\text{Multiply} & 1 & 1728 & 844078 & -1399 & \\
\text{By} & 275^3 & 275^2 & 275 & 1 & \\
\hline
 & 20796875 + & 13763750 + & 232121450 & -1399 &
\end{array}$$

Hence

$$\frac{1399}{1399+232121450}=\frac{1}{165920+}=\text{say to }\frac{1}{165925}=\frac{1}{25.6637}$$

is the 4th term.

Whence the root is $2+\frac{1}{11}+\frac{1}{11.25}+\frac{1}{25.6637}$ &c. $=$ 2.094551481364. The last three decimals are incorrect. The accuracy of the remaining nine our author verifies by means of another process characterized by the use of the *superior* limit.

In my original Essay, by the process described in the first article of this paper, the same root was found correct to 18 decimals, at the same number of steps as Budan employs to obtain these 9.

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5. *Holdred's Method, as arranged by Nicholson; with an improvement.*

Investigated in the following manner, which is more simple than any I have seen adopted for the purpose, this method furnishes a remarkable proof, how long a very plain deduction from very familiar premises may remain undiscovered. It is, in fact, nothing else than the common formula of transformation separated into horizontal and vertical processes.

Take, for example, Halley's Speculum Analyticum, in which $a+e$ represents two successive portions of z, the root of the given (or transformed) equation; whence

$$\begin{array}{lcccccc}
cz = & \cdots & \cdots & \cdots & \cdots & ce & +\,ca \\
dz^2 = & \cdots & \cdots & \cdots & de^2 & +\,2dae & +\,da^2 \\
fz^3 = & \cdots & \cdots & fe^3 & +\,3fae^2 & +\,3fa^2e & +\,fa^3 \\
gz^4 = & \cdots & ge^4 & +\,4gae^3 & +\,6ga^2e^2 & +\,4ga^3e & +\,ga^4 \\
hz^5 = & he^5 & +\,5hae^4 & +\,10ha^2e^3 & +\,10ha^3e^2 & +\,5ha^4e & +\,ha^5 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
\therefore \mathrm{N} = & \mathrm{H}e^5 & +\,\mathrm{G}e^4 & +\,\mathrm{F}e^3 & +\,\mathrm{D}e^2 & +\,\mathrm{C}e & +\,\mathrm{S}a
\end{array}$$

Each term in this line is the sum of the column above it.

¶ The first object, in practice, is to substitute a for z in the equation, or, in other words, to find $\mathrm{S}a$. Now if we discard all those quantities which do not conduce to this effect, and at the same time prepare the way for finding C, D, &c. which are also functions of a, the remaining portion of the table will

h	g ha	f ga ha^2	d fa ga ha^3	c da fa^2 ga^3 ha^4		No. 1.
				s	sa	
H	G He	F Ge He^2	D Fe Ge^2 He^3	C De Fe^2 Ge^3 He^4		No. 2.
n ma	p na	q pa	r qa	s ra		No. 3.
H	G &c.	F &c.	D &c.	C &c.		

stand in this form (see No. 1. in the margin); coeffs. c, d, &c. being first put down and the other quantities formed from them by continual multiplication by a.

If now the figurate factors be applied to the several terms in this table, as they stand in the complete one, the sums of the columns will be $(c+2da+3fa^2+4ga^3+5ha^4=)$C, $(d+3fa+6ga^2+10ha^3=)$ D, &c. And these being the new coefficients for the expression in e, may be placed as the former coeffs. c, d, &c. were, in order that the terms Ce, De^2, &c. may be formed from them by continual multiplication by e (see No. 2. in the margin). By this means, continuity is maintained; for e, being introduced in the same manner as a, prepares in the same manner for the introduction of the next portion of the root.

Remark. Figurate factors of the next lower order may be used, and the sums, viz. $(d+2fa+3ga^2+4ha^3=)r$, $(f+3ga+6ha^2=)q$ &c. be placed in line with s. The work would then proceed as in No. 3. in the Margin; C being $= s + ra$ and so of the rest.

In this remark I may possibly have been anticipated: I regard it as the highest improvement which Mr. Holdred's method admits of.

Bath, December 29, 1821. W. G. HORNER.

No. VI.

THE SAME SUBJECT, CONCLUDED.

Final Improvement in the Third Stage of Solution.

What remains to be adduced on the subject of equations in their third stage, may be arranged under the following particulars, which may be regarded as additional improvements on Newton's Method*.

1. The quotient is only taken as far as one digit

2. Since the first digits of all the divisors throughout the solution (at least after one root-digit, generally speaking, has been found) are constant, those of any one divisor are sufficient for seeking any succeeding root figure.

3. Hence the pure coefficients of the transformees need not be found, but only the once-corrected coefficients. This is done by placing the first correction obtained from the *new* root-figure under the last obtained from the *preceding*; but as many *periods* in advance toward the right, as are equal to the distance in digits, by which the latter root-figure stands in advance of the former; and then aggregating the column.

4. When the resolvend has extended to about one digit more than the remainder of the root is desired to contain, the contracted operation may commence. This consists in cutting off, from the previous first-corrections and all quantities beneath them, as many digits as are equal to the differences of the respective periods and the index of the equation, for every digit by which the root advances. The quantities so cut off are ne-

---

* The mode of illustration selected in this and the preceding paper, has struck me from an early date, as a plain and popular introduction to the *practice* of my algorithm. But, I really cannot conjecture, in what respects "the plain and sober methods of Newton and Raphson" could contribute (as has been surmised in a quarter for which I feel very high respect—*M. Rev. Dec.* 1820) any thing material to my *theory;* since they do not furnish the slightest indication towards entireness, or facility, or continuity, of transformation, which are the grand characteristics of the new method. The mode of developement adopted in my Essay was that which actually led me to the results there stated; and I trust it is not presumptuous to affirm that the public, far from having seen any more clear and simple developement of the principle there assumed, has seen *no other* developement which led to the same results. Indeed I am not apprized of any attempt of the kind having been made. For, after all, it ought not to be forgotten, that although my theorems have been applied hitherto in one particular direction only, viz. their bearing on numerical solution, they are in fact as universal as analysis itself.

glected, except so far as they are of use in adjusting the last digits of the products concerned in the operation.

EXAMPLE. *The same as in* No. V.

N. B. The mode of transformation here used, is the same variety (A) of our general method which was employed in the last No. *E.g.* Under 131, 13302 &c. stands one principal correction, the *same* as stands above the black line; and under 5831, 59656604 &c. stands one secondary correction, equal to the *square* of the former. And in summing up the columns, the number next above the black line is included. Such is the constant mode of procedure in Cubics, according to that variety.

```
  13.       57...
   .1       181
 --------------
  131       5831              — 6...   (.1028323542
    1..        1....             5831
    |02     26604              -------
 ------------------           — 169
  138|02   5965660|4 ........ 119313208
     |2.          |4          ---------
|.1|33|07...   106|4|56       — 49686792
           596938|5|76....... 47755086
                |3|99         ---------
         -------------        — 1931706
         59704|9|02 ......... 1791147
                 3            --------
        --------------        — 140559
        5|9|7|0|5|32 ........ 119411
                              -------
                              — 21148
                                17912
                              -------
                               — 3236
                                 2985
                                -----
                                — 251
                                  239
                                 ----
                                 — 12
                                   12
                                   --
```

Depression of Equations.

From the Theory of Transformation given in No. IV. it directly follows, that BY THE VERY PROCESS OF SOLVING an equation according to our method, it actually is DEPRESSED ONE DEGREE. For, since every set of corrected coefficients obtained in diminishing the roots by r, was shewn to be the quotient of the preceding line by $x - r$, we infer:

1. That if r be an exact root of the equation, the last of the once-corrected coefficients will be zero, and the rest of them will constitute a formula one degree lower, and containing the remaining roots.

2. That if r be an approximate root, the aforesaid last coefficient will have undergone a proportionate diminution, and the remainder constitute a formula depressed one degree, and from which an equal approximation toward the other roots may be obtained.

And since, in completing the transformations, the formulæ in question are operated upon precisely as they would be, if they had been equations of an inferior order, whose roots we were diminishing by r, it is obvious

3. That if in any transformee, the last term, either totally or to a certain extent, vanishes, the remaining terms may be regarded, and operated upon, as a complete transformee, containing the remaining exact or proportionably approximate, roots diminished by r.

Cor. The last mentioned remarkable and useful property is common to all methods of solution by *entire* transformation, and consequently to all those explained in No. V.; a fact which I believe has never been noticed before.

The utility of these remarks will be appreciated from the following examples, selected from various classes of equations.

Example 1. The complete solution of an equation, all whose roots, or all but two, are integral, such as that of $x^4 - 55x^2 - 30x + 504 = 0$, becomes, even without the aid of Newton's Divisors, an operation of the utmost facility.

$$
\begin{array}{llllll}
1 & 0 & -\,55 & -\,30 & +\,504 & (1 \\
 & 1 & 1 & -\,54 & -\,84 & \\
\hline
1 & 4 & -\,49 & -\,136 & +\,420 & (1 \\
 & 1 & 5 & -\,44 & -\,180 & \\
\hline
1 & 8 & -\,31 & -\,218 & +\,240 & (1 \\
 & 1 & +\,9 & -\,22 & -\,240 & \\
\hline
1 & 12 & -\,1 & -\,252 & +\,0 & (\sqrt{\tfrac{252}{12}} = \sqrt{21} > 4 \\
 & 4 & 64 & 252 & & \\
\cline{2-3}
 & 16 & 63 & & & \\
 & 4 & 16 & & & \\
\hline
1 & 24 & 143 & & 0 & (-\,12 \\
 & -\,12 & -\,144 & & & \\
\cline{2-2}
 & 12 & & & & \\
\hline
1 & 0 & -\,1 & (\sqrt{1} = \pm 1 & & \\
\hline
\end{array}
$$

Hence the roots are $(1+1+1=)\ 3$; $(3+4=)\ 7$; $(7-12\pm 1=)\ -4\ \&\ -6$.

N. B. The biquadratic has been reduced by compendium (B) of the Initial Solution. It might, however, have been reduced as the succeeding cubic is, by case (B) of the second stage; for the root must obviously be nearly $= \sqrt{\frac{504}{55}} = \sqrt{9} + > 3$, and the once-corrected coefficients would have evinced that 3 was a root; and the transformation, when completed according to the remaining directions of Prob. III., would have conducted to the same cubic formula $1 + 12 - 1 - 252$ which now appears in the work. Or, without completing the transformations; since we know from the first remark under this Article, that the formula $1 + 3 - 46 - 168$ of once-corrected coefficients contains the remaining roots; and since the inferior limit of those roots manifestly lies near $\sqrt{46}$ or 7, which, on applying the usual trial of first-corrections, would appear to be actually a root; the entire solution might be of the very concise form annexed. The first mode of solution is subject to no embarrassment, but expert calculators will certainly prefer the other.

$$
\begin{array}{llllll}
1 & 0 & -\,55 & -\,30 & +\,504 & (\sqrt{\tfrac{504}{55}} > 3 \\
 & 3 & 9 & -\,138 & -\,504 & \\
\hline
1 & 3 & -\,46 & -\,168 & +\,0 & (\sqrt{46} < 7 \\
 & 7 & 70 & 168 & & \\
\hline
1 & 10 & 24 & & 0 & (-5 \pm \sqrt{5^2 - 24} = -\,4\ \&\ -\,6 \\
\hline
\end{array}
$$

EXAMPLE 2. To determine whether the equation $6y^4 - y^3 - 21y^2 + 3y + 20 = 0$ has integral or finite fractional roots, and to depress the equation.

If such a fractional root exists, the numerator will measure the last term and the denominator the first. Accordingly this problem has been employed by Newton in exemplification of his method of divisors, and it is for the purpose of illustrating the advantage accruing from connecting our system of transformation with that method, that the problem is quoted in this place.

$$\begin{array}{rrrrr}
6 & -1 & -21 & +3 & +20\,(1 \\
 & 6 & 5 & -16 & -13 \\
\hline
6 & 23 & 12 & -18 & +7\,(1 \\
 & 6 & 29 & 41 & 23 \\
\cline{5-5}
 & & & & 30 \\
\cline{5-5}
\end{array}$$

Half-corrections for the Neg. Series

$$\begin{array}{rrrrl}
1 & 0 & -3 & 0\,(1 & \\
\hline
1 & 3 & 0 & -2\,(1 & \therefore\ 7 - 2 \times 2 = 3 \\
\hline
1 & 6 & 9 & 2 & \therefore\ 30 + 2 \times 2 = 34 \\
\hline
\end{array}$$

Results of substituting ……	2,	1,	0,	− 1,	− 2
are found above to be ……	30,	7,	20,	3,	34
Divisors *decreasing* by 3 are	10,	7	4	3	− 2

Consequently $\frac{-4}{3} = -1\frac{1}{3}$ *may* be a root

$\therefore$ change the first transformee into its negative

$$\begin{array}{rrrrr}
6 & 23 & 12 & -18 & +7 \\
 & 2 & 6 & 0 & -4 \\
\hline
6 & 25 & 18 & -18 & +3\,(\tfrac{1}{3} \\
 & 2 & 9 & 9 & -3 \\
\hline
6 & 27 & 27 & 9 & 0
\end{array}$$

N. B. The results of substituting 1 and 2 are obtained by one entire and one partial application of the compendium (B). From these, the results of the negative substitutions − 1 and − 2 are derived by means of corrections, the principle of which is too obvious to require particular explanation. They consist in performing an equal number of transformations on the alternate terms only of the given equation, the signs being changed and the intervals filled with ciphers; and then doubling the results. The last part of the work shews the advantage gained by connecting the concluding transformations with the nearest point of the preceding. And the final result shews at the same time that $-1\frac{1}{3}$ is a root and that the formula $2 + 9 + 9 - 3$, contains roots, which when augmented by 1 and the signs changed, give the remaining roots of the equation.

Because these researches after integral and fractional roots will generally serve no other purpose than to prove that the equation has no such roots, I have been the more careful to reduce them to the most expeditious manœuvres that could be contrived; in which kind nothing appears preferable to the process here adopted. But, as the straight forward way of first clearing the roots from denominators, and then regularly approximating to their numerators is more intelligible to tyroes, and, even in the most unpromising case of a large compound first coefficient, is capable of some peculiar facilities, I shall pursue that mode in regard of Newton's next example.

Example 3. Let the proposed equation be $24a^5 - 50a^4 + 49a^3 - 140a^2 + 64a + 30$.

	24 −	50 +	49 −	140 +	64 +	30
Mult. by	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
	6 −	25 +	49 −	280 +	256 +	240
Mult. by	$\frac{1}{2}$	1	2 −	4	8 +	16
	3 −	25 +	98 −	1120 +	2048 +	3840
Mult by	$\frac{1}{3}$	1	3	9	27	81
	1 −	2.5 +	2,94 −	10,080 +	5,5296 +	3,11040(1,0
		1,0 −	1,50 +	1,440 −	8,6400 −	3,11040

N. B. The previous reductions are effected by multiplying by the smallest common series of powers of any prime divisor of the first term, which will give integral products. The divisors so employed prove to be 2, 2, 3; whence the roots of the finally reduced equation are ($2 \times 2 \times 3 =$) 12 times greater than of the given one. After cutting off one period from each term, and beginning the initial approximation, the very first line shews that 10 is a root of the reduced equation, and consequently $\frac{10}{12} = \frac{5}{6}$ is a root of the proposed; and also that the roots of $1 - 15 + 144 - 8640 - 31104$, when divided by 12, or rather those of ($1 - \frac{1}{3} \times 15 + \frac{1}{9} \times 144 - \frac{1}{27} \times 8640 - \frac{1}{81} \times 31104 =$) $1 - 5 + 16 - 320 - 284$, when divided by 4, will give the remaining roots of the proposed.

Example 4. To illustrate the 2nd remark on the subject of depression. In the Phil. Trans. one root of $x^3 - 7x + 7 = 0$ was found $= 1.356895867$; and, when the approximation had advanced as far as 1.3568, the once-corrected coefficients were 4.0688 and − 1.48053696. Find the remaining roots by a quadratic equation.

8	325504		
4 0 6 8 8	− 1.48053696		
8, &c.	64... &c.		
95867	86634		
|4|0|7|0|49 &c.	2035		
	326		
	24		
	3		
		1.3568	
4.0 7 0 4 9 5 8|67	− 1.47689106(	− 0.335221472	= N
3	131114876		
		1.692021472	Pos. roots
4 3 7 0 4 9 5|8|7	− 16574280	1.356895867	
3	14101488		
		− 3.048917339	Neg. root
4 7 0 0 4 9|5|9	− 2472742		
5	2367748		
4 7 3 5 4|9|6	− 104994		
2	94814		
4 7 4 0|7|0	− 10180		
2	9482		
|4|7|4|0|9	− 698		
	474		
Or 4.070495867			
.670442944 = 2N	224		
	190		
− 4.740938811			
1.692021472	38		
	34		
.048917339 Neg. root			
	1		

N. B. In the first place the given coefficients are preceded and followed by the corrections which must have been so situated in the cubic process. Then the remaining part 95867 of the root is inserted in its proper place in the first column; and the products found by multiplying the sum of the first column by it, in a contracted operation, occupy the next portion of the second column. Thus the sums of the 1st and 2nd columns are the once-corrected co-efficients which follow next after the given pair, on the supposition that the portion 95867 is annexed to the root at once. Of course the 2nd of those sums is the proof-divisor to which that portion of the root is due; which shews that the use we have made of that portion is by anticipation, and that it does not, at the period we use it, form a part of the root. For this reason, the root of the quadratic equation, of which the newly corrected coefficients form the principal terms, is added to the portion 1.3568 alone, to form the positive root of $x^3 - 7x + 7$.

The preceding example was introduced for the purpose of explaining the mode and rationale of working even in the most intricate cases. But it must be evident, that in regard of the particular equation here solved, it would have been easier to get the required roots from a new operation on the original data. The comparative cumbrousness of the depressed process arises partly from the contractions employed in the previous cubic, and partly from the extended form of the new coefficients. Where neither of these circumstances occur, the superiority of the process for depression is very conspicuous, as in the following

EXAMPLE 5. By my method of Continuous Approximation, Mr. Nicholson finds a root of $x^3 - 7035x^2 + 15262754x = 100007\ 30880$ to be $x = 3456$; and the numbers at the foot of the preparatory operations are 3321 and 2448680. Required the other two roots of the equation.

```
 . . . .       . . . .
  3321        2448680   (— 1105
—1          — 2821         3450 = 3456 omitting the last figure.
———————     ———————————    ————
  2321         1276        2345
— 1          — 1221     — 1111 = 1116 deducting 5 the last figure in 1105
———————      —————————  ————————
  1221           5580      1234
  —05          — 5580      ————
  ————           ==
  1116
  ————
```

Hence the roots required are 2345 and 1234.

Equal Roots.

1. Reasoning precisely as in the foregoing article, we perceive that if *two* roots of the equation are each exactly equal to

the quantity r by which we are diminishing the roots, not only the last of the once-corrected coefficients will vanish, but the last of the twice-corrected also; and that the remaining terms of the latter set will constitute a formula *two* degrees lower than the original equation and containing all its remaining roots. And in short, if any assigned number of roots are each $= r$, just so many successive courses of partially corrected coefficients will end in zero; and the significant terms of the last such course constitute an entire formula containing all the remaining roots. And consequently that if the transformation be completed, the result will be a formula of the same reduced dimensions as the last mentioned, and containing all the remaining roots of the equation diminished by r.

The concluding observation applies to all the methods specified in the Corollary of the foregoing article.

Example. Solve the equation $x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$.

```
1 − 6 + 12 − 10 + 3(1
    1 −  5     7 − 3
    1 −  4     3
    1 −  3
Or thus,
1 − 6 + 12 − 10 + 3(1
    1 −  5     7   3
1 − 2    0     0   0
```

N. B. Without even completing the transformation by Prob. I., it appears from the first solution in the margin, and compendium (*D*) in Art. IV, that the last coefficient will in three successive courses be 0, whence it follows that 1, 1, 1, are three roots of the equation; and the last course 1 − 3 + 0 shews that 3 is the other root. In the same manner it appears from the second solution, that 1, 1, 1, and (2 + 1 =) 3 are the roots.

Example 2. Solve the Equation $x^4 - 9x^2 + 4x + 12 = 0$. (See Mr. Hellins's Paper in Phil. Trans. for 1782.)

```
1   0 −  9     4    12(1       Or 1   0 − 9   4   12 (2, since 3 > 1 < (√9 =) 3
    1    1 −   8 −   4            1   2 − 5 − 6    0
1   4 −  3 −  10     8(1          1   4   3    0(− 2 ± √(2² − 3) = − 1 & − 3
    1    5     2 −   8
1   8   15     0     0(− 4
  − 4 − 16
    4
1   0 −  1(± 1
```

N. B. The first solution gives the roots (1 + 1 =) 2, 2, (2 − 4 ± 1 =) − 1 and − 3; the same as are produced directly by the second solution. In the latter the corrections are applied mentally, and the process that of the Elementary Rule in No. IV.

Case 2. When the repeated root consists of several digits, the solution is attended with some peculiar circumstances, which have never, to my knowledge, been sufficiently scrutinized.

It is well known that equal roots, and roots differing by minute quantities, are among the causes which produce the greatest embarrassment in approximating by the ordinary methods, and in fact render them nearly ineffectual whenever they occur. A few very simple considerations will, I believe, place the source of this embarrassment in a clear light, and at the same time prove how effectually it is obviated by the entireness of transformation on which our method is grounded.

(1) Every equation may be regarded as the product of two inferior equations; one, containing the single or repeated root toward which we are actually approximating; and the other involving the remaining roots.

(2). During the process of indefinite approximation, the root in question becomes, abstracting signs, the least root of the transformed equation. Regarding the left hand period in each coefficient as integers, the root is a fraction < 1, and the farther the approximation proceeds, the smaller the fraction becomes.

(3). Now, if $x^p \pm ax^{p-1} \pm bx^{p-2} \ldots\ldots \pm h$ is multiplied by $x^q \pm krx^{q-1} \pm tr^2x^{q-2} \ldots$ &c. in which r is a small fraction, the first $p + 1$ terms of the product will have the same signs, and nearly the same coefficients as the former or constant factor, and the last $q + 1$ terms will have the same signs and nearly the same coefficients as $\pm h$ times the latter or variable factor.

This is manifest on actually multiplying the two formulæ, and retaining only that portion of each term in the product, which does not include r, or which includes the lowest power of r.

(4). If therefore, the variable factor is $=(x - r)^\alpha$, the last $\alpha + 1$ terms will vary but little from $\pm h(x - r)^\alpha$; and the first of those terms, as well as all that precede it, will undergo merely fractional variations during the process of the work.

(5). Hence, when an equation has α extensive or incommensurable equal roots, the following *indications* will be observed as the work proceeds: simultaneous and regular diminution in the last α terms; the preceding terms constant in regard of signs, and, to an increasing extent, in their first digits also; the last $\alpha + 1$ signs alternately + and —, or — and +; that mutual relation among the last $\alpha + 1$ terms which the expansion of $h(x - r)^\alpha$ indicates.

(6). From that law of expansion, also, the *value* of the approximation r may be readily found, and in various ways. *E.g.*

(A) If the last term but one be made the search divisor, the dividend must be α times the last term. (B) The αth term, reckoning backward, may be the dividend; and the search-divisor be α times the preceding term. (C) The last term being the dividend, and $\alpha + 1$th term, reckoning backward, the search-divisor, the αth root of the quotient may be tried for r.

The ordinary rules, which are only adapted to the case $\alpha = 1$, of course fail when applied to other cases; but, by attending to the indications and directions here made out, the new method is divested of perplexity, and the transformations may proceed, under every circumstance, without interruption.

EXAMPLE 3. Solve the equation $x^5 - 330x^4 + 35974x^3 - 1260072x^2 - 3733263x = 13676310$.

```
—3|30 + 3|59 74 — 1|260 072 — |373 3263 — |..13676310 (111
 1|   — 2|30    + 1|297 4    + |378 28    — |....463
 1|   — 1|30    —  |..26     + |347 28      |
+1|7|0 +  |39|74 —  |267|872 + |.347|2337 — |..137|22610
 1|        18|0       57|74   — 210|132          137|1017
 1|        19|0       76|74   — 138|392              |
+2 2|0| +117 74|   — 36 652| + 3|7097|    —    |12440
    1 |    2 21|     11|995| — 2 4657|    +     12440
    1 |    2 22|     12 217| — 1 2440|
    1 |    2 23|     12 440|(—110
         — 1 10      12 430
         ------
           1 13
```

$1 \quad + \quad 3 \quad + \quad 10 \left(\frac{-3 \pm \sqrt{3^2 - 40}}{2} = \frac{-3 \pm \sqrt{-31}}{2}\right.$

The roots are 111, 111, 111, and $-\frac{1}{2}(3 \pm \sqrt{-31})$.

N. B. In the first transformee the peculiar relation of the signs appears; the leading ones having obviously assumed a constant form, and the last four regularly alternating. But the terms are not yet prepared for *regular* simultaneous diminution. The reason of this appears in the sequel; for it turns out that 11, the portion of the root which yet remains to be evolved, is greater than any coefficient in the constant factor $x^2 + 3x + 10$. Accordingly in the second transformee, whose root 1 is less than either of those coefficients, we not only observe the indication of the signs to be confirmed, but every trial recommended in the 6th remark on the supposition of three equal roots, gives the same quotient 1; and when this is employed as the approximation, the tendency to constancy in the first term, and regular simultaneous diminution in the three last, become very prominent.

The same facilities would have characterized the further progress of the solution, had the root been extended. But, since the last terms were manifestly vanishing, as we proceeded to the *third* transformation, the compendium (B of No. III.) which had been adopted as usual at the preceding steps, was there abandoned for the regular track of Prob. I. for the sake of taking advantage of the remarks at the head of this article.

7. Even in an earlier stage of solution, the existence of one particular single or repeated root will be distinctly marked, whenever that root happens to be much greater, abstracting the signs, than any other root of the equation. In this event, the conclusions of the 3rd and subsequent remark will be reversed. The *first* portion of the product will be nearly $= x^p (x - r)^n$ and the rest will not vary much from $(-r)^n$ times the constant factor. Consequently the first $\alpha + 1$ signs will alternate; and the annexed coefficients will exhibit an approach to the law of the Binomial Theorem, and will predominate in value, when the αth period in every term is accounted for units.

When these indications occur, the precepts of the 6th remark may be applied to the first $\alpha + 1$ terms and a good approximation obtained.

For instance, in the last example, the binomial indications in the first four terms of the given equation, and the insignificance of the remaining terms when the 3rd or even the 2nd period is accounted units, are manifest. At the same time the quotients $\frac{33}{3}$, $\frac{360}{32}$, $\frac{3780}{360}$ and $\sqrt[4]{12360}$, all produce 11 as the nearest value of the root in two periods. Whence

$1 - 33|0 + 359|74 - 1260|078 - 373|3063 + 18676810\ (11|1$
$\quad 11| - 242| + 1295|14 + 385|748 + 1866881$
$\quad - 22|0 + 117|74 + 35|068 + 12|4217 - 12440$
$\quad 11 - 121 - 3586 - 8712$
$\quad 11 \quad 0 \quad - 3586$
$\quad 11 \quad 121$
$\quad 11$
$1 - 220 + 11774 - 86658 + 37997 - 12440$

and the rest as the last step in the former solution.

(8.) Again, in diminishing the roots of $p + p_1 x + p_2 x^2 + \ldots$ $p_n x^n = 0$ by r, the last coefficients become

$$P = p + p_1 r + p_2 r^2 + p_3 r^3 + p_4 r^4 \ldots\ldots = fr$$
$$P_1 = p_1 + 2p_2 r + 3p_3 r^2 + 4p_4 r^3 \ldots\ldots = Dfr$$
$$P_2 = p_2 + 3p_3 r + 6p_4 r^2 \ldots\ldots = D^2 fr$$

. . . .

$$\therefore P_{\alpha-1} = p_{\alpha-1} + \alpha p_\alpha r + \beta p_{\alpha+1} r + \gamma p_{\alpha+2} r^2 \ldots = D^{\alpha-1} fr$$

as will be manifest from the principle laid down in No. IV. more directly than as usually deduced from the Binomial Theorem. For the law of the literal parts is evident, and the figurate coefficients are necessarily formed in dividing successively by $x - r$, in the mode there enjoined.

Now we have already seen, that when r is a root repeated (for example) α times, the new coefficients P, $P_1, \ldots P_{\alpha-1}$ all become 0. Consequently r is a root of every one of the formulæ just now exhibited, and the first of which is the same as the original equation. And since the latter contains that root α times, and the new coefficients were produced by successive divisions by $x - r$, depressing the original one degree at each step, it is plain that $P_1 = 0$ contains r as a root $\alpha - 1$ times; $P_2 = 0$, $\alpha - 2$ times; $\ldots P_{\alpha-1} = 0$, just *once*.

Since, then, $x - r$ is a common measure of all these equations, including the original one, it may be found by a well-known rule, and the equation depressed. And this is the practice commonly prescribed by authors.

(9.) But another, and a useful reflection concerning these derived equations appears to have escaped observation. It is this; when we are in possession of a distinct approximation to the repeated root (and without this we are seldom sure of the existence of such a root), the further extraction of it may be obtained from either of the derived equations; and afterwards, since every transformee is the very same function of x as the formula from which it is derived, we may change the residual formula into one which shall contain all the remaining roots of the equation, by reversing the process of derivation, or in other words, dividing the successive coefficients by the same figurate numbers as the original coefficients were multiplied by. Thus, in the example already given:

$$\begin{array}{lllll}
 & 1 - 330 & + 35974 & - 1260072 & \\
\times \text{ by } & 5 \quad\; 3 & \quad \frac{3}{2} & \quad \frac{1}{2} & \\
\hline
 & 5 - 99|0 & + 359|61 & - 630|036 & (11|0 \\
 & \quad\; 55| & - 484| & \quad 611|71 & \\
\hline
 & \quad - 44|0 & + \; 55|61 & & \\
 & \quad\;\; 55 & \quad 605 & & \\
\hline
 & 5 + 660 & + 17661 & - \; 18326 & (1 \\
\hline
 & 5 + 675 & + 18996 & + \; 0 & \\
\times \text{ by } & \frac{1}{5} \quad\; \frac{1}{3} & \quad \frac{2}{3} & & \\
\hline
 & 1 + 225 & + 12664 & (- 111 & \\
 & \quad - 111 & - 12664 & & \\
\hline
 & \quad\; 114 & & & \\
\hline
 & 1 + \quad 3 & + \quad 10 & & \\
\hline
\end{array}$$

N. B. As it appeared, from the indication mentioned in the 7th remark, that the equation probably had 3 equal roots, whose first digits were 11, the several terms beginning at the 3rd from the end, and proceeding backward have been multiplied by the 3rd figurate series 1, 3, 6, 10, or rather by the halves of these numbers to produce the smallest integers. The extraction of the repeated root 111 being completed, the residual formula was multiplied by the reciprocals of the same series. The result is a formula containing the remaining roots of the equation, diminished by the roots already found. After augmenting them, therefore, by that quantity, the same quadratic factor $x^2 + 3x + 10$ appears, which was found by the first method.

Roots whose difference is very small.

When two or more roots of an equation agree in their first digit or digits, and are at the same time smaller or greater than any other root, abstracting the signs, the evolution of the portion in which they agree will necessarily be attended with indications very similar to those which we have traced to the influence of equal roots. Consequently they present the same resistance to the Newtonian and similar methods; a fact which Lagrange has not failed to dilate upon. But they also share in the facilities pointed out in our 5th and 7th remarks above; to such a degree at least, that an approximation may thence be generally obtained, which shall at once place the remaining solution within the reach of the ordinary methods.

I shall illustrate this observation by means of the problem which Lagrange, in the paper alluded to, selects as opprobrious to Newton's method, and an example of his own. The initial solution has been given already, in our most cautious manner, in No. II.

Example. Let the equation $x^3 - 7x + 7 = 0$ be proposed.

$$\begin{array}{rrrrl} 1 & 0 & -7 & +7 & (1 \\ \hline 1 & +3 & -4 & +1 & (.6 \\ & & .6 & 2.16 & -1.104 \\ \hline & & 36 & -1.84 & \\ & & 6 & 86 & \\ \hline 1 & +48 & +68 & -104 & \end{array}$$

N. B. In the first transformee we perceive the indications of the 5th remark on equal roots, pretty clearly made out. The last two terms have diminished together. The last three signs alternate. The first two terms have obtained their permanent signs; and cannot very greatly alter in value, since the root is obviously < 1. Accordingly we find from the 6th remark, $\sqrt{\frac{1}{3}} = .58$, $\frac{2 \times 1}{4} \doteq .5$, $\frac{4}{2 \times 3} \doteq .67$, suggesting .6 as a good approximation; which proves to be correctly the second digit in the greatest affirmative root.

Roots otherwise related.

No other relation between the roots of an equation has occurred to my recollection, as requiring any particular explanation in order to prevent the calculator from being detained among the processes of the initial stage, when the means of more rapid and yet safe advance are attainable. With regard to the cases of equal roots with contrary signs, roots in arithmetical progression, &c. for which particular modes of depression have been invented, I only think it requisite to observe, that whatsoever convenience may be proposed in the adoption of those rules will be considerably improved by the aid of our system of transformation.

No. VII.

SUPPLEMENTARY.

The principle which has introduced so much facility into the Transformation of Equations, appears to result from the combination of these two:

1. Detaching the literal parts from the coefficients, the law which the former observe directs us in *arranging* and disseminating the latter.

2. Inverse or analytical operations are performed by means of such as are direct or synthetic.

Several examples of the known or possible influence of such a combination, will occur to the reader. Thus, Subtraction in algebra is always effected by Addition; Division, in our transformations, is placed in the compact form of Multiplication; Evolution is coincident with Involution, &c. And the enquiry arises: Whether it is not, in every case, possible to find a required *inverse function by a direct process* similar to that which would be employed in verifying it?

It is with the wish of rendering the practicability of this idea a little more apparent, that I am induced to throw together the contents of this paper.

SYNTHETIC DIVISION.

In our transformations we have had occasion to use binomial divisors only, We shall now exhibit the rule in a general form. Premising therefore that the coefficient of the divisor is assumed to be unity;—a condition readily satisfied in every case by dividing each term in both factors by the actual first coefficient; and that the first coefficient of the quotient is therefore the same as that of the dividend, and is to be placed at a suitable distance *below* it; the remaining terms of the quotient are to be found by the following

RULE. Multiply each term of the quotient as soon as it is entered, by the second and subsequent terms of the divisor with their signs changed. Place the products in separate lines, and in due degrees of advance, under the successive terms of the dividend. Under the column, which contains the first of these products, place its amount, for a new quotient term. Proceed thus until the quotient terminates or is sufficiently extended.

DEM. For, on this arrangement, the identical quantities which must be *added* to the coefficients of the quotient in the ordinary process of multiplying it by the divisor to reproduce the dividend, are here subtracted from the latter to produce the former; as they ought to be.

EXAMPLE 1. Divide $4x^4 - 9a^2x^2 + 6a^3x - a^4$ by $2x^2 - 3ax + a^2$.

Arranging the coefficients according to the powers of a, and proceeding as above, we have

$$\begin{array}{rrrrrl} -1 & +6 & -9 & +0 & +4 & (3-2 \\ & -3 & +9 & +6 & +0 & \\ & & +2 & -6 & -4 & +0 \\ \hline -1 & +3 & +2 & +0 & +0 & \\ \hline\hline \end{array}$$

N. B. The 2nd and 3rd coefficients of the divisor are, for better recollection, written, with signs changed, on the right of the dividend. The quotient stops after 3 terms are found; wherefore these, united to the proper literal parts, give $-a^2 + 3ax + 2x^2$ or $2x^2 + 3ax - a^2$ for the exact quotient.

EXAMPLE 2. Reduce $\frac{x^4}{(x^2 - x + 1)(x^2 - 2x + 3)}$ to an infinite series.

$$\begin{array}{rrrrrrrl} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \&c. \ (1-1 \\ & 1 & 1 & 0 & -1 & -1 & 0 & \\ & & -1 & -1 & 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & -1 & -1 & 0 & 1 & \&c. \ (2-3 \\ & 2 & 6 & 6 & -8 & -36 & -48 & \\ & & -3 & -9 & -9 & +12 & +54 & +72 \\ \hline 1 & 3 & 3 & -4 & -18 & -24 & +7 & \&c. \\ \hline\hline \end{array}$$

Hence the series required is $1 + \frac{3}{x} + \frac{3}{x^2} - \frac{4}{x^3} - \frac{18}{x^4} - \frac{24}{x^5} + \frac{7}{x^6}$ &c.

EXAMPLE 3. Expand $\frac{6x^3}{x+1 \cdot 2x-3 \cdot 3x+5} = \frac{6^3x^3}{6x+6 \cdot 6x-9 \cdot 6x+10} =$ say to $\frac{y^3}{y+6 \cdot y-9 \cdot y+10}$.

$$\begin{array}{rrrrrrl} 1 & & & & & & (-6 \\ & -6 & +36 & -216 & +1296 & -7776 & \&c. \\ \hline 1 & -6 & +36 & -216 & +1296 & -7776 & \&c. \ (9 \\ & 9 & +27 & +567 & +3159 & +40095 & \\ \hline 1 & +3 & +63 & +351 & +4455 & +32319 & \&c. \ (-10 \\ & -10 & +70 & -1330 & +9790 & -142450 & \\ \hline 1 & -7 & +133 & -979 & +14245 & -110131 & \&c. \\ \hline\hline \end{array}$$

Answer $1 - \frac{7}{6x} + \frac{133}{6^2x^2} - \frac{979}{6^3x^3} +$ &c.

Its Conformity with Multiplication.

The conformity of this mode of Division with the usual process of Multiplication will be apparent at once, if we proceed to prove any of the foregoing examples; the 1st for instance. For here, in multiplying the quotient $-a^2 + 3ax + 2x^2$ by the divisor $a^2 - 3ax + 2x^2$, the work, omitting the literal parts would appear in the annexed form, of which the other is a simple transposition.

$$\begin{array}{rrrrrl}
-1 & +3 & +2 & 0 & 0 & (-3+2 \\
 & +3 & -9 & -6 & & \\
 & & -2 & +6 & +4 & \\
\hline
-1 & +6 & -9 & +0 & +4 &
\end{array}$$

The advantage accruing from attention to this conformity may be recognized in the following pair of examples, which Mr. Nicholson (in *Math. Comp.* No. 20.) has solved by the method of Combinations.

EXAMPLE 4. To decompose x^4 in the form $\text{A}x_2 x_3 x_4 x_5 + \text{B}x_2 x_3 x_4 + \text{C}x_2 x_3 + \text{D}x_2 + \text{E}$; where $x_m = x + m$.

$$\begin{array}{rrrrrl}
1 & 0 & 0 & 0 & 0 & (-2 \\
\hline
1 & -2 & +4 & -8 & +16 & (-3 \\
 & -3 & 15 & -57 & & \\
\hline
1 & -5 & +19 & -65 & (-4 & \\
 & -4 & 36 & & & \\
\hline
1 & -9 & +55 & (-5 & & \\
 & -5 & & & & \\
\hline
1 & -14 & & & &
\end{array}$$

N. B. Here, as in the Transformation of No. IV, the successive coefficients E, D, &c. are the remainders, or *coefficients which immediately follow the absolute terms*, in dividing the required formula, or its equivalent x^4, by the successive unknowns $x + 2$, $x + 3$, &c. Consequently, as x^4 is of the 4th order, and ∴ has its absolute term in the 5th place, and as the order is reduced one degree by each division, we infer that the coefficients E, D, &c. are the 5th, 4th, &c. in their respective sets. Checking ∴ the divisions at those distances respectively, the last term in each quotient is a coefficient of the required formula, and its place in its present line indicates its place in that formula. Hence the solution is $x^4 = x_2 x_3 x_4 x_5 - 14 x_2 x_3 x_4 + 55 x_2 x_3 - 65 x_2 + 16$.

EXAMPLE 5. To decompose $\frac{1}{x^3}$, in the form $\frac{1}{x_1 x_2 x_3} + \frac{\text{A}}{x_1 x_2 x_3 x_4} + \frac{\text{B}}{x_1 \ldots x_5} + \frac{\text{C}}{x_1 \ldots x_6} +$ &c.

N. B. Here also the coefficients 1, A, B, &c. are those which immediately follow the absolute terms, in multiplying $\frac{1}{x^3}$ successively by $x + 1$, $x + 2$, &c. This would be evident, if the formula were proposed in the integral form with negative indices, thus $x^{-3} = x_1^{-1} x_2^{-1} x_3^{-1} +$ &c. Now the coefficient of x^{-1} is manifestly the 3rd from the end in each set. Neglecting ∴ all but the last three in every set, the first of each reserved

$$\begin{array}{rrrrrrrl}
1 & 0 & (1 & & & & & \\
 & 1 & & & & & & \\
\hline
1 & 1 & & (2 & & & & \\
 & 2 & 2 & & & & & \\
\hline
1 & 3 & 2 & (3 & & & & \\
 & 3 & 9 & 6 & & & & \\
\hline
 & 6 & 11 & 6 & (4 & & & \\
 & & 24 & 44 & 24 & & & \\
\hline
 & & 35 & 50 & 24 & (5 & & \\
 & & & 175 & 250 & 120 & & \\
\hline
 & & & 225 & 274 & 120 & (6\ \&c. & \\
 & & & & 1350 & 1644 & \&c. & \\
\hline
 & & & & 1624 & \&c. & &
\end{array}$$

portion is a coefficient of the required formula, properly situated in regard of its distance from the actual first term. Hence we have

$$\frac{1}{x^3} = \frac{1}{x_1x_2x_3} + \frac{6}{x_1x_2x_3x_4} + \frac{35}{x_1x_2\ldots x_5} + \frac{225}{x_1\ldots x_6} + \frac{1624}{x_1\ldots x_7} + \&c.$$

The great advantages of operating in this way, seem to be (1.) that it is the most natural as well as the easiest way, and (2.) that it is quite unrestricted; for if x_1, x_2 &c. instead of being simple functions of x, proceeding in regular order, were taken with arbitrary coefficients and of any dimension, or of different dimensions and in any order indiscriminately, the method of operating would be equally apparent and guided by the same laws.*

Its Use in Equations.

The rule we have laid down for Synthetic Division agrees in substance with the well-known method of constructing a recurring series; the scale of relation, as it is called, being the same quantity which in our examples appears on the right hand of the dividend. Hence our rule serves every purpose for which such series are constructed; but with several additional recommendations; particularly, a perspicuous arrangement, well adapted, as we have seen, for continuous repetition, and directly illustrating the connexion of opposite parts of analysis; and an appropriate situation among elementary processes, tending to familiarize subjects which had previously been accounted abstruse. For instance, the rule for finding the successive *sums of the powers of the roots* of a proposed equation, may be restored to its legitimate form, thus;

Multiply each coefficient by the index of the annexed unknown quantity. Divide the whole result by the given formula. The coefficients of the quotient, taken in order, will be the sums of the 0, 1, 2, &c. *powers of the roots.*

N. B. 1. The sums of the reciprocals of the same powers will be found, if the factors are reversed, or if the equation itself is reversed in the first instance. 2. If a common denominator is subjoined to the index-multipliers, to obtain the least integral products, the required sums will appear in the quotient diminished in the same ratio 3. Whatever function of *one* order inferior to the divisor is assumed for the dividend, will produce the sums of certain constant multiples of the 0, 1, 2, &c. powers; if inferior by *two* dimensions, it produces constant multiples of the 1, 2, &c. powers; &c. In short in every term of the quotient, the index of the unknown quantity and that of the power implicated with the coefficient, taken together, constantly make − 1.

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* [Had the "Collection of Examples in Differential and Integral Calculus" been extant when this was written I should have selected my illustrations of the transformation of *Factorials* from Part 3, Section 2; to which the reader is recommended to apply our processes.]

Example 6. Find the sums of the successive powers of the roots of $x^3 - 3x^2 + 4 = 0$.

[*Euler*, Int. Inf. § 346: *Lagrange*, Res. des Equ p. 137.]

$$\begin{array}{lcccccccccc}
 & 1 & -3 & 0 & 4 & & & & & & \\
\times \text{ by} & \frac{3}{3} & \frac{2}{3} & \frac{1}{3} & \frac{0}{3} & & & & & & \\
\hline
 & 1 & -2 & 0 & 0 & (3 + 0 - 4 & & & & & \\
 & & 3 & 3 & 9 & 15 & 33 & 63 & 129 & & \\
 & & & & -4 & -4 & -12 & -20 & -44 & -84 & -172 \text{ \&c.} \\
\hline
 & 1 & 1 & 3 & 5 & 11 & 21 & 43 & 85 \text{ \&c.} & &
\end{array}$$

These results, multiplied by 3, give the successive sums required. *E.g.* The sum of the 7th powers is $3 \times 85 = 255 = 2^7 + 2^7 - 1^7$.

A property connected with these sums, has suggested some ingenious methods of approximating to the greatest or least root of an equation. For, it was said, if any root be greater than the rest, its powers will exceed the powers of any other root, and indeed of all the others together, when the power is high; and the more so, the higher we advance. This is true, whether the root be positive or negative. And if that root be the *least* which the equation has, what is here said of the roots will hold concerning their *reciprocals*.

Hence **Newton** infers that the $2m^{th}$ root of the sum of the $2m^{th}$ powers will be greater than the greatest root of the equation, but in high powers will make a near approach to the correct value. He was aware of the exception which the admixture of imaginary roots might occasion, but seems to have overlooked the possibility of two or more *equal* roots exceeding the rest; in which case, whether they have the same or different signs, it would be requisite to divide the sum of the powers by the number of such roots previous to extraction.

Euler, following D. Bernoulli, makes the first terms in the series of sums arbitrary or empirical, which sometimes expedites the convergency, but as frequently retards it, and, as Lagrange shews, must be hazardous in the event of equal roots. Otherwise, their principle of taking the quotient of two alternate terms, or the square root of the quotient of two alternate terms*, for an approximation, is preferable to Newton's, as it neither requires the extraction of complex roots, nor any alteration to be made in the actual state of the series.

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* It is curious that **Euler**, who touched upon this property of the alternate terms in his *Introductio ad An. Inf.* § 348, has afterwards overlooked it, in his *Algebra*, § 796; where in consequence of this oversight he brands the method in question as useless, in a case where its application is peculiar easy and certain.

We may add, that should even the alternate terms be useless, the mth root of the quotient of two coefficients which are m terms apart may be adopted, as in the case of $x^m = N$, or in general of $m - 1$ successive zeroes in the same formula.

Example 7. Find the greatest root of the equation in Ex. 6.

The values of $\frac{1}{1}$, $\frac{3}{1}$, $\frac{5}{3}$, $\frac{11}{5}$, $\frac{21}{11}$, &c. are alternately less and greater than 2, but approximate nearer to it at every step. Whence we conjecture 2 to be a root; which, on trial proves to be correct.

Moreover, if we remark that the sums 3, 3, 9, 15, 33, 65, &c. do not manifest a proportionate convergency toward the value of 2^0, 2^1, 2^2, 2^3, &c. but exceed that ratio, more and more nearly in the proportion of 2 : 1 as we advance, we shall be led to conclude that 2 is a *duplicate* root; which also proves correct.

This mode of detecting equal roots, is founded upon the remark made above on an oversight of Newton's. We shall now shew its use when pairs of roots neutralize each other.

Example 8. Find the roots of $x - x^2 - 5x + 5 = 0$.
[*Euler*, U. S. § 342.

$$
\begin{array}{lccccccccccl}
 & 1 & -1 & -5 & +5 & (& 1 & +5 & -5 & & & \\
\times\text{ by} & 3 & 2 & 1 & 0 & & & & & & & \\
\cline{2-5}
 & 3 & -2 & -5 & 0 & & & & & & & \\
 & & 3 & 1 & 11 & 1 & 51 & 1 & 251 & & & \\
 & & & 15 & 5 & 55 & 5 & 255 & 5 & 1255 & & \\
 & & & & -15 & -5 & -55 & -5 & -255 & -5 & -1255 & \&c. \\
\hline
 & 3 & 1 & 11 & 1 & 51 & 1 & 251 & 1 & \&c. & &
\end{array}
$$

Here the smallness of the sums of odd powers proves that the equation has one or more pairs of roots nearly equal; but with contrary signs. The value of $\frac{11}{3}$, $\frac{51}{11}$, $\frac{251}{51}$ &c. converge toward 5, whence we conjecture $\sqrt{5}$ to be a root of the equation. And since 5^0, 5^1, 5^2 &c. are less than 3, 11, 51 &c. in the ratio, more and more nearly of 2 : 1, we infer that $-\sqrt{5}$ is also a root. In the sums where these destroy each other, the constant power 1 remains. Therefore the roots are $\sqrt{5}$, $-\sqrt{5}$ and 1.

With the dividend, then, which we have adopted from Lagrange, no embarrassment can arise from *equal* roots whether with the same or contrary signs, if we pursue the track marked out in these two examples. As for *imaginary* roots, since the same author shews, that they will not ultimately frustrate the convergency, unless the product of a pair of them exceeds the square of the root we would approach, we conclude *a fortiori*, that whenever the cube of that root exceeds the absolute term of the equation, we are perfectly secure of convergency sooner or later. In fact, when this is the case, and the indications of equal roots (*see* No. VI.) are absent, we may very correctly choose an arbitrary dividend. None appears preferable to the first term of the equation itself; which has this useful property, that the first two terms in the result give Newton's or Raphson's approximation; the second and third Halley's or Delagny's. And as the convergency proceeds very equably, we thus have a standard whereby to estimate it.

Example 9. Find the root of $x^3 - 2x = 5$.

N. B. The introductory transformations evince the root to be very nearly $2\frac{1}{10}$. As the cube of this number exceeds the absolute term 5, the equation is at first in a form adapted for convergency, though slow, on account of the smallness of the excess. The root of the first transformee is ∴ nearly $1\frac{1}{10}$; and since $(1\frac{1}{10})^3 < 6$, this will *not* converge, but its reciprocal would. That of the 2nd is nearly $\frac{1}{10}$, ∴ that of its reciprocal is nearly 10. As 10^3 far exceeds $\frac{1}{1}$, the quotient of the first by the last term, we see that this reciprocal equation, viz. 1 — 10 — 6 — 1, is adapted for rapid convergency. And so the event proves.

$$\begin{array}{rrrrrrl}
1 & 0 & -2 & -5\,(1 \\ \hline
1 & 8 & 1 & -6\,(1 \\ \hline
1 & 6 & 10 & -1 \\ \hline\hline
1 & 0 & 0 & 0\;(& \underline{10\quad 6} & \underline{\quad 1} \\
 & 10 & 100 & 1060 & 11210 & 118560 \\
 & & 6 & 60 & 636 & 6726 \\
 & & & 1 & 10 & 106 & \&c. \\ \hline
1 & 10 & 106 & 1121 & 11856 & 125392 & \&c.
\end{array}$$

$$\therefore 1 + 1 + \frac{11856}{125392} = 2.094551486$$ correct in the 8th decimal place.

It is now, I trust, evident that any equation, whose roots are not all imaginary, may be readily reduced by our initial process to a state suited to this mode of solution.

Reciprocal Equations.

Those equations whose roots are equally the values of x and of $\frac{1}{x}$, will furnish a beautiful exemplification of the principle stated in the beginning of this paper, and will be better understood and more easily solved by using the following *General Rule*, than by the methods now in use. ¶ In changing a reciprocal equation into one whose roots are the sums of those of the former taken in pairs: ¶ 1. The 1st and 2nd coefficients remain unaltered. ¶ 2. As soon as a pair of the new coefficients have been entered, multiply each by the 2nd and following terms(1) of that power of a binomial whose index appertains to the unknown which is to be annexed to the new coefficient in question(2); and place the products, with signs changed, and intermingled in regular alternation below the succeeding terms of the given equation. ¶ 3. Under the columns which contain the first two of these products enter their amounts, which will be two more new coefficients. ¶ 4. Proceed thus until the set of coefficients is completed. ¶ N. B. (1) As the latter or duplicate portion of these sets of products will merely cancel the latter half of the given formula, these portions of the work may be neglected. (2) The given formula is supposed to be of even dimensions, or to be reduced to that state by dividing by $x + 1$; and then the highest power in the required, will be half of the highest in the given formula.

DEM. For the former equation is made from the latter by substituting $x + x^{-1}$ for x, or rather $x^2 + 1$ for x, to avoid fractions; and therefore $(x^2 + 1)^m$ for x^m; which is equivalent to *adding* to the alternate terms the products of the coefficients of x^m by the expansion of $(1 + 1)^m - 1$. Consequently the reverse operation is rightly performed by *subtracting* the same quantities. And, since the latter half of the former equation is constructed entirely of the duplicate portion of the said products, it will be necessarily cancelled by the reverse operation.

EXAMPLE 10. Find the roots of $2x^7 - 13a^2x^5 - 13a^5x^2 + 2a^7 = 0$. (*Bonnycastle*, vol. i. p. 183.)

$$\begin{array}{l}
\underline{2 \quad 0 - 13 \quad 0 \quad 0 - 13 \quad 0 \quad 2} \;(-1 \\
2 - 2 - 11 + 11 - 11 - 2 + 2 + 0 \\
\underline{\quad - 6 + 4} \\
2 - 2 - 17 + 15 \;\left(> \sqrt{\frac{17}{2}} \text{ say } 3 \right. \\
\underline{\; - 6 \quad 12 - 15} \\
\underline{\underline{2 \quad 4 - 5 \quad 0}} \;\left(\frac{-2 \pm \sqrt{2^2 + 2 \times 5}}{2} = \frac{-2 \pm \sqrt{14}}{2} \right.
\end{array}$$

$$\therefore\; z^2 - 3z + 1 \left\{ \frac{3 \pm \sqrt{5}}{2} \right.$$

$$\text{and } z^2 + \frac{2 \pm \sqrt{14}}{2} z + 1 \left(- \frac{2 \pm \sqrt{14}}{4} \pm \frac{\sqrt{2 \pm 4\sqrt{14}}}{4} \right.$$

N. B. The given equation being of odd dimensions, has one root $x = -a$; wherefore we begin by dividing by $x + a$. The quotient is of the 6th order, ∴ the required formula will be of the 3rd, ∴ its first two terms $2 - 2$ appertains to the quantities whose indices are 3 and 2. The coefficients of binomials raised to these powers are 1, 3, 3, 1 and 1, 2, 1; which, when the first term and the duplicate terms are omitted, leave simply 3 and 2 to be multiplied by 2 and -2 respectively. The products 6 and -4 are treated as the rule directs, and the transformation in this instance, is completed The resulting cubic is solved as being in the second stage; and the roots treated in the usual manner, produce the above results, which shew that the roots of the equation, besides $-a$, are $\frac{1}{2}(3 \pm \sqrt{5})a$, and $-\frac{1}{4}(2 \pm \sqrt{14} \mp \sqrt{2 \pm 4\sqrt{14}})a$.

COR. Binomial equations may be either treated by the above Rule, or by a separate rule easily modified upon it.

ARTICLE VI.

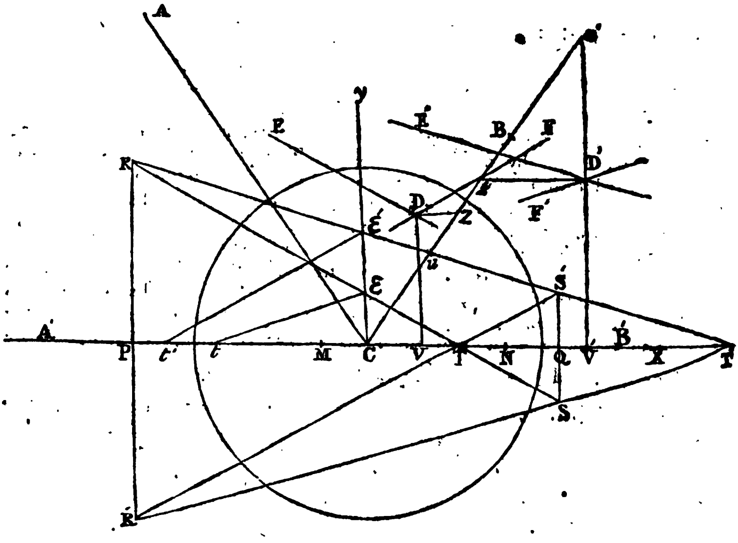
Investigation of two Geometrical Porisms.

By THOMAS GALLOWAY, *A. M. Royal Military College.*

In the First Volume of the Repository, Mr. Ivory has given an elegant Geometrical Analysis of one of those curious Porisms which were published without Solutions by Dr. Matthew Stewart at the end of his *General Theorems.* I propose in what follows to investigate two more of them, to which, so far as I know, no solutions have hitherto been published, although the *General Theorems* have occupied the attention of several very distinguished Mathematicians. I have followed the algebraic method of investigation, but the constructions to which I have been led are remarkably simple and symmetrical; and both Propositions afford a good specimen of the facility with which the solutions of geometrical questions, rather above the ordinary degree of difficulty, can be obtained by the aid of the modern analysis.

PORISM I.

The Porism which I shall first consider is thus enunciated by Dr. Stewart: "Let there be a circle given by position, and two points A, B given; two straight lines DE, DF may be found, such, that if from the points A, B there be drawn AG, BG to any point G in the circumference of the circle, and from the point G there be drawn GH, GK perpendiculars to DE, DF, the sum of the squares of GH, GK will be to the rectangle AGB as the rectangle AGB to a certain given space: that is, the rectangle AGB will be a mean proportional between the sum of the squares of GH, GK and a certain given space."



Analysis.

Let C be the centre of the circle, join AC, BC, bisect the angle ACB by the straight line CY, and through C draw CX perpendicular to CY. This last straight line CX is assumed as the axis of x, the point C being the origin of the rectangular co-ordinates x y.

1st. Let β denote the angle which each of the straight lines CA, CB makes with the axis CX; let also CA $= a$, CB $= b$, then x and y being the co-ordinates of a point G we have the following expressions

$$\text{AG}^2 = (a \cos\beta + x)^2 + (a \sin\beta - y)^2$$
$$\text{BG}^2 = (b \cos\beta - x)^2 + (b \sin\beta - y)^2.$$

If the point G is confined to the circumference of a circle of which the radius is r, then $x^2 + y^2 = r^2$, whence, by expanding and substituting,

$$\text{AG}^2 = a^2 + r^2 + 2a \cos\beta x - 2a \sin\beta y$$
$$\text{BG}^2 = b^2 + r^2 - 2b \cos\beta x - 2b \sin\beta y.$$

Multiplying these two expressions together, substituting $r^2 - y^2$ for x^2, and arranging the terms, there results

$$\left.\begin{aligned}\text{AG}^2 \times \text{BG}^2 = {} & (a^2 + r^2)(b^2 + r^2) - 4abr^2 \cos^2\beta \\ & - 2x \cos\beta \left\{ b(a^2 + r^2) - a(b^2 + r^2) \right\} \\ & - 2y \sin\beta \left\{ b(a^2 + r^2) + a(b^2 + r^2) \right\} \\ & + 4aby^2.\end{aligned}\right\} \ldots\ldots (\text{A})$$

In order to render this expression somewhat simpler, let

$$p = \frac{a^2 + r^2}{2a}, \qquad q = \frac{b^2 + r^2}{2b},$$

then

$$2ab(p - q) = b(a^2 + r^2) - a(b^2 + r^2)$$
$$2ab(p + q) = b(a^2 + r^2) + a(b^2 + r^2)$$
$$4ab \cdot pq \quad = (a^2 + r^2)(b^2 + r^2).$$

Hence, by substitution,

$$\left.\begin{aligned}\text{AG}^2 \times \text{BG}^2 = {} & 4ab(pq - r^2 \cos^2\beta) \\ & - 4ab \cos\beta (p - q)x \\ & - 4ab \sin\beta (p + q)y \\ & + 4aby^2.\end{aligned}\right\} \ldots\ldots\ldots (1)$$

2d. Let h, l be the co-ordinates of a point D, through which a straight line passes making an angle φ with the axis CX; the length of a perpendicnlar on that line, from a point of which the co-ordinates are x, y is

$$(y - l)\cos\phi - (x - h)\sin\phi,$$

and the square of this expression, when $r^2 - y^2$ is substituted for x^2, is

$$\begin{aligned}
&y^2\left\{\cos^2\phi - \sin^2\phi\right\} - 2xy\cos\phi\sin\phi \\
&- 2x\left\{h\sin^2\phi - l\cos\phi\sin\phi\right\} \\
&- 2y\left\{l\cos^2\phi - h\cos\phi\sin\phi\right\} \\
&+ l^2\cos^2\phi - 2hl\cos\phi\sin\phi + (h^2 + r^2)\sin^2\phi.
\end{aligned}$$

If we now suppose another straight line to pass through the same point D, making an angle ψ with the axis CX, the square of a perpendicular on it from the point G will evidently be the same expression as above, having ψ written throughout instead of ϕ; and the sum of the squares of the perpendiculars on both lines will still be the same expression if $\cos^2\phi + \cos^2\psi$ be put instead of $\cos^2\phi$, $\sin^2\phi + \sin^2\psi$ instead of $\sin^2\phi$, and $\cos\phi\sin\phi + \cos\psi\sin\psi$ instead of $\cos\phi\sin\phi$.

Let $\phi + \psi = \text{A}$, $\phi - \psi = \text{B}$,
then, by the Arithmetic of Sines

$$\begin{aligned}
\cos^2\phi - \sin^2\phi + \cos^2\psi - \sin^2\psi &= 2\cos\text{A}\cos\text{B}, \\
\cos\phi\sin\phi + \cos\psi\sin\psi &= \sin\text{A}\cos\text{B} \\
\cos^2\phi + \cos^2\psi &= 1 + \cos\text{A}\cos\text{B}, \\
\sin^2\phi + \sin^2\psi &= 1 - \cos\text{A}\cos\text{B}.
\end{aligned}$$

On substituting these values the sum of the squares of the two perpendiculars will be expressed as follows:

$$\begin{aligned}
&2y^2\cos\text{A}\cos\text{B} - 2xy\sin\text{A}\cos\text{B} \\
&- 2x\left\{h(1 - \cos\text{A}\cos\text{B}) - l\sin\text{A}\cos\text{B}\right\} \quad \ldots\ldots (2) \\
&- 2y\left\{l(1 + \cos\text{A}\cos\text{B}) - h\sin\text{A}\cos\text{B}\right\} \\
&+ l^2(1 + \cos\text{A}\cos\text{B}) - 2hl\sin\text{A}\cos\text{B} + (h^2 + r^2)(1 - \cos\text{A}\cos\text{B}).
\end{aligned}$$

3rd. The Proposition to be demonstrated amounts to this, namely, that such values may be assigned to the indeterminate quantities h, l, A, B in this last expression, that on multiplying it by a certain constant quantity, it shall become identical with the expression for $\text{AG}^2 \times \text{BG}^2$, whatever be the values of x and y, provided $x^2 + y^2 = r^2$.

Now in order that the two expressions may be equal, independently of x and y, it is necessary that the coefficients of y^2,

xy, x, y, and also the terms containing only constant quantities, in the two expressions, be separately equal to each other. On comparing the two expressions (1) and (2) I observe that the last contains a term multiplied by xy to which there is no corresponding term in the first; the coefficients of xy must therefore necessarily be zero, that is, we must have

$$\sin \text{A} \cos \text{B} = 0;$$

but this equation can only exist either by supposing $\sin \text{A} = 0$ or $\cos \text{B} = 0$. The supposition of $\cos \text{B} = 0$ is, however, inadmissible, inasmuch as the term multiplied by y^2 would thereby vanish, which is obviously impossible. It follows, therefore, that $\sin \text{A} = 0$, whence $\varphi + \psi = 0$, and therefore $\varphi = -\psi$; an equation which shews that the two straight lines to be found must make equal angles with the axis CX.

Since $\sin \text{A} = 0$, it follows that $\cos \text{A} = 1$, therefore, making c^2 the constant quantity which multiplies the expression (2), the comparison of the terms of the two expressions (1) and (2) gives the four following equations of condition to determine h, l, φ, and c^2.

I. $c^2 \cos \text{B} = 2ab$

II. $c^2 l(1 - \cos \text{B}) = 2ab\,(p - q) \cos \beta$

III. $c^2 l(1 + \cos \text{B}) = 2ab\,(p + q) \sin \beta$

IV. $c^2 l^2 (1 + \cos \text{B}) + c^2 (h^2 + r^2)(1 - \cos \text{B}) = 4ab(pq - r^2 \cos^2 \beta)$.

4th. From equation I. there results

$$c^2(1 - \cos \text{B}) = c^2 - 2ab, \quad c^2(1 + \cos \text{B}) = c^2 + 2ab;$$

substituting these values in II. and III. and squaring them both we have

$$c^2 h^2 (1 - \cos \text{B}) = \frac{4a^2b^2\,(p - q)^2 \cos^2 \beta}{c^2 - 2ab},$$

$$c^2 l^2 (1 + \cos \text{B}) = \frac{4a^2b^2\,(p + q)^2 \sin^2 \beta}{c^2 + 2ab},$$

hence equation IV. becomes

$$\frac{4a^2b^2\,(p + q)^2 \sin^2 \beta}{c^2 + 2ab} + \frac{4a^2b^2\,(p - q)^2 \cos^2 \beta}{c^2 - 2ab} + r^2(c^2 - 2ab)$$
$$= 4ab\,(pq - r^2 \cos^2 \beta).$$

If therefore any values can be assigned to c^2 which will satisfy the conditions of the Problem, they must be such as on substitution will render this equation identical. Now, since $2ab\,(2\cos^2 \beta - 1) = 2ab \cos 2\beta$, I observe that it may be put under the following form,

$$r^2(c^2+2ab\cos 2\beta)=4ab\,.\,pq-4a^2b^2\left\{\frac{(p+q)^2}{c^2+2ab}+\frac{(p-q)^2}{c^2-2ab}\right\},$$

and by multiplying both sides by $c^4-4a^2b^2$, and making the proper reductions, it becomes

$$\begin{aligned} &r^2(c^4-4a^2b^2)(c^2+2ab\cos 2\beta)\\ &=c^4\,.\,4ab\,.\,pq-c^2\,.\,4a^2b^2(p^2+q^2-2pq\cos 2\beta)\\ &-4a^2c^2\,.\,4ab\,.\,pq-4a^2b^2\,.\,2ab\left\{(p^2+q^2)\cos 2\beta-2pq\right\}\\ &=\left\{c^2\,.\,4abpq-4a^2b^2(p^2+q^2)\right\}(c^2+2ab\cos 2\beta), \end{aligned}$$

whence, dividing by $c^2+2ab\cos 2\beta$,

$$r^2(c^4-4a^2b^2)=c^2\,4abpq-4a^2b^2(p^2+q^2)$$

or,
$$c^4-c^2\,.\,\frac{4abpq}{r^2}=-\frac{4a^2b^2(p^2+q^2-r^2)}{r^2},$$

an equation which is remarkable as being independent of the angle β.

On completing the square we have

$$c^4-c^2\,.\,\frac{4abpq}{r^2}+\frac{4a^2b^2p^2q^2}{r^4}=\frac{4a^2b^2}{r^4}\left\{p^2q^2-p^2r^2-q^2r^2+r^4\right\},$$

but $p^2q^2-p^2r^2-q^2r^2+r^4=(p^2-r^2)(q^2-r^2)$

therefore

$$c^2-\frac{2abpq}{r^2}=\pm\frac{2ab}{r^2}\sqrt{(p^2-r^2)(q^2-r^2)}$$

whence

$$c^2=\frac{2ab}{r^2}\left\{pq\pm\sqrt{(p^2-r^2)(q^2-r^2)}\right\}.$$

Now since $p^2=\frac{(a^2+r^2)^2}{4a^2}$, it follows that $p^2-r^2=\frac{(a^2-r^2)^2}{4a^2}$;

and similarly, $q^2-r^2=\frac{(b^2-r^2)^2}{4b^2}$

therefore $\sqrt{(p^2-r^2)(q^2-r^2)}=\frac{(a^2-r^2)(b^2-r^2)}{4ab}$; and since also $pq=\frac{(a^2+r^2)(b^2+r^2)}{4ab}$, therefore

$$c^2=\frac{1}{2r^2}\left\{(a^2+r^2)(b^2+r^2)\pm(a^2-r^2)(b^2-r^2)\right\},$$

hence, $$c^2=\frac{a^2b^2+r^4}{r^2}:$$

or $$c^2=a^2+b^2.$$

Two values of c^2 have thus been found, each of which will furnish a separate solution. Taking first $c^2 = \frac{a^2b^2+r^4}{r^2}$, and substituting it in equation (I.) we find

$$\cos B = \frac{2abr^2}{a^2b^2+r^4},$$

but, observing that $B = 2\varphi$, and that therefore $\cos B = \cos^2\varphi - \sin^2\varphi$, we deduce

$$\cos^2\varphi = \frac{(ab+r^2)^2}{2(a^2b^2+r^4)}, \quad \sin^2\varphi = \frac{(ab-r^2)^2}{2(a^2b^2+r^4)},$$

and hence $\text{tang.}\,\varphi = \frac{ab-r^2}{ab+r^2}$.

Again, $$1-\cos B = \frac{(ab-r^2)^2}{a^2b^2+r^4},$$

therefore $c^2(1-\cos B) = \frac{(ab-r^2)^2}{r^2}$;

and substituting this value in equation (II.), there results

$$h = \frac{2ab(p-q)r^2}{ab-r^2}\cos\beta;$$

but $$p-q = \frac{b(a^2+r^2)-a(b^2+r^2)}{2ab} = \frac{(a-b)(ab-r^2)}{2ab},$$

therefore $$h = \frac{(a-b)r^2}{ab-r^2}\cos\beta.$$

In like manner $c^2(1+\cos B) = \frac{(ab+r^2)^2}{r^2}$; and by substituting this value in equation (III.), we have

$$l = \frac{2ab(p+q)r^2}{ab+r^2}\sin\beta = \frac{(a+b)r^2}{ab+r^2}\sin\beta.$$

Taking now the second value of c^2, viz. a^2+b^2, and substituting it in equation (I.) we find

$$\cos B = \cos^2\varphi - \sin^2\varphi = \frac{2ab}{a^2+b^2},$$

whence

$$\cos^2\varphi = \frac{(a+b)^2}{2(a^2+b^2)}, \quad \sin^2\varphi = \frac{(a-b)^2}{2(a^2+b^2)},$$

and therefore

$$\text{tang.}\,\varphi = \frac{a-b}{a+b}.$$

By forming the values of $c^2(1 - \cos \mathrm{B})$, $c^2(1 + \cos \mathrm{B})$, and substituting them respectively in equations (II.) and (III.) we find in this case,

$$k = \frac{2ab(p-q)}{(a-b)^2} \cdot \cos\beta = \frac{ab - r^2}{2ab} \cos\beta.$$

$$l = \frac{2ab(p+q)}{(a+b)^2} \sin\beta = \frac{ab + r^2}{2ab} \sin\beta.$$

The result of this investigation is, therefore, that there are two systems of equations which determine the position of the straight lines which the Theorem affirms may be found, or which determine φ, k and l, and consequently that there are two systems of straight lines, which satisfy the conditions of the Porism. Accenting the letters φ, k, and l, in the second case, these two systems of equation are as follows:

1st. When $c^2 = \dfrac{a^2b^2 + r^4}{r^2}$,

then $$\operatorname{tang}\varphi = \frac{ab - r^2}{ab + r^2},$$

$$k = \frac{r^2(a-b)}{ab - r^2} \cos\beta,$$

$$l = \frac{r^2(a+b)}{ab + r^2} \sin\beta.$$

2nd. When $c^2 = a^2 + b^2$,

then $$\operatorname{tang}\varphi' = \frac{a-b}{a+b},$$

$$k' = \frac{ab - r^2}{a-b} \cos\beta,$$

$$l' = \frac{ab + r^2}{a+b} \sin\beta.$$

5th. It now only remains to construct the equations which have been furnished by the Analysis.

Having joined the two points A, B with C the centre of the circle, draw CY bisecting the angle ACB, and draw CX perpendicular to CY. On CX take CA′ equal to CA, CB′ = CB (observing that the points A′ and B′ are on the same side of the axis CY as A and B respectively). On CA′ take a point M such that CA′ . CM $= r^2$, and on CB′ a point N such that CN . CB′ $= r^2$. Then since CA′ $= a$ and CM $= \dfrac{r^2}{a}$, we have MA′ $= \dfrac{a^2 - r^2}{a}$, and similarly $\mathrm{NB}' = \frac{b^2 - r^2}{b}$. Bisect MA' in P, and NB' in Q; then $\mathrm{MP} = \frac{a^2 - r^2}{2a}$, and $\mathrm{NQ} = \frac{b^2 - r^2}{2b}$: also $\mathrm{CP} = a - \mathrm{MP} = \frac{a^2 + r^2}{2a} = p$, and $\mathrm{CQ} = b - \mathrm{NQ} = \frac{b^2 + r^2}{2b} = q$, hence $\frac{a^2 + r^2}{2a} + \frac{b^2 + r^2}{2b} = \mathrm{PQ}$.

To construct the angle φ we have

$$\text{tang. } \varphi = \frac{ab - r^2}{ab + r^2} = \frac{(a + b)(ab - r^2)}{(a + b)(ab + r^2)} =$$

$$\frac{b(a^2 - r^2) + a(b^2 - r^2)}{b(a^2 + r^2) + a(b^2 + r^2)}.$$ But it is obvious

$$\frac{b(a^2 - r^2) + a(b^2 - r^2)}{2ab} = \frac{a^2 - r^2}{2a} + \frac{b^2 - r^2}{2b} = \mathrm{MP} + \mathrm{NQ},$$

and that $\frac{b(a^2 + r^2) + a(b^2 + r^2)}{2ab} = \frac{a^2 + r^2}{2a} + \frac{b^2 + r^2}{2b} = \mathrm{PQ}$;

hence tang. $\varphi = \frac{\mathrm{MP} + \mathrm{NQ}}{\mathrm{PQ}}$. If therefore at the points P, Q straight lines be drawn perpendicular to CX, and PR taken on the first equal to CM, QS on the second equal to QN (R and S being on opposite sides of the axis CX) then R and S being joined, and T being the point in which RS meets the axis CX, the angle PTR, or QTS will be equal to φ. Let the lines PR and QS be prolonged on the opposite side of the axis CX, and make $\mathrm{PR}' = \mathrm{PR}$, $\mathrm{QS}' = \mathrm{QS}$, then R′ and S′ being joined, the straight line R′S′ will also meet the axis in T, and make with it an angle equal to φ. Hence the two straight lines required must be respectively parallel to RS and R′S′.

The construction of the angle φ' is entirely similar. We have

$$\text{tang. } \varphi' = \frac{a - b}{a + b} = \frac{(a - b)(ab + r^2)}{(a + b)(ab + r^2)} = \frac{b(a^2 - r^2) - a(b^2 - r^2)}{b(a^2 + r^2) + a(b^2 + r^2)}.$$

But $\frac{b(a^2 - r^2) - a(b^2 - r^2)}{2ab} = \frac{a^2 - r^2}{2a} - \frac{b^2 - r^2}{2b} = \mathrm{MP} - \mathrm{NQ}$;

and, as was noticed already $\frac{b(a^2 + r^2) + a(b^2 + r^2)}{2ab} = \mathrm{PQ}$,

therefore tang. $\varphi' = \frac{\mathrm{MP} - \mathrm{NQ}}{\mathrm{PQ}} = \frac{\mathrm{PR} - \mathrm{QS}'}{\mathrm{PQ}}$. It is evident

The perpendiculars GH, GK; GH′, GK′ are omitted for the sake of avoiding confusion in the figure.

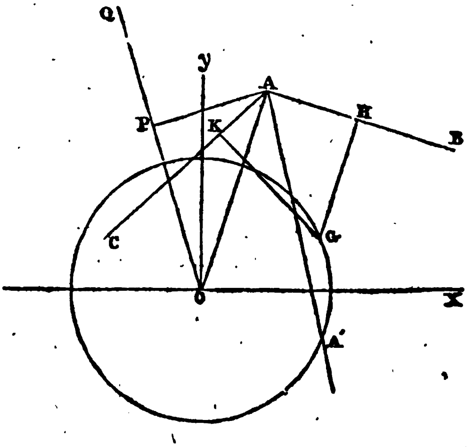
It will be observed that when the points A and B are at equal distances from the centre, the two straight lines RT′, R′T′ become parallel, and the point T′ and consequently D′ is thrown to an infinite distance. In this case the points T and *t* coincide with the centre C, and the straight line *tt* with the axis; the construction therefore fails in determining the line C*t* or *t*. But this inconvenience is easily obviated; for (as is evident on inspecting their analytical values) the rectangle C*t* × C*t*′ is equal to the square of the radius, and therefore, since C*t*′ is still found as in the general case, C*t* is also given in magnitude.

PORISM II.

"Let there be a circle given by position, and let AB, AC be two straight lines given by position, and let the angle BAC be equal to two angles of an equilateral triangle; two straight lines DE, DF may be found that will be given by position, such that if from any point G in the circumference of the circle within the angle BAC, there be drawn GH, GK perpendicular to AB, AC, and likewise there be drawn GL, GM perpendicular to DE, DF, the sum of the cubes of GH, GK will be equal to a solid whose base is the sum of the squares of GL, GM, and altitude a given line."

Analysis.

1st. Through the point A let AA′ be drawn bisecting the angle BAC, and through O the centre of the circle, OP parallel to AA′, and also AP perpendicular to OP.



Let OP $= a$, AP $= b$, the radius $= r$, and the angle BAA′ $= k$; O being the origin of the rectangular coordinates, and x taken on OP. This being premised we have

$$\text{GH} = (a - x)\sin k - (y - b)\cos k$$
$$\text{GK} = (a - x)\sin k + (y - b)\cos k$$

and therefore

$$\text{GH}^3 + \text{GK}^3 = 2(a - x)^3\sin^3 k + 6(a - x)\sin k(y - b)^2\cos^2 k$$
$$= 2(a - x)\sin k\left\{(a - x)^2\sin^2 k + 3(y - b)^2\cos^2 k\right\},$$

but by hypothesis $k = 60°$, therefore

$$\sin^2 k = \tfrac{3}{4},\ \cos^2 k = \tfrac{1}{4},$$

hence, by reason of $x^2 + y^2 = r^2$,

$$\text{GH}^3 + \text{GK}^3 = \tfrac{3}{4}\sqrt{3}(a - x)\left\{a^2 + b^2 + r^2 - 2ax - 2by\right\},$$

or, putting $a^2 + b^2 + r^2 = 2p^2$,

$$\frac{\text{GH}^3 + \text{GK}^3}{\tfrac{3}{4}\sqrt{3}} = 2ap^2 - 2(p^2 + a^2)x - 2aby + 2bxy + 2ax^2.$$

2nd. I might now proceed to equate the different terms of the second member of this equation with those found in § 2. of the analysis of the last Porism, after multiplying the latter by a constant quantity d; but the intimate connexion of the two Porisms will be better exemplified by referring the unknown quantities to another system of rectangular co-ordinates.

Let the angle AOP be bisected by the straight line OY, and let OX be drawn perpendicular to OY. Assuming these two straight lines as the axis of the new co-ordinates x', y', and making the angle AOX $= \gamma$, we shall have the two following equations expressing the values of the old co-ordinates in terms of the new,

$$y = y'\cos\gamma + x'\sin\gamma$$
$$x = y'\sin\gamma - x'\cos\gamma,$$

whence

$$x^2 = y^2\sin^2\gamma + x'^2\cos^2\gamma - 2x'y'\sin\gamma\cos\gamma$$
$$xy = y'^2\sin\gamma\cos\gamma - x'^2\sin\gamma\cos\gamma + x'y'(\sin^2\gamma\cos^2\gamma),$$

but cos AOP $= \sin^2\gamma - \cos^2\gamma = \dfrac{a}{\sqrt{a^2 + b^2}}$, therefore, putting $\sqrt{a^2 + b^2} = m$

$$\sin\gamma = \sqrt{\frac{m + a}{2m}}$$

$$\cos\gamma = \sqrt{\frac{m - a}{2m}}$$

$$\sin\gamma\cos\gamma = \frac{\sqrt{m^2 - a^2}}{2m} = \frac{b}{2m}.$$

By the substitution of these values we find

$$2ax^2 = ay'^2 + \frac{a^2}{m}y'^2 + ax'^2 - \frac{a^2}{m}x'^2 - 2x'y'\,\frac{ab}{m}$$

$$2bxy = \frac{b^2}{m}y'^2 - \frac{b^2}{m}x'^2 + 2x'y'\,\frac{ab}{m}$$

therefore adding and substituting r^2 for $x'^2 + y'^2$

$$2ax + 2bxy = ar^2 + my'^2 - mx'^2$$

or, by taking $r^2 - y'^2$ for x'^2,

$$2ax + 2bxy = -(m-a)r^2 + 2my'^2.$$

In like manner we have

$$2(p^2+a^2)x = 2y'(p^2+a^2)\sin\gamma - 2x'(p^2+a^2)\cos\gamma$$

$$2aby = 2y'ab\cos\gamma + 2x\,ab\sin\gamma,$$

but

$$b\cos\gamma = b\sqrt{\frac{m-a}{2m}} = (m-a)\sqrt{\frac{m+a}{2m}} = (m-a)\sin\gamma.$$

and

$$b\sin\gamma = b\sqrt{\frac{m+a}{2m}} = (m+a)\sqrt{\frac{m-a}{2m}} = (m+a)\cos\gamma.$$

therefore by substituting and making the proper reductions,

$$2(p^2+a^2)x + 2aby = 2x'(am-p^2)\cos\gamma + 2y'(am+p^2)\sin\gamma.$$

By reason of these equations the expression for the sum of the cubes of GH and GK will be transformed into the following

$$\left.\begin{aligned}\frac{\mathrm{GH}^3+\mathrm{GK}^3}{\frac{3}{4}\sqrt{3}} = 2ap^2 - \frac{b^2r^2}{m+a} - 2x'(am-p^2)\cos\gamma \\ - 2y'(am+p^2)\sin\gamma + 2my'^2.\end{aligned}\right\}\ldots\ldots\mathrm{B}.$$

3rd. The equations expressing the conditions of the problem will now be obtained by making the terms of this expression respectively proportional to those found in § 2. of the analysis of the preceding Porism, and as the term in $x'y'$ is again wanting, the angle A, as was the case before, is zero. Hence the four equations of condition (d being a coefficient to be determined):

(I.) $d\cos\mathrm{B} = m$

(II.) $dh(1-\cos\mathrm{B}) = (am-p^2)\cos\gamma$

(III.) $dl(1+\cos\mathrm{B}) = (am+p^2)\sin\gamma$

(IV.) $dl^2(1+\cos\mathrm{B}) + d(h^2+r^2)(1-\cos\mathrm{B}) = 2ap^2-(m-a)r^2.$

The first gives $\cos\mathrm{B} = \dfrac{m}{d}$, whence

$$d(1-\cos\mathrm{B}) = d-m,\quad d(1+\cos\mathrm{B}) = d+m.$$

therefore squaring the second and third and substituting

$$dh^2(1 - \cos \text{B}) = \frac{(am - p^2)^2}{d - m} \cos^2 \gamma,$$

$$dl^2(1 + \cos \text{B}) = \frac{(am + p^2)^2}{d + m} \sin^2 \gamma.$$

By restoring the values of $\cos^2 \gamma$ and $\sin^2 \gamma$ the sum of the terms on the right hand side of these two last equations may be expressed as follows,

$$\frac{d\left\{a^2m + p^4 + 2a^2p^2\right\} - a\left\{2m^2p^2 + a^2m^2 + p^4\right\}}{d^2 - m^2}$$

and by substituting this value of $dh^2(1 - \cos \text{B}) + dl^2(1 + \cos \text{B})$ in equation (IV.) and muliplying by $d^2 - m^2$ we find

$$\begin{aligned} &d\left\{a^2m^2 + p^4 + 2a^2p^2\right\} - a\left\{2m^2p^2 + a^2m^2 + p^4\right\} \\ &+ r^2d^3 - mr^2d^2 - m^2r^2d + m^3r^2 \\ &= (2ap^2 - mr^2 + ar^2)d^2 - m^2(2ap^2 - mr^2 + ar^2); \end{aligned}$$

or, on arranging the terms, and reducing,

$$0 = \begin{cases} r^2d^3 - a(2p^2 + r^2)d^2 + \left\{a^2m^2 + p^4 + 2ap^2 - m^2r^2\right\}d \\ \quad - a\left\{a^2m^2 + p^4 + r^2m^2\right\} \end{cases}$$

an equation which is divisible by $d - a$, and which becomes on performing the division,

$$0 = r^2d^2 - 2ap^2d + m^2(a^2 - r^2) + p^4$$

This being an equation of the second degree shews that the present Porism, like the former, may in general be satisfied in two different ways, and it may be remarked, that it can be satisfied in two ways only; for although the cubic equation in d to which the analysis conducted, may be satisfied by making the factor which was separated, viz. $d - a$ equal to zero, yet the supposition of $d - a = 0$ is inconsistent with the conditions of the problem as will be manifest on substituting a for d in equation (1). The same remark applies to the factor $c^2 + 2ab \cos \beta$ in the analysis of the first Porism.

By transposing and completing the square, and observing that $p^4 - m^2r^2 = \left(\frac{m^2 - r^2}{2}\right)^2$, we have

$$r^2d^2 - 2ap^2d + \frac{a^2p^4}{r^2} = \left(\frac{m^2 - r^2}{2}\right)^2\left(\frac{a^2 - r^2}{r^2}\right),$$

and consequently

$$rd - \frac{ap^2}{r} = \pm \frac{(m^2 - r^2)\sqrt{a^2 - r^2}}{2r},$$

whence, dividing by r, and restoring the value of p^2,

$$d = \frac{a(m^2 + r^2) \pm (m^2 - r^2)\sqrt{a^2 - r^2}}{2r^2}.$$

4th. Let n be a quantity such that $a = \frac{n^2 + r^2}{2n}$; this gives $\sqrt{a^2 - r^2} = \frac{n^2 - r^2}{2n}$, and by the substitution of these values that of d becomes

$$d = \frac{(n^2 + r^2)(m^2 + r^2) \pm (n^2 - r^2)(m^2 - r^2)}{4nr^2}.$$

Placing an accent over d when the lower sign is taken, and also over the corresponding values of h, l, φ in order to distinguish the two cases, we shall have

$$d = \frac{m^2n^2 + r^4}{2nr^2}, \text{ and } d' = \frac{m^2 + n^2}{2n}.$$

The equation $\cos \text{B} = \frac{m}{d}$ will give the directions of the straight lines DE, DF. Because $\text{B} = 2\varphi$, therefore $\cos^2\varphi - \sin^2\phi = \frac{m}{d}$ whence

$$\sin^2\varphi = \frac{d - m}{2d}, \cos^2\varphi = \frac{d + m}{2d}$$

$$\text{tang}\,\varphi = \sqrt{\frac{d - m}{d + m}}.$$

Now $d - m = \frac{(mn - r^2)^2}{2nr^2}$, and $d + m = \frac{(mn + r^2)^2}{2nr^2}$ therefore

$$\text{tang}\,\varphi = \frac{mn - r^2}{mn + r^2}.$$

Also $d' - m = \frac{(n - m)^2}{2n}$, and $d' + m = \frac{(n + m)^2}{2n}$ therefore

$$\text{tang}\,\varphi' = \frac{n - m}{n + m}.$$

To determine the point D, or its co-ordinates h and l, we have

$$h = \frac{am + p^2}{d - m}\cos\gamma$$

$$l = \frac{am + p^2}{d + m}\sin\gamma;$$

but $am = \frac{(n^2+r^2)m}{2n}$, and $p^2 = \frac{m^2+r^2}{2} = \frac{(m^2+r^2)n}{2n}$,
therefore

$$am - p^2 = \frac{(n^2+r^2)m-(m^2+r^2)n}{2n} = \frac{(mn-r^2)(n-m)}{2n},$$

hence

$$\frac{am-p^2}{d-m} = \frac{(n-m)r^2}{mn-r^2},$$

and also

$$\frac{am-p^2}{d'-m} = \frac{mn-r^2}{n-m},$$

consequently

$$h = \frac{(n-m)r^2}{mn-r^2}\cos\gamma, \text{ and } h' = \frac{mn-r^2}{n-m}\cos\gamma.$$

In the same way we find

$$\frac{am+p^2}{d+m} = \frac{(n+m)r^2}{mn+r^2}$$

and also

$$\frac{am+p^2}{d'+m} = \frac{mn+r^2}{n+m}$$

consequently

$$l = \frac{(n+m)r^2}{mn+r^2}\sin\gamma, \text{ and } l' = \frac{mn+r^2}{n+m}\sin\gamma.$$

Now it is evident that if the letters n, m and γ are changed into a, b, and β these values of φ, h, l are identically those found in the former Porism. It is therefore obvious that the former construction will also apply in this case. It is only necessary to produce the line OP to a point Q taken such that OQ $= n$; or, which is the same thing, till PQ is equal to the tangent to the circle from the point P, (for then OP $= \frac{OQ^2+r^2}{2OQ}$); and from the points Q and A to construct the straight lines DE, DF and D′E′, D′F′ exactly in the same manner as they were before constructed from A and B.

It is essential to remark that this Porism is subject to a limitation not adverted to in the enunciation given by its illustrious author. Supposing r to be less than a, the roots of the equation

$$r^2d^2 - 2ap^2d + \frac{a^2p^4}{r^2} = \left(\frac{m^2-r^2}{2}\right)^2\left(\frac{a^2-r^2}{r^2}\right)$$

become impossible; it follows, therefore, that a cannot be less than r, consequently the position of the two straight lines AB, AC is not entirely arbitrary; it must be such that on drawing OP and AP as directed, the point P may not fall *within* the circle. The equation $a = \frac{n^2 + r^2}{2n}$ also gives imaginary values of n when r is less than a; and it implies, moreover, that the point P bisects the distance between two conjugate poles Q and Q′, or two points in the straight line OP taken so that the rectangle QOQ′ may be equal to the square of the semidiameter.

I have given the solutions of these two Porisms independently of each other, but, as is evident from the result of the investigation, the second might have been deduced from the first without having recourse to the four equations of condition. By substituting $\frac{n^2 + r^2}{2n}$ for a in the equation marked B, multiplying by $2n$, and changing n, m, γ into a, b, β respectively, the resulting expression becomes identical with the second member of the equation marked A in the first analysis; whence we infer that $GH^2 + GK^2$, multiplied by a constant quantity, is equal to $QG^2 \times AG^2$.

ARTICLE VII.

An Extension of a Theorem in the doctrine of Central Forces.

By the Rev. Mr. C. Ff. BROMHEAD, *Trinity College, Cambridge.*

THEOREM. If v be the velocity of a body at any point acted upon by any number of forces F, F′, &c. in the same plane, and c, c', &c. be the chords of the circle of curvature drawn through the centres, then will

$$v^2 = \tfrac{1}{2} Fc + \tfrac{1}{2} F'c' + \&c.$$

For, take R the radius of curvature, p and r, p' and r', &c. the perpendiculars upon the tangent, and the distances of the body from each of the centres of force, then we have,

$\frac{v^2}{R}$ = the force which retains the body in the circle of curvature

= the whole quantity of force perpendicular to the tangent of the curve described.

$$= \frac{\mathrm{F}p}{r} + \frac{\mathrm{F}'p'}{r'} + \text{\&c. by resolution of forces}$$

$$= \frac{\mathrm{F}c}{2\mathrm{R}} + \frac{\mathrm{F}'c'}{2\mathrm{R}} + \text{\&c. by similar triangles}$$

$$\therefore v^2 = \tfrac{1}{2}\mathrm{F}c + \tfrac{1}{2}\mathrm{F}'c' + \text{\&c.}$$ *Q. E. D.*

COROLLARY, If the body move in a resisting medium we shall have the same expression for the square of the velocity as in a non-resisting medium, for the retarding force acts in the direction of the body's motion, and in that case there is no chord of curvature.

ARTICLE VIII.

On the Solution of Equations.

By the Rev. Mr. S. HAWKES, *Trinity College, Cambridge.*

1. **PROB.** To ascertain the value of the expression $ax^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ when p is substituted for x,

$$\begin{aligned} \text{Let } p_1 &= ap + a_1 \\ p_2 &= p_1 p + a_2 \\ p_3 &= p_2 p + a_3 \\ \vdots\ & \quad \vdots \\ p_n &= p_{n-1} p + a_n \end{aligned}$$

Let the $n-1$ quantities p_1, p_2, … p_{n-1} be exterminated from these n equations, and the resulting equation will be

$$p_n = ap^n + a_1 p^{n-1} + \ldots + a_{n-1} p + a_n$$

i. e. p_n is the value of the given expression when p is substituted for x. *Q. E. I.*

2. *Schol.* If the value to be given to x is negative, as $-p$, then p may be substituted after changing the alternate signs; and p_n or $-p_n$ will be the value required, according as n is an even or an odd number.

3. *Cor.* If $p_n = 0$ then is p a root of the equation $ax^n +$ &c. $= 0$; and conversely.

4. **THEOR.** $a, p_1, p_2, \ldots p_{n-1}$ are the coefficients of the quotient and p_n is the remainder when $ax^n + a_1x^{n-1} + \ldots + a_n$ is divided by $x - p$.

For

$$(x-p)\left\{ax^{n-1}+p_1x^{n-2}+\ldots+p_{n-2}x+p_{n-1}\right\}$$

$$=ax^n+p_1x^{n-1}+\ldots+p_{n-1}x$$

$$-apx^{n-1}-\ldots-p_{n-2}px-p_{n-1}p$$

$$=ax^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n-p_n$$

(Since $p_1-ap=a_1,\ldots p_{n-1}-p_{n-2}p=a_{n-1}$, and $-p_{n-1}p=a_n-p_n$)

Therefore, by transposition,

$$ax^n+a_1x^{n-1}+\ldots+a_n=(x-p)\left\{ax^{n-1}+p_1x^{n-2}+\ldots+p_{n-1}\right\}+p_n$$

therefore $\left\{ax^n+a_1x^{n-1}+\ldots+a_n\right\}\div(x-p)$

$$=ax^{n-1}+p_1x^{n-2}\ldots+p_{n-1}+\frac{p_n}{x-p}$$

that is $ax^{n-1}+p_1x^{n-2}+\ldots+p_{n-1}$ is the quotient and p_n is the remainder. *Q. E. D.*

5. *Cor* If p is a root of the equation ax^n+ &c. $=0$ then is the expression ax^n+ &c. divisible by $x-p$ without remainder. For, in this case, the remainder $p_n=0$. *Q. E. D.*

6. *Def.* For the convenience of reference, the equation $ax^n+a_1x^{n-1}+$ &c. $=0$, the expression $ax^n+a_1x^{n-1}+$ &c. and the coefficients a, a_1, a_2 &c. may be called *primitive*; and the equation $ax^{n-1}+p_1x^{n-2}+$ &c. $=0$, the expression $ax^{n-1}+p_1x^{n-2}+$ &c. and the coefficients a, p_1, p_2 &c. may be called *reduced.*

7. *Theor.* If the primitive coefficients are all integral, and any one of the reduced coefficients is fractional, p_n is also a fraction.

Let p_m be the first of the reduced coefficients which is fractional; then its value $pp_{m-1}+a_m$ is also fractional; and, p_{m-1} and a_m being integers, p is a fraction, and its denominator is either the same with that of p_m, or is a multiple of it: therefore pp_m is also fractional; and so, consequently, is pp_m+a_{m-1}, or p_{m+1} that is the *next*, and therefore similarly, *every* subsequent reduced coefficient is fractional; and therefore p_n must also be a fraction. *Q. E. D.*

8. *Cor.* 1. Hence it appears, that if the equation $ax^n+a_1x^{n-1}$ + &c. $=0$ have only integral coefficients; and, during the substitution of p for x a fractional reduced coefficient occurs, then p will not be a root of the equation.

For, the ultimate result, instead of being zero, will be a finite fraction.

9. *Cor.* 2. If the equation $x^n + a_1x^{n-1} + \&c. = 0$ have its coefficients all integral it cannot have a fractional root.

For in this case, since $a = 1$, p_1 is equal to $p + a_1$ and is therefore fractional if p is a fraction, and consequently p will not then be a root.

10. *Cor.* 3. If the equation $ax^n + a_1x^{n-1} + \&c. = 0$, with integral coefficients, have a fractional root p; then is the denominator of the root a divisor of the first coefficient a.

For otherwise ap would not be integral, and therefore p_1 which is equal to $ap + a_1$ would be a fraction, and p could not be a root.

11. *Cor.* 4. If p is a root of the equation $ax^n + a_1x^{n-1} + \&c. = 0$ which has all the coefficients integral, then, if p is integral, it is a divisor of the last term a_n, and if it is fractional, its numerator is a divisor of a_n.

For in both cases p_{n-1} is integral; but since $p_n = 0$ therefore $pp_{n-1} + a_n = 0$, therefore $a_n \div p = -p_{n-1}$ an integer.

12. *Rule.* The rule of operation derived from the preceding reasonings is this:

To find the value of $ax^n + a_1x^{n-1} + \&c.$ for an assigned value of x.

(1.) Set down the expression with all its terms.

(2.) Multiply the first coefficient by the value assigned to x, and add the product to the second coefficient.

(3.) Multiply the sum by the value assigned to x, and add the product to the third coefficient.

(4.) Continue the process till all the coefficients are employed. The last result is the required value of the expression.

13. *Schol.* When one or more terms are wanting, it may sometimes, instead of employing them with the coefficient zero, be more convenient to multiply at once the reduced coefficient preceding the hiatus by that power of p the index of which is the difference of the indices of x before and after the hiatus, and to add the product to the primitive coefficient following the hiatus, for the next reduced coefficient.

14. **Example 1.** Required the value of

$$\begin{array}{llllll} 2x^5 - & 13x^4 - & 25x^3 + & 8x^2 + & 87x + & 73 \quad \text{when } x = 7 \\ \hline 2 & 1 & -18 & -118 & -739 & -5100 \\ 14 & 7 & -126 & -826 & -5173 & \end{array}$$

Here, under the first term, is set down the first coefficient 2, under 2 is set down 2 × 7 or 14; under the second term is set down the sum of 14 and — 13, or 1, and under that 1 × 7 or 7; under the third term the sum of 7 and — 25, or 18, and under that — 18 × 7 or — 126, &c. till at last — 5100, which falls under the last term is attained; and that is the value of the expression when $x = 7$.

Example (2). To find the value when $x = -3$.

Set the expression down with its alternate signs changed

$$\begin{array}{lcccccc} \text{thus} & 2x^5 + & 13x^4 - & 25x^3 - & 8x^2 + & 87x - & 73 \\ \text{and substitute 3 thus} & 2 & 19 & 32 & 88 & 351 & 980 \\ & 6 & 57 & 96 & 264 & 1053 & \end{array}$$

The last result is 980, which, having its sign changed because 5 the highest index of x is an odd number, gives — 980 for the value required.

15. **Example** 3. From the first example and article 4, it results that $\{2x^5 - 13x^4 - 25x^3 + 8x^2 + 87x + 73\} \div (x - 7)$

$$= 2x^4 + x^3 - 18x^2 - 118x - 739 - \frac{5100}{x - 7}.$$

The coefficients of the quotient being the numbers placed next under the terms of the primitive expression in Example (1).

Example (4). The process in Example (2) supposed x changed into $-x$, therefore, to rechange $-x$ into x, the signs of the alternate terms in the quotient must be changed, which will give $\{2x^5 - 13x^4 - 25x^3 + 8x^2 + 87x + 73\} \div (x + 3)$

$$= 2x^4 - 19x^3 + 32x^2 - 88x + 351 - \frac{980}{x + 3}.$$

16. *General Scholium.* The applications of the method of substitution given in article 12, are

(1.) To ascertain the value of an expression ax^n + &c. for a given value of x.

(2). To divide such an expression by a given simple expression.

(3). To find limits between the roots of an equation.

(4). To discover, by trial, the roots of an equation.

The advantage of the method in its first three applications will best become known by trial. In its second application it admits easily of extension to *dividing by a polynomial expression.*

In its fourth application, although extending to the investigation of incommensurable roots, it is principally of use for finding commensurable roots whether integral or not; for which it is general.

The facility with which it makes any proposed subtitution for x is such, that unless the number of divisors of the last term is very great, it will be needless to limit the number of trials by Sir Isaac Newton's method: and, even when it is found expedient to exclude before trial, divisors which are not roots, the ease with which *several* results may be obtained, corresponding to different values of x, will facilitate the use of more extended arithmetical progressions, by which more of the divisors which are not roots may be excluded than when 1, 0, — 1 alone are substituted.

The advantage of this method, besides that of the *facility* of its application, are

(1). That it extends to fractional roots with even greater ease than to integral, both because the numbers occurring in the course of substitution will be smaller; and, because if the fraction which is being substituted is not a root, that fact will generally soon be declared by the occurrence of a fractional reduced coefficient.

(2). That the process of substitution by which a root is discovered, gives, at the same time, the coefficients of the reduced equation which contains the other roots.

The method, and its advantages will best be learnt from examples.

17. EXAMPLE 5. Proposed for solution the equation

$$x^6 - 20x^5 + 63x^4 - 53x^3 - 5x^2 + 48x - 84 = 0.$$

Trials	x^6	$\pm$	$20x^5$	$+$	$63x^4$	$\pm$	$53x^3$	$-$	$5x^2$	$\mp$	$48x$	-84	*Roots*
1	1	$-$	19		44	$-$	9	$-$	14		34	-50	
-1	(1	$\pm$	21	$+$	84	$\pm$	137	$+$	132	$\pm$	84)	0	-1
-1	1		22		106		243		375		459		
2	(1	$\pm$	19	$+$	46	$\pm$	45	$+$	42		0)		2
	2	$-$	38		92	$-$	90		84				
2	(1	$\pm$	17	$+$	12	$\pm$	21)		0				2
	2	$-$	34		24	$-$	42						
3	1	$-$	14	$-$	30	$-$	111						
	3	$-$	42	$-$	90								
7	1	$-$	10	$-$	58	$-$	427						
	7	$-$	70	$-$	406								
21	1		4		96		1995						
	21		84		2016								

Therefore the only integral roots are $-1, 2, 2$, the remaining roots are those of the equation

$$x^3 - 17x^2 + 12x - 21 = 0.$$

18. *Observations on Example 5.*

(1). Since negative as well as positive substitutions are requisite, the equation is prepared by writing above its alternate signs their opposites; the upper set of signs being used for the negative substitutions, the lower for the positive.

(2). When ± 1 is being substituted, one row of numbers is sufficient for the process.

(3). Immediately that a root is found the row of reduced coefficients is enclosed in parentheses for convenience of the eye in making the future subtitutions in it instead of in the primitive equation. Also the changed signs are placed over or under the alternate terms, according as the root found was positive or negative; that the lower signs may, as at first, always be employed in making the positive substitutions, the upper always in making the negative.

(4). Unless the root found is not a divisor of the last term of the reduced expression, it is made the first substitution, because there may be equal roots.

(5). In this example no negative substitution is tried after the discovery of the root -1; for the expression when prepared for negative subtitutions has then all its terms positive and consequently the reduced coefficients will go on encreasing to the end.

Indeed, in every substitution, directly that a reduced coefficient is obtained which has the same sign with all the following primitive coefficients, that substitution need be carried on no farther; nor need it, when the value being substituted exceeds unity, even if there should follow some coefficients of a different sign, provided that they do not together amount to the reduced coefficient obtained; for, if unity were being substituted the result would exceed zero, and, *a fortiori*, it cannot be zero when a number greater than zero is being substituted. This remark will be useful.

19. EXAMPLE 6.

Trials	x^7	$35x^6$	$+334x^5$	$\pm 362x^4$	$-8591x^3$	$\mp 33805x^2$	$-9600x$	± 72000	*Roots*
1	1	—34	300	—66	—8653	25152	15552	—56448	
—1	1	36	370	732	—7859	—41664	—51264	—20736	

Trials		*Roots*
2	1 — 33 268 174 — 8243 17319 25038 — 21024 2 — 66 536 348 — 16486 34638 50076	
—2	1 37 408 1178 — 6235 — 46275 — 102150 — 132300 2 74 816 2356 — 12470 — 92550 — 204300	
3	(1 ± 32 + 238 ∓ 352 — 7535 ∓ 11200 + 24000) 0 3 — 96 714 1056 — 22605 33600 72000	3
3	1 — 29 151 805 — 5120 — 4160 11520 3 — 87 453 2415 — 15360 — 12480	
—3	1 35 343 677 — 5504 — 27712 — 59136 3 105 1029 2031 — 16512 — 83136	
4	1 — 28 126 856 — 4111 — 5244 3024 4 — 112 504 3424 — 16444 — 20976	
—4	1 36 382 1176 — 2831 — 22524 — 66096 4 144 1528 4704 — 11324 — 90096	
5	(1 ± 27 + 103 ∓ 867 — 3200 ± 4800) 0 5 — 135 515 4335 — 16000 — 24000	5
5	(1 ± 22 — 7 ∓ 832 + 960) 0 5 — 110 — 35 4160 4800	5
5	1 — 17 — 92 — 372 — 900 5 — 85 — 460 — 1860	
—5	(1 ± 27 + 128 ∓ 192) 0 5 135 640 — 960	— 5
—5	1 32 288 1248 5 160 1440	
6	1 — 21 2 204 6 — 126 12	
—6	1 33 P 6 198	
7	1 — 20 — 12 104 7 — 140 — 84	
—7	1 34 P 7 238	
8	(1 ± 19 — 24) 0 8 — 152 — 192	8

and ± 1, ± 2, ± 3 being none of them roots of the quadratic $x^2 - 19x + 24 = 0$, neither can ± 24, ± 12, ± 8; therefore the only integral roots of the equation are 3, 5, 5, —5, 8. The other two roots being those of the quadratic $x^2 - 19x + 24$.

20. *Schol.* When a substitution is terminated abruptly, the cause is denoted by a P, if, according to the remark at the end of article 18, the result must be positive, or by an N if it must be negative, and by an F if the substitution is stopped by the occurrence of a fraction.

21. EXAMPLE 7.

Trials	$42x^5 \pm 365x^4 + 237x^3 \mp 1738x^2 + 639x \pm 455$						*Roots*
1	42	— 323	— 86	1652	P		
—1	(42	± 407	+ 644	∓ 1094	— 455	0)	—1
$\frac{1}{2}$	42 21	— 386 — 193	451	F			
$-\frac{1}{2}$	42 21	428 214	858 429	— 665	F		
$\frac{1}{3}$	42 14	— 393 — 131	513 171	1265	F		
$-\frac{1}{3}$	42 14	421	F				
$\frac{1}{6}$	42 7	— 400	F				
$-\frac{1}{6}$	42 7	414 69	713	F			
$\frac{1}{7}$	42 6	— 401	F				
$-\frac{1}{7}$	42 6	413 59	703	F			
$\frac{1}{14}$	42 3	— 404	F				
$-\frac{1}{14}$	42 3	410	F				
$\frac{1}{21}$	42 2	— 405	F				

Trials						*Roots*
$-\frac{1}{21}$	42	409	F			
	2					
$\frac{1}{42}$	42 —	406	F			
	1					
$-\frac{1}{42}$	42	408	F			
	1					
5	42 —	197 —	341	N		
	210 —	985 —	1705			
-5	42	617	P			
	210	3085				
$\frac{5}{2}$	42 —	302 —	111	F		
	105 —	755				
$-\frac{5}{2}$	42	512	1924	3716	8835	
	105	1280	4810	9290		
$\frac{5}{3}$	42 —	337	F			
	70					
$-\frac{5}{3}$	42	477		F		
	70					
$\frac{5}{6}$	42 —	372	334	F		
	35 —	310				
$-\frac{5}{6}$	42	442	F			
	35					
$\frac{5}{7}$	42 —	377	F			
	30					
$-\frac{5}{7}$	42	437	F			
	30					
$\frac{5}{14}$	42 —	392	504	1274	0	$\frac{5}{14}$
	15 —	140	180	455		

The lower row divided by 5 gives for the reduced equation $3x^3 \pm 28x^2 + 36x \mp 91 = 0$.

Trials	$3x^3 \pm 28x^2 + 36x \mp 91$				*Roots*
7	3 —	7 —	13	0	7
	21 —	49 —	91		

So the roots are -1, $\frac{5}{14}$, and 7, the remaining ones being those of the quadratic $3x^2 - 7x - 13 = 0$.

22. EXAMPLE 8.

Trials	$60x^7\mp1301x^6+775x^5\pm1673x^4+2785x^3\pm625x^2-332x\mp14=0$	*Roots*
1	60 1361 2136 464 3249 P	
− 1	60 — 1241 — 466 1206 P	
$\frac{1}{2}$	60 1331 F 30	
—$\frac{1}{2}$	60 — 1271 F 30	
$\frac{1}{3}$	60 1321 F 20	
—$\frac{1}{3}$	60 — 1281 348 — 1788 3381 1752 252 0 (20 ∓ 427 + 116 ± 596 1127 ± 584 + 84)	—$\frac{1}{3}$
$\frac{1}{4}$	20 432 224 — 540 992 — 336 0 (5 ∓ 108 + 56 ± 135 + 248 ± 84)	$\frac{1}{4}$
$\frac{1}{5}$	5 109 F 1	
− $\frac{1}{5}$	5 — 107 F 1	
2	5 118 P 10 236	
—2	(5 ∓ 98 — 140 ∓ 145 — 42) 0 10 — 196 — 280 — 290 — 84	—2
—2	5 — 88 N 10	
$\frac{2}{5}$	5 100 — 100 105 0 2 40 — 40 42	$\frac{2}{5}$
	Dividing the lower by 2 ($x^3\mp20x^2-20x\mp21=0$)	
3	1 23 P 3	
—3	1 — 17 N 3	
7	1 27 P 7	

Trials		*Roots*
−7	$\frac{1}{7} - 13$ N	
21	$\frac{1}{21}$ P	
−21	$\left(\frac{1}{21} \pm \frac{1}{21} + \frac{1}{21}\right)$. 0	−21

Therefore the commensurable roots are $\frac{1}{3}$, $\frac{1}{4}$, -2, $\frac{2}{5}$ and -21, the remaining two roots being those of the quadratic $x^2 - x + 1 = 0$.

ARTICLE IX.

Addition to Article VI.

By Thomas Galloway, *A. M. Royal Military College.*

Before proceeding with the demonstration of the remaining Porisms proposed by Dr. Stewart, I shall make one or two observations on what has been done in Article VI.

1st. The two points D and D′ (see the first figure in that Article) may be determined in a manner which renders their construction somewhat more elegant in a geometrical point of view. On CA take CA″ a third proportional to CA and the radius, and on CB take CB″ a third proportional to CB and the radius: join AB″, A″B; the intersection of these two lines will give the point D, and the intersection of the straight lines passing through AB and A″B″ will determine the point D′. For, denoting CA″ by a' and CB″ by b', the equation of the straight line which passes through the points A and B″, the co-ordinates of which are respectively $a \cos \beta$, $a \sin \beta$ and $b' \cos \beta$, $b' \sin \beta$ will be

$$y - b' \sin \beta = \frac{(a - b') \sin \beta}{(a + b') \cos \beta} (x + b' \cos \beta),$$

and, similarly, the equation of A″B will be

$$y - a' \sin \beta = \frac{(b - a') \sin \beta}{(b + a') \cos \beta} (x - a' \cos \beta).$$

At the point D these equations will be both true at the same time; therefore eliminating x and putting $\frac{r^2}{a}$, $\frac{r^2}{b}$ instead of a' and b', we find for the co-ordinates of D,

$$x = \text{CV} = \frac{a-b}{ab-r^2}\cos\beta, \quad y = \text{DV} = \frac{a+b}{ab+r^2}\sin\beta,$$

the same values which were deduced from the equations expressing the conditions of the problem. The co-ordinates of D′ are found exactly in the same manner.

It is easy to see from this construction that D is the identical point determined by Mr. Ivory in his investigation of another of these Porisms (See the First Volume of the Repository); and and it is singular that Dr. Stewart in enunciating that proposition should have limited its position to the interior of the circle. Mr. Ivory's method of solution will apply to both cases with equal facility.

The angle φ, or the inclination of the straight lines DE, DF to the axis, being such that $\tan\varphi = \frac{ab-r^2}{ab+r^2} = \frac{a-b'}{a+b'}$; and the tangent of the angle made by the line AB″ or A″B with the axis being $\frac{a-b'}{a+b'}\tan\beta$, we shall have, on denoting this last angle by α, the equation $\tan\varphi\tan\alpha = \tan\beta$, which furnishes a very simple method of constructing the lines DE, DF.

2nd. The reduction of the second Porism to the first may be effected in the following manner:

Suppose G, A (fig. 2nd.) to be joined, and let the angle GAA′ be denoted by ψ; we shall have

$$\text{GH} = \text{AG}\sin(60^\circ - \psi) = \text{AG}\left(\tfrac{1}{2}\sqrt{3}\cos\psi - \tfrac{1}{2}\sin\psi\right),$$
$$\text{GK} = \text{AG}\sin(60^\circ + \psi) = \text{AG}\left(\tfrac{1}{2}\sqrt{3}\cos\psi + \tfrac{1}{2}\sin\psi\right);$$

therefore, on taking the cubes and adding,

$$\text{GH}^3 + \text{GK}^3 = \text{AG}^3\tfrac{3}{4}\left(\sqrt{3}\cos^3\psi + \sqrt{3}\cos\psi\sin^2\psi\right) = \tfrac{3}{4}\sqrt{3}\,\text{AG}^3\cos\psi.$$

Let us now suppose a straight line to be drawn through G perpendicular to AA′ or OQ, to meet OQ in M; then $\text{AG}\cos\psi = \text{PM}$, whence, on multiplying the above equation by OQ,

$$(\text{GH}^3 + \text{GK}^3)\,.\,\text{OQ} = \tfrac{3}{4}\sqrt{3}\,\text{AG}^2\,.\,\text{PM}\,.\,\text{OQ}.$$

Now since the rectangle OQ, QQ′ is equal to the square of the radius, and P bisects QQ′, it is a well known property of the circle, and has been demonstrated by Dr. Stewart both in his "General Theorems" and "Tracts," that PM . OQ = QG², consequently

$$(\text{GH}^3 + \text{GK}^3)\,.\,\text{OQ} = \tfrac{3}{4}\sqrt{3}\,\text{AG}^2\,.\,\text{QG}^2.$$

Although this demonstration of the Porism may appear very simple compared with that given in Art. VI. Yet as it is obviously applicable only in the case when P is withont the circle, and as no such limitation is mentioned in Dr. Stewart's enunciation, it was desirable to have a direct investigation, independently of the other proposition; otherwise it might have been matter of doubt whether a system of straight lines, satisfying the specified conditions, could not have been determined in some other manner, when the point P, or even A, is situated *within* the circle. The analytic investigation sets this question at rest.

Porism III. (See Fig. 3.)

"Let there be a circle given by position, and let there be two straight lines AB, AC given by position; and let a, b be two given magnitudes: a point D may be found, such, that if through the point D there be drawn any right line meeting the circle in E, F, and from the point E there be drawn EG, EH perpendicular to AB, AC, and from the point F there be drawn likewise FK, FL perpendicular to AB, AC, the square of EG together with the space to which the square of EH has the same ratio that a has to b will be to the square of FK together with the space to which the square of FL has the same ratio that a has to b, as the square of ED to the square of DF."

In order to connect this Porism with the first in Article VI. it will be convenient to premise the following Lemmas:

1st. Let AB, AC be two straight lines making given angles with the axis AX. Through any point o in AX draw a perpendicular to AX meeting AB in m and AC in n; take op and oq each equal to a mean proportional between om and on; through A, p and A, q draw AD and AE, and from any point P let there be drawn PB, PC, PD, PE respectively perpendicular to AB, AC, AD, AE; then $\text{PB}^2 + \frac{a}{b}\text{PC}^2 = \frac{a+b}{2b}(\text{PD}^2 + \text{PE}^2)$ where $a : b$ is a given ratio.

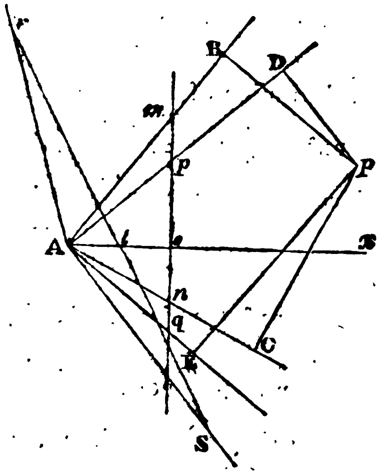


Fig. I.

Let the angle BAX = α, CAX = β, and DAX or EAX = γ. Then, (x and y being the co-ordinates of P,)

$$\text{PB} = x \sin\alpha - y \cos\alpha,$$
$$\text{PC} = x \sin\beta + y \cos\beta,$$

whence

$$\text{PB}^2 = x^2 \sin^2\alpha + y^2 \cos^2\alpha - 2xy \sin\alpha \cos\alpha,$$
$$\text{PC}^2 = x^2 \sin^2\beta + y^2 \cos^2\beta + 2xy \sin\beta \cos\beta.$$

On multiplying the second of these two last equations by $\frac{\sin\alpha \cos\alpha}{\sin\beta \cos\beta}$, adding them together, and observing that $\sin\alpha \cos\beta + \cos\alpha \sin\beta = \sin(\alpha + \beta)$, we shall have

$$\text{PB}^2 + \frac{\sin\alpha \cos\alpha}{\sin\beta \cos\beta} \cdot \text{PC}^2 = \frac{\sin(\alpha+\beta)}{\sin\beta \cos\beta} \cdot (x^2 \sin\alpha \sin\beta + y^2 \cos\alpha \cos\beta).$$

Now, since by construction $op^2 = om \cdot on$, that is $\tan^2\gamma = \tan\alpha \tan\beta$, we shall have also $\cos^2\gamma = \frac{\cos\alpha \cos\beta}{\cos(\alpha - \beta)}$, $\sin^2\gamma = \frac{\sin\alpha \sin\beta}{\cos(\alpha - \beta)}$, therefore by substituting and reducing,

$$\text{PB}^2 + \frac{\sin\alpha \cos\alpha}{\sin\beta \cos\beta} \cdot \text{PC}^2 = \frac{\sin(\alpha+\beta) \cos(\alpha-\beta)}{\sin\beta \cos\beta} (x^2 \sin^2\gamma + y^2 \cos^2\gamma).$$

Let us now put $\frac{a}{b} = \frac{\sin\alpha \cos\alpha}{\sin\beta \cos\beta}$. This gives $\frac{a+b}{b} = \frac{\sin\alpha \cos\alpha + \sin\beta \cos\beta}{\sin\beta \cos\beta} = \frac{\sin(\alpha+\beta) \cos(\alpha-\beta)}{\sin\beta \cos\beta}$; therefore by the substitution of these values the above equation becomes

$$\text{PB}^2 + \frac{a}{b}\,\text{PC}^2 = \frac{a+b}{b}\,(x^2 \sin^2\gamma + y^2 \cos^2\gamma),$$

Now since DAX = EAX = γ, we have farther

$$\text{PD} = x \sin\gamma - y \cos\gamma$$
$$\text{PE} = x \sin\gamma + y \cos\gamma$$

therefore

$$\text{PD}^2 + \text{PE}^2 = 2(x^2 \sin^2\gamma + y^2 \cos^2\gamma).$$

and consequently

$$\text{PB}^2 + \frac{a}{b}\,\text{PC}^2 = \frac{a+b}{2b}\,(\text{PD}^2 + \text{PE}^2).$$

Because $2 \sin\alpha \cos\alpha = \sin 2\alpha$, and $2 \sin\beta \cos\beta = \sin 2\beta$, therefore $\frac{a}{b} = \frac{\sin 2\alpha}{\sin 2\beta}$. Through A draw A$r$ making an angle XAr = ½ XAB, and As making an angle XAs = ½ XAC, make Ar = As, and let rs meet AX in t. Then $\frac{rt}{st} = \frac{\sin \frac{1}{2}\alpha}{\sin \frac{1}{2}\beta} = \frac{a}{b}$; hence, when the straight lines AB, AC, and the ratio of a to b is given, and it is required to determine AD and AE, it is only necessary to construct an isosceles triangle rAs, having the angle at the vertex double of the given angle BAC, and to divide the base into two parts in t so that $rt : st :: a : b$, the point t will determine the position of the axis AX, and consequently that of the straight lines AD and AE.

2d. Let AB, AC be two straight lines at right angles to each other, and let P be a given point. It is required to draw a straight line through P, meeting AB and AC in B and C so that the difference between the squares of PB and PC shall be equal to a given space.

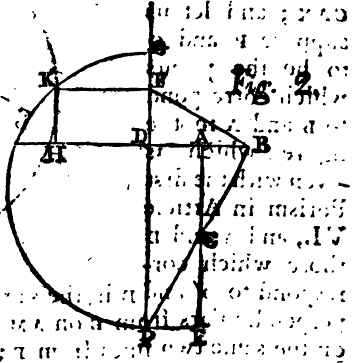


Fig. 2.

Through P draw PD and PE perpendicular to AB and AC, and through B draw BF perpendicular to PB meeting PD in F. Let PD = l, PE = h, PF = x, and let r^2 represent the given space. We have then

$$PB^2 = lx, \quad DB^2 = (x - l)l,$$

and by similar triangles

$$PC^2 : PE^2 :: PB^2 : BD^2$$

that is $$PC^2 : h^2 :: lx : (x - l)l,$$

therefore $$PC^2 = \frac{h^2x}{x - l},$$

consequently

$$PB^2 - PC^2 = r^2 = lx - \frac{h^2x}{x - l}$$

whence $$r^2x - r^2l = lx^2 - l^2x - h^2x$$

and $$x\left\{\frac{r^2 + l^2 + h^2}{l} - x\right\} = r^2.$$

This equation gives the following construction. On PD take PG equal to $\frac{r^2 + l^2 + h^2}{l}$, on PG describe a semicircle, make

DH = r, draw HK perpendicular to DB, meeting the semicircle in K, and draw KF parallel to DB. By this means the point F is determined, and the intersection of another semicircle described on PF with the line AB will determine the point B.

To proceed with the investigation of the present Porism, let AX be drawn as directed in the First Lemma, so that sin 2BAX : sin 2CAX :: a : b, and AM and AN so that the tangent of MAX or NAX shall be a mean proportional between the tangents of BAX CAX; and let us suppose P and Q to be the points which correspond to B and A in the figure which is given with the first Porism in Article VI., and A and D those which correspond to D′ and D in the same figure. Let m and m' represent perpendiculars from E on AM and AN, and n, n' perpendiculars on the same two lines from F; and join PE, QE, PF and QF.

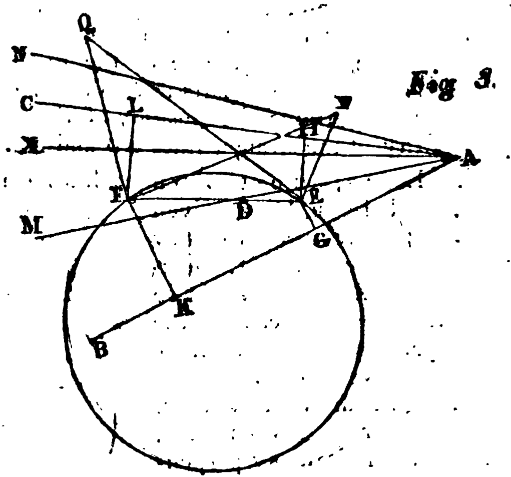


Fig. 3.

In consequence of the property demonstrated in the first Lemma, we have

$$EG^2 + \frac{a}{b} EH^2 : FK^2 + \frac{a}{b} FL^2 :: m^2 + m'^2 : n^2 + n'^2.$$

and it has been demonstrated in the first Porism in Art. VI. that

$$m^2 + m'^2 : n^2 + n'^2 :: PE^2 \times QE^2 : PF^2 \times QF^2;$$

it has also been shewn by Mr. Ivory in the first volume of the Repository, that

$$PE^2 \times QE^2 : PF^2 \times QF^2 :: DE^2 : DF^2,$$

therefore

$$EG^2 + \frac{a}{b} EH^2 : FK^2 + \frac{a}{b} FL^2 :: DE^2 : DF^2,$$

which is the property enunciated.

It is obvious from what has already been demonstrated that the same proposition is true of any straight line drawn through A as well as through D.

It now remains only to shew in what manner the points P and

Q, and thence D, may be found when the circle and the two straight lines AM, AN, are given by position.

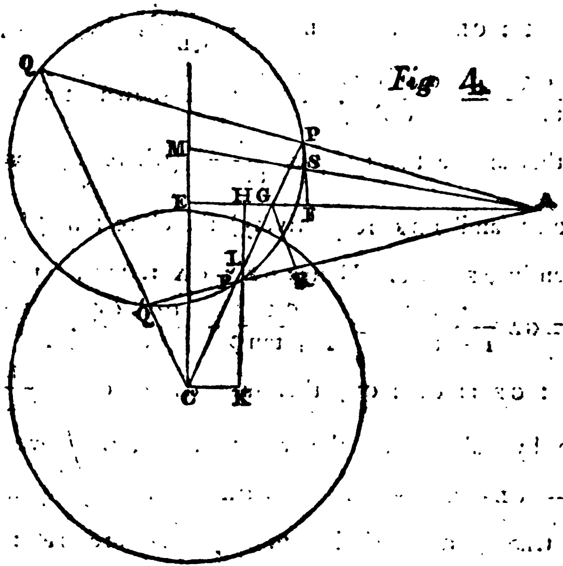


Fig 4.

For this purpose let us suppose the thing done; and that P′ and Q′ are points, such, that (C being the centre of the given circle, and r its radius) $CP' = \frac{r^2}{CP}$, and $CQ' = \frac{r^2}{BQ}$.

Let φ represent the given angle EAM, then we know from the solution of the first Porism that $\tan\varphi = \frac{CQ - CP}{CQ + CP}$; whence $\frac{CP}{CQ} = \frac{1 - \tan\varphi}{1 + \tan\varphi}$; consequently the problem amounts to this: Having given the points A and C and the straight lines AE and CE (the perpendicular to AE), it is required to find a quadrilateral figure PQ Q′P′ inscribable within a circle, and such that $CP \cdot CP' = r^2$, that CP is to CQ in a given ratio, and that its opposite sides shall intersect each other in A and C, and make equal angles respectively with AE and CE.

Let AE and AM intersect CE in E and M, draw PF perpendicular to AE, make EH equal to EM, draw HK parallel to EC and CK parallel to AE meeting HK in K, let G and L be the points in which CP meets AE and CG, draw GR parallel to CQ, and let AM intersect PR in S.

Because EH or CK = EM, therefore EA = $\frac{\text{CK}}{\tan\varphi}$; but EG : GC :: CK : CL, therefore EG = $\frac{\text{GC . CK}}{\text{CL}}$, and consequently AG = $\frac{\text{CK}}{\tan\varphi} - \frac{\text{GC . CK}}{\text{CL}}$. Now it was shewn at the beginning of this article that $\tan\varphi = \frac{\tan\alpha}{\tan\beta}$, (where α, β denote the angles PAF and PGF respectively) therefore $\tan\varphi = \frac{\text{FG}}{\text{FA}}$, consequently GF = FS. Hence GF : GA :: $\tan\varphi$: $1 + \tan\varphi$, and GF = GA $\frac{\tan\varphi}{1+\tan\varphi} = \frac{\text{CK}}{1+\tan\varphi} - \frac{\text{GC . CK}}{\text{CL}} \cdot \frac{\tan\varphi}{1+\tan\varphi}$. But GP : GF :: CL : CK, therefore GP = GF . $\frac{\text{CL}}{\text{CK}}$, and therefore by substituting the value of GF, we have GP = $\frac{1}{1+\tan\varphi}(\text{CL} - \text{GC}\tan\varphi)$. Now GR is evidently equal to GP, and the triangle GRP′ similar to CPQ, therefore GP′ : GP :: $1 + \tan\varphi$: $1 - \tan\varphi$, whence GP′ = $\frac{1}{1-\tan\varphi}(\text{CL} - \text{GC}\tan\varphi)$. From this we have

$$\text{CP} = \text{CG} + \text{GP} = \frac{1}{1+\tan\varphi}(\text{CG} + \text{CL})$$

$$\text{CP}' = \text{CG} - \text{GP}' = \frac{1}{1-\tan\varphi}(\text{CG} - \text{CL})$$

therefore $\text{CP} \times \text{CP}' = r^2 = \frac{1}{1-\tan^2\varphi}(\text{CG}^2 - \text{CL}^2)$.

From this equation it appears that the position of CP will be determined by drawing CG as directed in the 2d Lemma, so that $\text{CG}^2 - \text{CL}^2 = r^2(1 - \tan^2\varphi)$; and the point G being determined, the remainder of the construction is obvious, on account of GF = FS. The point D is the intersection of the two straight lines which join P, Q′ and Q, P′.

PORISM IV.

Let there be a circle given by position, and AB, AC two straight lines given by position, and let the angle BAC be equal to two angles of an equilateral triangle, and let the circle be contained within the angle BAC; a point D may be found, such, that if through the point D there be drawn any line meeting the circle in E, F, and from the point E there be drawn EG, EH perpendicular to AB, AC, and likewise there be drawn FK, FL perpendicular to AB, AC, the sum of the cubes of EG, EH will be to the sum of the cubes of FK, FL as the square of DE to the square of DF."

This Porism is an obvious consequence of the second which was demonstrated in Article VI. and that in the First Volume, to which we have already alluded. For, having found the point Q as in the second Porism, and thence the point D, we shall have by reason of the property there demonstrated,

$$\mathrm{EG}^3 + \mathrm{EH}^3 : \mathrm{FK}^3 + \mathrm{FL}^3 :: \mathrm{QE}^2 \times \mathrm{AE}^2 : \mathrm{QF}^2 \times \mathrm{AF}^2$$

but, as was demonstrated by Mr. Ivory,

$$\mathrm{QE}^2 \times \mathrm{AE}^2 : \mathrm{QF}^2 \times \mathrm{AF}^2 :: \mathrm{DE}^2 : \mathrm{DF}^2$$

therefore,

$$\mathrm{EG}^3 + \mathrm{EH}^3 : \mathrm{FK}^3 + \mathrm{FL}^3 :: \mathrm{DE}^2 : \mathrm{DF}^2,$$

This is the last of the five Porisms announced by Dr. Stewart at the conclusion of the work to which we referred in the beginning of Art. VI. They are all intimately connected with each other, and their investigation discloses some highly curious properties of the circle and its inscribed quadrilateral figures. Elegant demonstrations of many of the "General Theorems" have been given by Dr. Small and Mr. Glenie in the Edinburgh Transactions; and also by Mr. Babbage in the first number of Brande's Journal of Science.

⁂ *Erratum.*—Page 104, line 17, *for* $\tan\phi \tan\alpha = \tan\beta$, *read* $\tan\phi \tan\beta = \tan\alpha$.

ARTICLE X.

Researches in the Geometry of Three Dimensions.

By Mr. DAVIES, *Bath.*

Section the First. *On the Contacts of the Sphere, Cylinder, Cone, and Wedge.*

Several foreign mathematicians have considered and resolved the Apollonian Problem of Tangencies, and its kindred one of Tangent-spheres. Solutions of the former on the supposed method of the Geometer of Perga, had been given by Marinus and Vieta, and the latter had been solved by its proposer, Fermat; and, except by Mr. Swale (1), I believe *their* methods have been, in substance at least, invariably retained by English writers down to the present day. These solutions, which are preserved in the works of the Rev. John Lawson, proceed on the system of gradual reduction, making the more general case depend upon some more simple one: but the foreign methods commence with the most general case of the problem, and consider all the others as particular cases of this. This proceeding, independent of its greater elegance, has the advantage of requiring exceedingly little work in the actual construction, and at the same time of opening a new and interesting road for geometrical enquiry. In the writings of those who have attended to the problem under this form—as Monge (2), Hachette (3), Gergonne (4), Durrande (5), Gaultier (6), and Steiner (7)—will be found as beautiful a series of speculations as can perhaps be boasted by any branch of geometry whatever, so purely elementary: and even as left by them the subject is far from being exhausted. May I beg, therefore, to offer my mite to the common stock?

Concerning the methods of investigation which have been generally employed by the mathematicians of the Continent, I know there is much difference of opinion in England; and for my own part, I look upon the Descriptive Geometry, or Geometry of Co-ordinates to be merely the construction of equations of three variables, and, therefore, as scarcely differing in principle from the Analytical Geometry. Nevertheless, its power is great, its methods elegant, and its results highly curious: and by means of it we can often condense a course of tedious investigation into a very narrow compass. I have, therefore, for the sake of brevity, generally adhered to the methods of reasoning employed by the Continental writers, and which are founded upon the doctrines of Monge, as first developed in his method of drawing tangent-planes to spheres, cones, and cylinders. So far, indeed, as no constructions are actually exhibited, it might be thought that the principles of the Descriptive Geometry are not essentially involved; but as these reasonings appear to have been one of the immediate fruits of that system, and as they certainly cannot be by any means referred to the Geometry of the Greeks, it is, if necessary to characterise them at all, most natural to consider them as a branch of the Descriptive Geometry. Yet, though I have employed this mode of reasoning, which in its application to these subjects is sufficiently simple, I have employed the common or perspective method of delineation, on account of its being so much more familiar to the eye of the English Geometer: and I may add that except where indispensible to the progress of investigation I have omitted the figures altogether, the delineations by the multiplicity of their lines rendering the figure more difficult to comprehend than it would be without them. Every subject of this kind, though simple enough when studied in models, becomes exceedingly complex when represented in projected lines, either perspective or descriptive.

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(1) Liverpool Apollonius, No. 1, p. 25—56.

(2) Géométrie Descriptive, Sect. II. Art. 42 ad finem,

(3) Supp. Géom. Descrip. Art. 24 to 40.
Corresp. sur l'E'cole Polytech. tom. I. p. 18—28, tom. II. p. 425—429, tom. III. p. 21, 22.
Journ. de l'E'cole Polytech. Cah. XVII. p. 129—136.

(4) Annales des Mathématiques, tom. IV. p. 349—359, tom. VII. p. 289—303.
See, also, the Mathematical Repository, vol. IV. p. 109—115.

(5) Annales des Math. tom. VI. p. 326—329, tom. XI. p. 1—67.

(6) Journal de l'E'cole Polytechnique, Cahier XVI. 124—to 214.

(7) Annales des Math. tome XVII. p. 285—315.
See, also, Cauchy, Français, "Un Abonne," & Petit, in the Annales, Correspondance, and Journal.

It is the property of all subjects of investigation that they gradually acquire a set of terms and notations peculiarly their own: and it has been the case in this. I shall, therefore, prefix the definitions of those I shall have occasion to use in each section, which have not yet been naturalized amongst us. For the present section, the following will suffice.

1. Circles and spheres are often denoted by a single letter inclosed in parentheses; the same is true of other surfaces: thus (A), or (G).

2. It is often found convenient to denote a cone by the letter at its centre (vertex) and some plane section of it; or by the centre and some figure which it envelopes: thus s″ (G).

3. A cylinder is conveniently represented by two of the surfaces which it envelopes: thus (G)(G′).

4. The *axe-radical* of two circles is that straight line from any point of which tangents being drawn to the two circles, they will be equal to one another: such is shewn to be the property of a certain line, in all our elementary works.

5. The *plane-radical of two spheres* is that plane from any point of which if tangents be drawn to the two spheres, will be equal. If we conceive two circles to revolve about the line joining their centres, they will generate the spheres, and at the

same time their axe-radical will generate the plane radical of the spheres.

6. The *plane-radical of two parallel right cylinders*, is a plane from any point of which tangents to the cylinders being drawn in planes perpendicular to these axes, their tangents will be equal. If we imagine two circles in the same plane to move parallel to themselves, their axe-radical will generate the plane-radical of the cylinders.

7. The *plane-radical of two concentric right cones*, is that plane from any point of which tangents being drawn to the cones, in planes perpendicular to their axes, these tangents will be equal. If we conceive that any point in the plane-radical of two spheres be made the centre of cones each enveloping one of the spheres, then the axe-radical of the spheres becomes that of the cones.

8. The *plane-radical* of two wedges of coincident edges is the same as that of two tangent-cones whose axes are in a plane perpendicular to the edges of the wedges.

9. *Three spheres* have an *axe-radical*: it is a line from which tangents to the three spheres being drawn, are equal to one another.

10. The centre of similitude of two figures is a point similarly related to them both, and is such that any line or plane drawn through it and the figures shall cut the figures into similar segments.

11. The centre of similitude of two circles is the point in which the tangents intersect, when the circles are not within each other: and if they be wholly without each other, there are two centres of similitude—the *direct* which lies beyond the less, and the *inverse*, which lies between them.

12. In spheres the same things take place and the same names are given.

13. In cones and cylinders, instead of enveloping cones and angles we have tangent planes: their intersections are lines—these are called lines of similitude, and are denominated *internal* or *external* under similar circumstances to the foregoing.

(I.)

Let two pairs of circles, (A), (B), and (G), (H) touch one another symmetrically, then the centre of similitude of either pair will be on the axe-radical of the other pair.

Let (A, B) be touched symmetrically by (G, H) in C, F, D, E; then O, D, S, being the centres of similitude of (B, G, H), are in one straight line; and F, E, S, being the centres of similitude of (A, G, H) are also in one straight line.

Fig. 1.

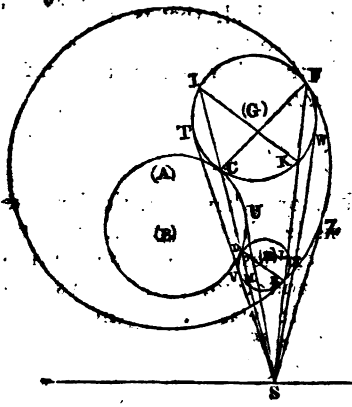
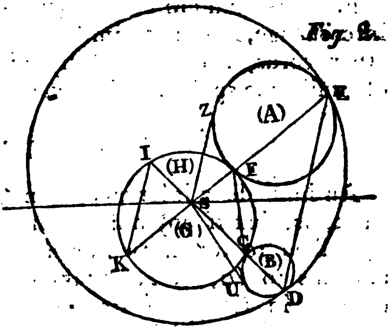


Fig. 2.



Join DE, CF, IK; and draw the tangents SV, SZ from S to the circles (B) and (A). Then, because S is the centre of similitude of (G, H), IK is parallel to DE; and, therefore, the angle EDS = ∠ KIS = ∠ CFS, and ∠ S common to the two triangles SED, SCF. Hence SE : SD :: SC : SF, or DS . SC = FS . SE. But DS . SC = SV^2, and FS . SE = SZ^2. Hence, SV = SZ, and S is a point in the axe-radical of (A, B).

In the same manner it may be shewn that the centre of similitude of (A, B) is upon the axe-radical of (G, H).

(II.)

As there are, generally, two centres of similitude of two circles, it is necessary before we proceed further, to ascertain to which of them the preceding theorem applies. However to avoid giving a tedious investigation I shall merely set down the result, the truth of which might be immediately verified by going through the particular cases.

1. Let (A), (B) be *directly* symmetrical: that is both interchangeable and similarly related to each other. This is the case when one of them does not fall *wholly* within the other.

Then if (G), (H) touch (A) in the same manner, to preserve the symmetry they must touch (B) in the same manner. (G) and (H) may in this case be said to be *directly* symmetrical; and the centre of similitude *direct* is that spoken of in the proposition.

If, on the other hand, (G), (H) touch (A) in a *different* manner, the symmetry can only be preserved by their touching (B) also in a *different* manner: in this case (G) and (H) are *inversely* symmetrical, and the centre of similitude *inverse* is that meant in pr. 1.

2. If (A) and (B) be *inversely* symmetrical, that is, capable of interchange, but not similarly related to one another,—or, in other words, if one be wholly within the other,—then the centre of similitude spoken of is always *direct.*

(III.)

An immediate inference from (I.), or rather, a particular case of it, is, that if any number of variable circles, (G, H, ...) be described to touch two permanent circles (A, B), the mutual tangents to (G, H, ...) taken two and two, will always intersect in the same straight line. It was this form of the general property that first suggested itself to my attention, and which I still think the most elegant case of it.

(IV.)

Whatever circles be drawn to touch two circles (A, B), or to cut them in points C, D which are upon an axis of similitude, the tangents to them from S will be of the same length.

(V.)

If the whole system of circles (A, B, G, H) and the axe-radical RS be supposed to move parallel to itself, the circles will generate a series of cylinders, similar and similarly situated, whilst the linear tangents TS, WS will generate tangent planes to the cylinders (G, H) and the axe-radical will generate the plane-radical of the cylinders (A, B), and it is in this plane that the intersection of the mutual tangent planes to (G, H) will always meet.

Hence, if variable cylinders touch two permanent cylinders, all being similar and similarly situated, the tangent planes to the variable cylinders will always intersect in the same plane, and in a line parallel to the axes of the cylinders.

(VI.)

If instead of circles in (III.), or indeed in (I.), we substitute ellipses similar and similarly placed, the same proposition will hold true respecting them; for such would be the result of any section made in the cylinders of (V.).

(VII.)

If instead of cylinders as in (V.) we had supposed cones described from any point, P′, in space, and having for their bases the circles (A, B, G, H): then the tangents TSP′, WSP′ would determine the tangent planes of the cones P′(G), P′(H) and P′RS would be the plane in which they intersect.

Whence, if two concentric permanent cones upon circular bases be touched by two other cones whose bases are also circular, and their centres in the same point as the first, then the tangent planes to these latter cones will always intersect in the same plane.

(VIII.)

Any system of sections of the cones of (VII.) will have the same property as the circles in (III.) or the cylinders in (V): and for a reason similar to that of (V.)

(IX.)

If upon any such system as that in (VIII.) a system of cones or cylinders with their corresponding planes be raised like those of (V.) and (VII.), these will have likewise those said properties: and any sections of these will also have the properties of (I, VI, VIII.); and so on infinitely.

To these series of cones some curious properties belong; but to insert them here would be to depart from the course marked out.

(X.)

If two variable spheres touch two permanent spheres; the variable cones which circumscribe these will have their vertices (or centres) in the same plane.

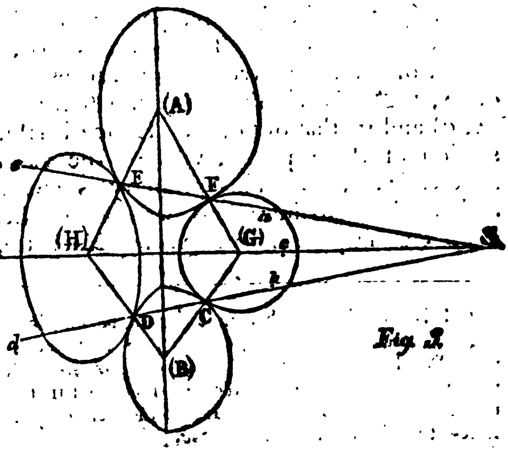


Fig. 2

Let (A, B) be the permanent spheres (G, H) the variable ones. Join HA, AG, GB, BH: these pass through the four points of contact C, D, E, F. Produce EF to meet HG in S and DC to meet it in S′ imagined for the moment to be different from S.

Because EAF, DBC are isosceles triangles we have

$$HS : SG :: HE : FG, \text{ and}$$
$$HS' : S'G :: HD : CG; \text{ but}$$
$$HE = HD \text{ and } FG = CG.$$
$$\therefore HS' : GS' :: HS : GS; \text{ or } S, S' \text{ coincide.}$$

But since $HS : GS :: HE : FG$, S is the centre of similitude of the spheres (G, H): and the centre of similitude of two spheres is the centre (or vertex) of their enveloping cone.

Again, let the lines EF, DC, GH cut the spheres again in *a*, *b*, *c*, and *e*, *d*, *f*. Then it may be shewn, as in (I.), that $ES . SF = se . sf = sb . sd = SC . SD$.

But $SE . SF$ and $SC . SD$ are the squares of the tangents from S to the spheres (A, B): and, therefore, S is in the plane-radical of the two spheres (A, B). Hence S is the centre of the enveloping cone, and in the plane-radical of (A, B), which proves the proposition.

Note. Precisely the same method of investigating the particular centre of similitude referred to, applies here as in (II.).

(XI.)

It is clear since E, F and D, C meet in S that the four points of contact of four spheres are in one plane, as was remarked by M. Hachette: and it may moreover be deduced with equal readiness from (X.) that they are in one circle.

(XII.)

If variable spheres touch *three* permanent ones, the centres of the variable enveloping cones, will always be in one straight line.

Let (A, B, C) be the three permanent spheres, and (G, H) two of the variable ones. Then because (G, H) touch (A, B) the vertex of the cone enveloping (G, H) will be in the plane-radical of (A, B).* For similar reasons, it will be in the plane-radical of (A, C) and in that of (B, C). Hence it is in the common intersection of these planes, viz. in the axe-radical of the three spheres (A, B, C). This property is mentioned by M. Dupin, and by him attributed (as well as another beautiful property very elegantly demonstrated by Hachette, Art 34, Supp. to

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* It is very obvious that in order to be enveloped by a cone, the spheres (G, H) must both envelope (A, B, C), or both touch them externally. But if instead of the vertex of the cone the centre of similitude be put, then the case where one of them as G includes the three, and H touches them externally may be included in the Theorem. Hence there are only three cases of symmetry possible—two *direct*, and one *inverse*.

Monge's Geom. Descr.) to M. Dupuis. See Corr. Polyt. tom. II. p. 422.

(XIII.)

Let two permanent right cones be touched by two variable right cones, all issuing from the same centre: then the tangent-planes to the variable cones will always intersect in the same plane.

Let the common centre s″ of the four cones be made the centre of a sphere, and let this sphere intersect the four cones in the four circles (*a*, *b*, *g*, *h*). If now spheres be described (A, B, G, H) whose contacts with the four cones are the circles (*a*, *b*, *g*, *h*,) it may be shewn as in (X.) that the variable intersections of the cone enveloping (G, H) is in the plane-radical of (A, B); that is in the plane-radical of the cones s″ (A), s″ (B). But the tangent planes of the cones s″ (G), s″ (H) repose upon the variable cone s (G, H) and also pass through s″: that is, they intersect in the line s″s, which is always in the plane-radical of s″ (A), s″ (B), since two of its points s″ and s are in that plane.

(XIV.)

The variable cone enveloping two variable spheres each of which touches two concentric right cones, or two parallel right cylinders, will have its vertex in the same plane.

For envelope the two spheres in two right cones s″ (G), s″ (H), or parallel right cylinders (G, G), (H, H) as the case may require: then the tangent planes to these variable cones or right cylinders will touch the cone in question s (G, H); and, therefore, its vertex will be in their line of intersection, that is, by (XIII.), in the plane-radical of the permanent cones or cylinders.

(XV.)

If the two cones s″ (A) and s″ (B) have their axes in the same plane as the centres of the spheres (G, H), and if a plane be made to pass along these axes, the section would give a figure like the annexed. Hence we learn that if from any point s″, two angles are drawn and any circles (G), (H) be described to touch the sides of the angles; then the intersection

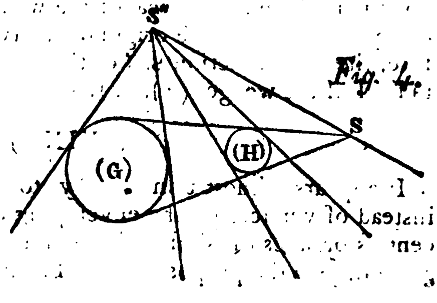


Fig. 4.

s of the tangents to (G, H) will always be in a straight line SS''. A property which admits of a neat demonstration by plane Geometry.

(XVI.)

If two variable right cylinders be inscribed in two wedges whose edges are coincident, the tangent planes of these cylinders will always meet in the same plane.

For if the figure in (XV.) move parallel to itself, it will generate such a system as that we speak of: hence the proposition is obvious.

(XVII.)

If two variable spheres be inscribed in two wedges of coincident edges, the variable envelope-cone will have its vertex in the same plane.

For if the spheres (G, H) be each enveloped in a cylinder the common tangent planes to those cylinders will rest upon the sides of the variable cone, and the vertex of this will be, therefore, in the intersection of those tangent planes, that is, by (XVI.), in the plane-radical of the wedges.

(XVIII.)

Let two concentric variable cones whose common centre is in the common edge of two wedges, be touched by tangent-planes; these tangent-planes shall intersect always in the same plane.

For in these two variable cones $S''(G)$, $S''(H)$ let any two spheres (G, H) be inscribed, and let them be enveloped by a third cone, $S(G, H)$. The tangent planes to the two former variable cones, $S''(G)$, $S''(H)$, will rest upon this cone also. The intersection of the two tangent planes will, therefore, be a line passing through the common centre S'' of the two first cones and the centre, S, of the third cone. But the centre of the third cone is in the plane-radical of the wedges (XVII.) which plane also passes through the origin of the two first cones; and, therefore, the line of section SS'' of the tangent planes is in the plane-radical of the wedges (A), (B).

(XIX.)

It appears almost unnecessary to remark, that, in all cases, instead of vertices of the enveloping cones we might have said centres or axes of similitude, and thereby have given greater generality to the propositions. The method of reasoning would in some cases have undergone slight modifications, and the statements would have required more expansion; on which account I have satisfied myself, generally, with the enunciation and demonstration of the most prominent and most elegant cases. The same may be said of the following propositions,—the concluding ones of the present section.

(XX.)

If upon the surface of a sphere, there be two permanent circles (*a*, *b*) and these be touched by two variable ones (*g*, *h*): then, great-circle tangents to these two variables will always intersect in the same great circle of the sphere.

For, to the centre of the sphere s'', draw cones resting upon the circles *a*, *b*, *g*, *h*: then the tangent planes of $s''(g)$, $s''(h)$ cut the sphere (s'') in great-circle tangents to the circles (*g*, *h*) and the plane-radical of the two cones $s''(a)$, $s''(b)$ in which the tangent planes intersect being a plane through s'' cuts the sphere in a great circle, the locus of the intersection of the great-circle tangents to (*g*, *h*).

(XXI.)

It may readily be perceived by comparing (XI.) with (XX.) that if two circles upon the sphere touch two others symmetrically, the four points of contact are upon another circle of the sphere. The same which takes place when the four circles are on a plane.

As a good deal of preliminary matter will be necessary ere we shew how far and under what circumstances, properties analogous to these are true of lines and surfaces of the second order situated in space, I shall make these the subject of another section.

(To be continued.)

ARTICLE XI.

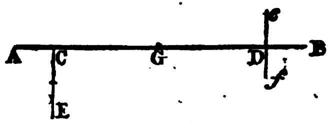
Dynamical Proposition and Problem.

By Mr. MASON.

I. *On the Progressive and Rotatory Motion of a Body in free space.*

The following proposition respecting the progressive and rotatory motion of a body, when impelled at any point by any impulse, may be of service; it has been proved by many writers of Mechanics; but its application does not seem to be generally understood. The proof following is taken from Francœur's Treatise of Mechanics.

Let AB be the body supposed on an horizontal plane, G its centre of gravity, C the point at which an impulse in direction EC perpendicular to AB is applied; Let P be the impulse.



Take GD = GC, and suppose two forces each = ½P to be applied at D in opposite directions, and parallel to EC, and also conceive P to be composed of two parts each = ½P. There are then four forces each = ½P acting on the body, but since ½P at D in direction *f*D and ½P also at D in direction *e*D opposite to *f*D must counteract each other's effect: the four forces produce the effect of P alone acting at C.

We will first consider the force ½P at D in direction *e*D and ½P at C; and here since G is situated with respect to D exactly the same as it is with respect to C, what effect soever ½P at C would produce on G the equal force ½P at D in direction *e*D would produce on G exactly an *equal* and *opposite* effect: consequently ½P at C in direction EC and ½P at D in direction *e*D destroy each other's effect on the centre of gravity G of the whole mass, it will therefore be undisturbed by these two forces: but since they are equal and at equal distances from G and tend to turn the body round G in the same direction, their effect to turn the body round G is the same as P alone would be at C if there were at G a fixed axis in the plane ECG, and this is the only effect of ½P at C and ½P at D in direction *e*D.

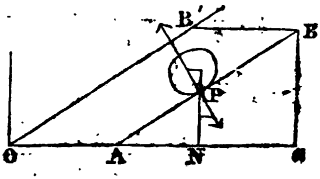
Again, the remaining ½P at C and the ½P at D in the direction *f*D, being equal and at equal distances on opposite sides of G balance each other about G, or do not turn it round G at all; but since they press the body in the same direction with forces each = ½P their effect on G is the same as if they acted at any other *equal* distances on contrary sides of G in the same directions, or if they were united at G and in the same direction, that is, their effects on G is the same exactly as if the single force P alone acted immediately on G in the direction EC.

It appears therefore that the force P at C produces a two-fold effect, it translates G in the same line and with the same velocity as if it acted on the whole mass collected at G; and it produces a rotatory motion round G exactly the same as P would do if there were a fixed axis at G perpendicular to the plane ECG.

We may therefore calculate these two effects on the body separately as if the other did not exist. Thus if m = mass of the body, P = momentum of the impulse and k = radius of gyration of the body with respect to G and GC = h, then the velocity of G = $\frac{P}{m}$ in direction of P's action and the angular ve-velocity round G = $\frac{P \cdot h}{m \cdot k^2}$. We have supposed the force P an impulsive force, but if we consider a succession of impulses at intervals ultimately vanishing, we shall have a continuous force, or a force acting incessantly, and the effects of any pressures on G are the same at whatever point in the body they are applied, and the effects of rotation are the same as if a fixed axis passed through G.

From the considerations above it appears that Mr. Lowry has fallen into an error in his solution to the ingenious problem (450) by Mechanicus, in the Repository. For the pressure of the prop on the rod produces the same effect on G as if it acted immediately on G in direction perpendicular to the rod, and this being resolved perpendicular and parallel to the horizon the vertical pressure takes so much from the weight of the body and G therefore descends more slowly than if there were no pressure; and the horizontal pressure of the prop drives G from a vertical line, the same as if it pressed at G, while the whole pressure of the prop turns the rod round G as much as if a fixed axis passed through G.

II. A sphere whose mass is P is placed on the slant side of a smooth prism whose mass is Q; the prism is at liberty to slide along a smooth horizontal plane, it is required to find the path of the point of contact of the prism and sphere, and the pressure of the sphere upon the same: the sphere being supposed to move from rest.



Let O be the place of A at the instant the point of contact was at B, AB = c, AO = b, and BC = a, P the point of contact at any time t; ON = x, NP = y, and OA = x_1, p = pressure on the prism. Therefore

$$\frac{d^2x}{dt^2} = -\frac{pa}{P \cdot c} \quad \text{....(1) the accelerative force on P in direction OC}$$

$$\frac{d^2x_1}{dt^2} = \frac{pa}{Q \cdot c} \quad \text{....(2) .... Q .. ..}$$

and $$\frac{d^2y}{dt^2} = -g + \frac{pb}{P \cdot c} \quad \text{....(3) .... P in direction NP.}$$

Multiply (1) by P and (2) by Q and add, and we have

$$P \cdot \frac{d^2x}{dt^2} + Q \cdot \frac{d^2x_1}{dt^2} = 0$$

$\therefore P \cdot \frac{dx}{dt} + Q \cdot \frac{dx_1}{dt} = c = 0 \; \therefore$ P and Q begin to move from rest.

$$\left.\begin{array}{l} \therefore Px + Qx_1 = c_1 \\ \text{and } Pb + 0 \quad = c_1 \end{array}\right\} \therefore P(b - x) - Qx_1 = 0 \; \ldots\ldots\ldots\ldots (4)$$

Again $NP = AN \cdot \frac{CB}{AC}$ or $y = (x - x_1) \cdot \frac{a}{b} \; \ldots \; \ldots\ldots (5)$

$\therefore x_1 = x - \frac{b}{a} \cdot y$; putting this value of x_1 in equation (4)

$\therefore P \cdot b - P \cdot x - Qx + Q \cdot \frac{b}{a} \cdot y = 0$, or $y = \frac{P+Q}{Q} \cdot \frac{a}{b} \cdot x - \frac{Pa}{Q}$ the equation to the locus of P which is therefore a straight line cutting OC at a point whose distance from $O = \frac{P}{Q} \cdot a$, and inclined to AC at an angle whose tangent is $\frac{P+Q}{Q} \cdot \frac{a}{b}$.

Again subtracting (2) from (1) we have

$$\frac{d^2x}{dt^2} - \frac{d^2x_1}{dt^2} = -p \cdot \frac{a}{c}\left(\frac{1}{P} + \frac{1}{Q}\right) = -p \cdot \frac{a}{c} \cdot \frac{P+Q}{PQ},$$

but $\frac{d^2x}{dt^2} - \frac{d^2x_1}{dt^2} = \frac{b}{a} \cdot \frac{d^2y}{dt^2}$, by equation (5)

$$\therefore -p \cdot \frac{a}{c} \cdot \frac{P+Q}{PQ} = -\frac{b}{a} \cdot g + \frac{p}{P} \cdot \frac{b^2}{ac}$$

$$\therefore \frac{b}{a} \cdot g = p \left\{\frac{a}{c} \cdot \frac{P+Q}{PQ} + \frac{b^2}{ac \cdot P}\right\} = p \cdot \frac{a^2 \cdot (P+Q) + Q \cdot b^2}{ac \cdot P \cdot Q}$$

$\therefore p = \frac{bc \cdot P \cdot Q \cdot g}{a^2 \cdot (P+Q) + Q \cdot b^2}$ the pressure required, which is therefore constant.

Secondly. Suppose the body P to be projected down the plane with a velocity it would acquire in falling through a height $= h$, $\therefore v = \sqrt{2gh}$.

Now as before $P \cdot \frac{dx}{dt} + Q \cdot \frac{dx_1}{dt} = c$

and here $\quad P\sqrt{2gh} \cdot \frac{b}{a} + 0 \quad = c$

$$\therefore P\sqrt{2gh} \cdot \frac{b}{a} - P \cdot \frac{dx}{dt} - Q \cdot \frac{dx_1}{dt} = 0$$

$$\therefore \mathrm{P}\sqrt{2gh}\,.\,\frac{b}{a}\,.\,t-\mathrm{P}\,.\,x-\mathrm{Q}\,.\,x_1\;=c_1$$

$$0-\mathrm{P}\,.\,b-0\qquad=c_1$$

$$\therefore \mathrm{P}\sqrt{2gh}\,.\,\frac{b}{a}\,.\,t+\mathrm{P}\,.\,(b-x)-\mathrm{Q}x_1\;=0\,\ldots\ldots\ldots(6)$$

$$\therefore \mathrm{P}\sqrt{2gh}\,.\,\frac{b}{a}\,.\,t=-\mathrm{P}\,.\,b+\mathrm{P}\,.\,x+\mathrm{Q}\,.\,x-\mathrm{Q}\,.\,\frac{b}{a}\,.\,y.$$

Also, as before $\frac{d^2x}{dt^2}-\frac{d^2x_1}{dt^2}=\frac{b}{a}\,.\,\frac{d^2y}{dt^2}$

and consequently $p=\frac{\mathrm{P}\,.\,\mathrm{Q}\,.\,bc\,.\,g}{a^2\,.\,(\mathrm{P}+\mathrm{Q})+b^2\,.\,\mathrm{Q}}$ the pressure;

$$\therefore \frac{d^2y}{dt^2}=-g+\frac{\mathrm{Q}\,.\,b\,.\,g}{a^2\,.\,(\mathrm{P}+\mathrm{Q})+b^2\,.\,\mathrm{Q}}=-\frac{a^2\,.\,(\mathrm{P}+\mathrm{Q})\,.\,g}{a^2\,.\,(\mathrm{P}+\mathrm{Q})+b^2\,.\,\mathrm{Q}}$$

$\therefore \frac{dy}{dt}=c_2-\frac{a^2\,.\,(\mathrm{P}+\mathrm{Q})\,.\,g\,.\,t}{a^2\,.\,(\mathrm{P}+\mathrm{Q})+b^2\,.\,\mathrm{Q}}$. and here $\sqrt{2gh}\,.\,\frac{a}{c}=c_2$

$$\therefore \frac{dy}{dt}=\sqrt{2gh}\,.\,\frac{a}{b}-\frac{a^2\,.\,(\mathrm{P}+\mathrm{Q})\,.\,g\,.\,t}{a^2\,.\,(\mathrm{P}+\mathrm{Q})+b^2\,.\,\mathrm{Q}}$$

$\therefore y+\mathrm{C}=\sqrt{2gh}\,.\,\frac{a}{c}\,.\,t-\frac{\frac{a^2}{2}\,.\,(\mathrm{P}+\mathrm{Q})\,.\,g\,.\,t^2}{a^2\,.\,(\mathrm{P}+\mathrm{Q})+b^2\,.\,\mathrm{Q}}$ and $\mathrm{C}=a$

$$\therefore a-y=\frac{\frac{a^2}{2}\,.\,(\mathrm{P}+\mathrm{Q})\,.\,g\,.\,t^2}{a^2\,.\,(\mathrm{P}+\mathrm{Q})+b^2\,.\,\mathrm{Q}}-\sqrt{2gh}\,.\,\frac{a}{c}\,.\,t$$

$$=\frac{\frac{a^2}{2}\,.\,(\mathrm{P}+\mathrm{Q})\,.\,g}{a^2\,.\,(\mathrm{P}+\mathrm{Q})+b^2\,.\,\mathrm{Q}}\,.\,\left\{\frac{(\mathrm{P}+\mathrm{Q})\,.\,x-\mathrm{Q}\,.\,\frac{b}{a}\,.\,y-\mathrm{P}\,.\,b}{\mathrm{P}\,.\,\sqrt{2gh}\,.\,\frac{b}{c}}\right\}^2$$

$$-\sqrt{2gh}\,.\,\frac{a}{c}\,.\,\frac{(\mathrm{P}+\mathrm{Q})\,.\,x-\mathrm{Q}\,.\,\frac{b}{a}\,.\,y-\mathrm{P}\,.\,b}{\mathrm{P}\,.\,\sqrt{2gh}\,.\,\frac{b}{c}}$$

$$\text{or } a-y=\frac{\frac{a^2}{2}\,.\,(\mathrm{P}+\mathrm{Q})\,.\,g}{a^2\,.\,(\mathrm{P}+\mathrm{Q})+b^2\,.\,\mathrm{Q}}\,.\,\frac{\left\{(\mathrm{P}+\mathrm{Q})x-\mathrm{Q}\,.\,\frac{b}{a}\,.\,y-\mathrm{P}\,.\,b\right\}^2}{\mathrm{P}^2\,.\,2gh\,.\,\frac{b^2}{c^2}}$$

$$-\frac{a}{b}\,.\,\frac{(\mathrm{P}+\mathrm{Q})\,.\,x+\mathrm{Q}\,.\,\frac{b}{a}\,.\,y-\mathrm{P}\,.\,b}{\mathrm{P}}$$

is the equation to the curve described by P which being of the second order belongs to a conic section, and it may be proved to be the eqnation of a parabola whose axis is inclined to the plane of the horizon at an angle whose tangent is $\frac{P+Q}{Q} \cdot \frac{a}{b}$.

We will now consider the case of a sphere rolling down the prism. Conceive a fine thread fixed to the prism at B and coiled round the sphere in the plane of the great circle passing through the point of contact, this will prevent sliding and may be considered as producing the effect of perfect friction, without which the body must slide.

Let T = the tension of the string; the other things being the same as in the first case.

Then $\frac{d^2x}{dt^2} = -\frac{pa}{P.c} + \frac{T.b}{P.c} = \left\{\begin{array}{l}\text{accelerating force on P in a}\\ \text{horizontal direction}\end{array}\right.$

$\frac{d^2x_1}{dt^2} = \frac{pa}{Q.c} - \frac{T.b}{Q.c} = \ldots\ldots\ldots\ldots\ldots\ldots Q\ldots\ldots$

$\frac{d^2y}{dt^2} = -y + \frac{pb}{P.c} + \frac{T.a}{P.c} = \left\{\begin{array}{l}\ldots\ldots\ldots\ldots\ldots P,\text{perp.}\\ \text{to the horizon}\end{array}\right.$

and BP being = the space through which a point in the surface of the sphere has revolved and which is $= \frac{c}{a} \cdot (a - y)$

$\therefore -\frac{c}{a} \cdot \frac{d^2y}{dt^2} = \frac{5T}{2P}$ = the accelerating force generating rotation round the centre of the sphere,

$\therefore \frac{a}{c} \cdot \frac{5T}{2P} = g - \frac{pb}{P.c} - \frac{T.a}{P.c} \therefore T = \frac{2P.c.g}{7a} - \frac{2b}{7a} \cdot p.$

Again $-\frac{5b}{2c} \cdot \frac{T}{P} = \frac{b}{a} \cdot \frac{d^2y}{dt^2} = \frac{d^2x}{dt^2} - \frac{d^2x_1}{dt^2} = -\frac{a}{c} \cdot p \cdot \frac{P+Q}{P}$

$+ \frac{b}{c} \cdot \frac{P+Q}{PQ} \cdot T$

$\therefore \frac{2a}{b} \cdot p \cdot \frac{P+Q}{2P+7Q} = T = \frac{2P.cg}{7a} - \frac{2b}{7a} \cdot p.$

Hence $p = \frac{P.b.c.g.(2P+7Q)}{7a^2.(P+Q) + b^2.(2P+7Q)}$, $\left\{\begin{array}{c}\text{a constant}\\ \text{quantity}\end{array}\right.$

and hence $T = \frac{2a.c.g.P.(P+Q)}{7a^2.(P+Q) + b^2.(2P+7Q)}$ = the tension, or the force necessary to prevent sliding.

If we multiply the first equation by P and the second by Q and add them we have $P \cdot \frac{d^2x}{dt^2} + Q \cdot \frac{d^2x_1}{dt^2} = 0$ as in the first case, the equation therefore to the path described by P is therefore $y = \frac{P+Q}{Q} \cdot \frac{a}{b} \cdot x - \frac{P.a}{Q}$.

The point therefore describes the same straight line whether it slide down the prism or rolls down, sliding being entirely prevented.

This problem has not been furnished for the Repository as an original one, but because it was proposed in the Gentleman's Diary for the year 1827, and two solutions are given in the Diary for 1828, both of which it appears are incorrect, the one making it a transcendental curve; and the other a straight line, but not the right straight line.

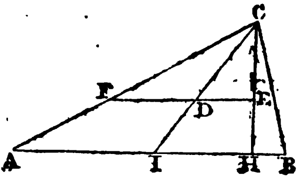
The problem according to the first and second cases is in Bernouilli's Works, but the method pursued in the solutions above differ widely I believe from Bernouilli's method, but I have no means of comparing them.

ARTICLE XII.

Suppose a wedge of a given weight to be laid on a horizontal plane, on which it may move without friction, then, if a sphere of a given weight were laid on the slant side of the wedge and then suffered to roll down, it would by its pressure cause the wedge to slide on the horizontal plane. Find the path described on a vertical plane by the point of contact in its descent.

To the Editor of the Repository.

SIR,—Perhaps some of your readers may recollect that I proposed the above question in the Gentleman's Diary for the year 1827. and to such I wish to observe that my accompanying solution in the succeeding Diary is incorrect, inasmuch as that I have therein made the motive forces of the sphere and wedge equal instead of their momenta; also by assuming the differential of the ordinate constant, it thereby became assimilated with a function of the time. On the hypothesis that the pressure on the wedge is constant, the solution might be as follows.



Let ABC be a vertical section of the wedge, C the point whence the motion of contact commences, CDI its path, and CD the part of it described in the time t; draw the vertical line CEH, and through D draw the horizontal one FDE; put $\angle$CAB $= a$; CE $= x$; ED $= y$; $w =$ weight of the sphere; W $=$ that of the wedge; then, EF $= x \,.\, \text{cot.}\, a$, and DF $= x \,.\, \text{cot.}\, a - y$; and, since the motion commences at C, the horizontal space described by the sphere by the time it has arrived at D is $= y$; also, the space described by the wedge in the same time is FD $= x \,.\, \text{cot.}\, a - y$; whence the velocity of the sphere at D in the direction ED is $\frac{dy}{dt}$; and its momentum, *supposing* *the whole mass of the sphere to act in that direction,* is $\frac{dy}{dt} \times w$; in like manner the velocity of the wedge at D is $\frac{d.(x.\cot.a-y)}{dt}$, and therefore its momentum in that direction is $\frac{d.(x.\cot.a-y)}{dt} \times W$; and since one of these generates the other they are evidently equal, therefore, multiplying both by dt we have

$$dy \times w = d.(x.\cot.a-y) \times W,$$

integrating gives

$$y \times w = (x.\cot.a-y) \times W;$$

that is ED : FD :: W : w, whence the path is a right line.

Were the pressure on the wedge constant the preceding solution would be correct, but I have some doubt respecting that, for it appears to me that it is a weight equivalent to only a part of the sphere's weight which acting horizontally puts the wedge in motion, and if this part be variable the pressure must be variable also. The most rational method of obtaining a correct solution to this question, appears to me, to be from *the resolution of forces.* Had you room in the next number of the Repository, perhaps the insertion of the above might induce some gentleman to give it a solution from that principle, which, Sir, would be a gratification to

Your most respectful,

Liverpool, 1828. S. JONES.

ARTICLE XIII.

A Dynamical Problem.

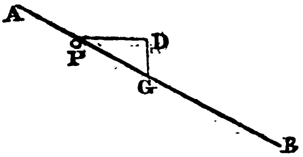
By Mr. W. S. B. WOOLHOUSE, *North Shields.*

A heavy, slender and uniform rod of a given length, (say 36 inches) is placed so as to balance itself over an extremely small pulley placed vertically, and an indefinitely small impulse is communicated at one end in direction of the rod, which causes the other end to descend while the pulley turns round on account of the increasing inclination of the rod to the horizon. It is required to investigate the nature of the curve described by its centre of gravity when it rolls upon the pulley; its lineal and angular motion when in any position; its position at the instant it quits the pulley and from thence the time of its uniform revolutions, and the magnitude and position of the parabola described by the centre of gravity after leaving the pulley.

SOLUTION.

SOLUTION.

Let AB represent a position of the rod at any instant; G its centre of gravity and P the pulley; PD parallel to the horizon and GD perp. to it: also let AG = GB = a; PG = x; ∠ DPG = θ; the pressure at P which acts perp. to the rod = P; the accelerative force of gravity = $2g$; the weight of the rod = W; and the time elapsed = t.



Were we to suppose the time to increase uniformly as usual, it would be impossible from thence to ascertain the nature of the curve described by G, for the resulting equation combining second differentials would not be integrable; in fact it would be inconsistent to attempt the integration of such an equation, for when the differentials of both quantities are variable, there must be some other equation to regulate those differentials. We shall therefore have θ to increase uniformly, so that $d\theta$ = constant.

The effect of the pressure in direction PD = P $\sin\theta$ and in direction GD = P $\cos\theta$; the velocity of G in direction PD = $\frac{d.(x\cos\theta)}{dt}$ and in direction DG = $\frac{d.(x\sin\theta)}{dt}$, consequently the acceleration of motion in direction PD = $\frac{d^2.(x\cos\theta)}{dt^2} - \frac{d^2t.d.(x\cos\theta)}{dt^3}$ and in direction DG = $\frac{d^2.(x\sin\theta)}{dt^2} - \frac{d^2t.d.(x\sin\theta)}{dt^3}$ and similarly the acceleration in angular motion = $-\frac{d\theta.d^2t}{dt^3}$. Therefore by the nature of forces we have the three following equations: $\frac{\text{P}}{\text{W}}.\sin\theta = \frac{d^2.(x\cos\theta)}{dt^2} - \frac{d^2t.d.(x\cos\theta)}{dt^3}$ .... (1)

$$2g - \frac{\text{P}}{\text{W}}.\cos\theta = \frac{d^2.(x\sin\theta)}{dt^2} - \frac{d^2t.d.(x\sin\theta)}{dt^3} \;\ldots\; (2).$$

$$\text{and } \frac{\text{P}}{\text{W}} = -\frac{a^2}{3x}.\frac{d\theta.d^2t}{dt^3} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (3).$$

By substituting the value of $\frac{\text{P}}{\text{W}}$ given by the 3d in the first and second and transposing, we have

$$\frac{d^2 . (x \cos \theta)}{dt^2} - \frac{d^2 t . d(x \cos \theta)}{dt^3} + \frac{a^2 d\theta}{3x} . \frac{d^2 t}{dt^3} . \sin \theta = 0 \ldots . (4)$$

and $$\frac{d^2 . (x \sin \theta)}{dt^2} - \frac{d^2 t . d . (x \sin \theta)}{dt^3} - \frac{a^2 d\theta}{3x} . \frac{d^2 t}{dt^3} . \cos \theta = 2g \ldots (5).$$

To the fomer of these multiplied by $2d . (x \cos \theta)$ add the latter multiplied by $2d . (x \sin \theta)$ and we have

$$\frac{2d . (x \cos \theta) . d^2 . (x \cos \theta)}{dt^2} - \frac{2d^2 t . (d . \overline{x \cos \theta})^2}{dt^3} +$$

$$\frac{2d . (x \sin \theta) . d^2 . (x \sin \theta)}{dt^2} - \frac{2d^2 t . (d . \overline{x \sin \theta})^2}{dt^3} - \frac{2a^2 . d\theta}{3x} . \frac{d^2 t}{dt^3}$$

$$. \left\{ \cos 0 . d . (x \sin \theta) - \sin \theta . d . (x \cos \theta) \right\} = 4g . d . (x \sin \theta)$$

or $\left(\text{since } \cos \theta . d . (x \sin \theta) - \sin \theta . d . (x \cos \theta) = x . d\theta . \right)$

$$\frac{2d . (x \cos \theta) . d^2 . (x \cos \theta)}{dt^2} - \frac{2d^2 t . (d . \overline{x \cos \theta})^2}{dt^3} +$$

$$\frac{2d . (x \sin \theta) . d^2 . (x \sin \theta)}{dt^2} - \frac{2d^2 t . (d . \overline{x \sin \theta})^2}{dt^3} - \frac{2a^2 . d\theta^2}{3} . \frac{d^2 t}{dt^3} =$$

$4gd . (x \sin \theta)$ which integrated we have

$$\frac{(d . \overline{x \cos \theta})^2}{dt^2} + \frac{(d . \overline{x \sin \theta})^2}{dt^2} + \frac{a^2}{3} . \frac{d\theta^2}{dt^2} = 4gx \sin \theta ;$$ but

$$(d . \overline{x \cos \theta})^2 + (d . \overline{x \sin \theta})^2 = dx^2 + x^2 d\theta^2$$

$$\therefore \frac{dx^2}{dt^2} + \left(\frac{a^2}{3} + x^2 \right) . \frac{d\theta^2}{dt^2} = 4gx \sin \theta \ldots\ldots\ldots\ldots\ldots (6).$$

By equations 4 and 5 we have

$$\frac{d^2 . (x \cos \theta)}{dt^2} - \left\{ d . (x \cos \theta) - \frac{a^2 d\theta . \sin \theta}{3x} \right\} . \frac{d^2 t}{dt^3} = 0,$$

and $$\frac{d^2 . (x \sin \theta)}{dt^2} - \left\{ d . (x \sin \theta) + \frac{a^2 d\theta . \cos \theta}{3x} \right\} . \frac{d^2 t}{dt^3} = 2g.$$

Multiplying the latter of these by

$$x . \left\{ d . (x \cos \theta) - \frac{a^2 d\theta . \sin \theta}{3x} \right\} = x . dx . \cos \theta - \left(\frac{a^2}{3} + x^2 \right) d\theta . \sin \theta ;$$

the former by

$$x \left\{ d . (x \sin \theta) + \frac{a^2 d\theta . \cos \theta}{3x} \right\} = x . dx . \sin \theta + \left(\frac{a^2}{3} + x^2 \right) d\theta . \cos \theta$$

and subtracting the two products, we get :

$$\frac{d^2 . (x \sin \theta)}{dt^2} . \left\{ x \ dx . \cos \theta - \left(\frac{a^2}{3} + x^2 \right) d\theta . \sin \theta \right\} - \frac{d^2 . (x \cos \theta)}{dt^2} .$$

$$. \left\{x \,.\, dx \,.\, \sin\theta + \left(\frac{a^2}{3} + x^2\right) d\theta \,.\, \cos\theta\right\} = 2g$$

$$. \left\{x \,.\, dx \,.\, \cos\theta - \left(\frac{a^2}{3} + x^2\right) d\theta \,.\, \sin\theta\right\}.$$

Or since by expansion $\cos\theta \,.\, d^2 \,.\, (x\sin\theta) - \sin\theta \,.\, d^2 \,.\, (x\cos\theta) = 2dx.d\theta$ and $\sin\theta \,.\, d^2 \,.\, (x\sin\theta) + \cos\theta \,.\, d^2 \,.\, (x\cos\theta) = d^2x - xd\theta^2$, we have

$$\frac{d\theta}{dt^2} \,.\, \left\{2x \,.\, dx^2 - \left(\frac{a^2}{3} + x^2\right)(d^2x - x \,.\, d\theta^2)\right\} = 2g$$

$$. \left\{x \,.\, dx \,.\, \cos\theta - \left(\frac{a^2}{3} + x^2\right) . \, d\theta \,.\, \sin\theta\right\} \ldots\ldots\ldots\ldots (7).$$

Multiplying this equation by $2dt^2 \,.\, x\sin\theta$ and equation 6 by $dt^2 \,.\, \left\{xdx \,.\, \cos\theta - \left(\frac{a^2}{3} + x^2\right) . \, d\theta \,.\, \sin\theta\right\}$ the right hand members become identical, therefore equating those on the left, we have

$$2dx \,.\, d\theta \,.\, \sin\theta \,.\, \left\{2x \,.\, dx^2 - \left(\frac{a^2}{3} + x^2\right)(d^2x - x \,.\, d\theta^2)\right\} =$$

$$\left\{dx^2 + \left(\frac{a^2}{3} + x^2\right)d\theta^2\right\} . \left\{x \,.\, dx \,.\, \cos\theta - \left(\frac{a^2}{3} + x^2\right)d\theta . \sin\theta\right\} \ldots (8).$$

Which is the equation to the curve described by G, $d\theta$ being constant.

In order to express it by integral values, instead of x substitute the indeterminate series $A\theta + B\theta^3 + C\theta^5 + D\theta^7 + \&c.$, and for dx substitute $d\theta \,.\, (A + 3B\theta^2 + 5C\theta^4 + 7D\theta^6 + \&c.)$, &c. &c. throwing every part into functions of θ and equate the coefficients of the several powers, and we obtain

$$x = \frac{a}{\sqrt{3}} \,.\, \left\{\theta + \frac{1\cdot 3}{3}\theta^3 + \frac{15\cdot 77}{72}\theta^5 + \frac{2\cdot 1551}{21}\theta^7 \;\&c.\right\}.$$

Or since the terms are nearly in geometrical progression

$$x = \frac{a}{\sqrt{3}} \,.\, \left\{\theta + \frac{1\cdot 3}{3} . \theta^3 + \frac{81\theta^5}{400 - 196\theta^2}\right\} \text{ say } = \frac{a}{\sqrt{3}} \,.\, \text{S}.$$

To determine the velocities when in any position we have

$$dx = \frac{a}{\sqrt{3}} . d\theta \,.\, \left\{1 + 1\cdot 3\,\theta^2 + \frac{405\theta^4}{400 - 196\delta^2} + \frac{31752\theta^6}{(400 - 196\theta^2)^2}\right\}$$

say $= \frac{a}{\sqrt{3}} \,.\, \text{R} \,.\, d\theta$, then by substitution in equation 6, we have

$$\left\{\frac{a^2}{3} . (\text{R}^2 + 1) + x^2\right\} . \frac{d\theta^2}{dt^2} = 4gx\sin\theta \;\therefore\; \frac{d\theta}{dt} = 2\sqrt{\frac{3gx\sin\theta}{a^2(\text{R}^2+1) + 3x^2}}$$

the angular velocity,

$\therefore \frac{dx}{dt} = \frac{a}{\sqrt{3}} \cdot \text{R} \cdot \frac{d\theta}{dt} = 2a\text{R} \cdot \sqrt{\frac{gx \sin\theta}{a^2(\text{R}^2 + 1) + 3x^2}}$ the velocity along the pulley.

At the instant the rod quits the pulley, the pressure P will vanish; and since $\frac{d\theta}{dt}$ the angular velocity becomes constant, we have $dt =$ constant and $d^2t = 0$. Therefore by equations 1 and 2, we have $d^2 . (x \cos\theta) = 0$ and $d^2 . (x \sin\theta) = 2g . dt^2$. From the latter multiplied by $\cos\theta$ deduct the former multiplied $\sin\theta$ and $\cos\theta . d^2 . (x \sin\theta) - \sin\theta . d^2 . (x \cos\theta) = 2g . dt^2 . \cos\theta$: but as before $\cos\theta . d^2 . (x \sin\theta) - \sin\theta . d^2 . (x \cos\theta) = 2dx . d\theta$
$\therefore dx . d\theta = 2g . dt^2 . \cos\theta$ .... (9).

Multiply by $4x \tan\theta$ and $4x . dx . d\theta . \tan\theta = 4gx . dt^2 . \sin\theta$; but by equation 6, $dx^2 + \left(\frac{a^2}{3} + x^2\right) d\theta^2 = 4gx . dt^2 . \sin\theta$, $\therefore dx^2 + \left(\frac{a^2}{3} + x^2\right) d\theta^2 = 4x . dx . d\theta . \tan\theta$. Divide by $\frac{a^2}{3} d\theta^2$ and $1 + \frac{3x^2}{a^2} + \frac{3dx^2}{a^2 . d\theta^2} = \frac{4x}{\frac{a}{\sqrt{3}}} . \frac{dx}{\frac{a}{\sqrt{3}} . d\theta} . \tan\theta$. Or $1 + \text{R}^2 + \text{S}^2 = 4\text{RS} . \tan\theta$, from which together with the values of R and S in terms of θ, we find $\theta = \cdot686682$; $\text{R} = 1\cdot940943$ and $\text{S} = \cdot867198$.

Hence it appears that the instant the rod quits the pulley, its inclination to the horizon will be 39° 20′ 38″ and the distance of the centre of gravity from the pulley $= x = \frac{a}{\sqrt{3}} . \text{S} = \frac{a}{\sqrt{3}} . \cdot867198 = \cdot500677 . a$, about half the semi-rod, in the present case being about 9·01218 inches.

By equation 9, $\frac{dx}{dt} . \frac{d\theta}{dt} = g \cos\theta$; but $\frac{dx}{dt} = \frac{a}{\sqrt{3}} . \text{R} . \frac{d\theta}{dt}$, $\therefore \frac{a}{\sqrt{3}} . \text{R} . \frac{d\theta^2}{dt^2} = g \cos\theta$ and $\frac{d\theta}{dt} = \sqrt{\frac{g \cos\theta . \sqrt{3}}{a\text{R}}} = \cdot830736 \sqrt{\frac{g}{a}}$, (or since $g = 193$ and $a = 18$ inches) $= 2\cdot720229$ the angular velocity per second; and the time of each revolution $= \frac{2\pi . dt}{d\theta} = \frac{2\pi}{\cdot830736} . \sqrt{\frac{a}{g}}$, in the present case being $= \frac{2\pi}{2\cdot720229}$ or $\frac{6\cdot2832}{2\cdot720229} = 2''\cdot3098$.

To determine the parabola described, we have $\frac{dx}{dt} = \frac{a}{\sqrt{3}} \cdot \mathrm{R} \cdot \frac{d\theta}{dt}$ and $x = \frac{a}{\sqrt{3}} \cdot \mathrm{S}$, $\therefore$ the velocity of G in the direction PD $= \frac{d.(x \cos\theta)}{dt} = \cos\theta \cdot \frac{dx}{dt} - x \sin\theta \cdot \frac{d\theta}{dt} = \frac{a}{\sqrt{3}} \cdot \frac{d\theta}{dt} \cdot \{\mathrm{R} \cos\theta - \sin\theta\} = 26{\cdot}891514$. This being equal to that acquired in descending freely down one-fourth of the parameter, we have the parameter $= \frac{26{\cdot}891514^2}{g} = \frac{723{\cdot}1535}{193} = 3{\cdot}7469$ inches. Also the velocity of descent, that is in direction DG $= \frac{d.(x \sin\theta)}{dt} = \sin\theta \cdot \frac{dx}{dt} + x \cos\theta \cdot \frac{d\theta}{dt} = \frac{a}{\sqrt{3}} \cdot \frac{d\theta}{dt} \cdot \{\mathrm{R} \sin\theta + \mathrm{S} \cos\theta\} = 53{\cdot}74468$, which being equal to that acquired in descending freely down the abscissa, we have the abscissa $\mathrm{X} = \frac{53{\cdot}74468^2}{4g} = \frac{2888{\cdot}4906}{772} = 3{\cdot}7416$ inches. Consequently the ordinate $\mathrm{Y} = \sqrt{\{3{\cdot}7469 \times 3{\cdot}7416\}} = 3{\cdot}74425$.

Hence the distance of the axis from the pulley towards the same side as the motion $= x \cos\theta - \mathrm{Y} = 3{\cdot}22536$ and the distance of the vertex below the pulley $= x \sin\theta - \mathrm{X} = 1{\cdot}9718$ inches. Consequently the magnitude and position of the parabola are completely determined.

ARTICLE XIV.

By the Rev. Mr. SAMUEL HAWKES.

"*From Développemens, &c.*" *by* REYNAUD & DUHAMEL.

THEOREM.

Whatever may be the values of the coefficients and of the index in the equation $x^n + a_1 x^{n-1} + \ldots + a_n = 0$, there always exists an expression of the form $p + q\sqrt{-1}$ which being substituted for x, will satisfy it; p and q being neither of them infinite, but either or both of them finite or equal to zero.

(1.) To avoid the concealed existence of imaginary quantities, put $u+v\sqrt{-1}$ for x, and $k_1+l_1\sqrt{-1}, k_2-l_2\sqrt{-1}, \ldots k_n+l_n\sqrt{+1}$ for the coefficients in their order. Let these substitutions be effected; and let U denote that function of u and v which forms the real part of the resulting expression, and V that which forms its imaginary part, so that $x^n+a_1 x^{n-1}+ \ldots\ldots +a_n=\mathrm{U}+\mathrm{V}\sqrt{-1}$.

Now will $x^n+a_1 x^{n-1}+\ldots\ldots+a_n=0$, whenever $\mathrm{U}=0$ and $\mathrm{V}=0$ together; and this will be whenever $\mathrm{U}^2+\mathrm{V}^2=0$ since neither U^2 nor V^2 can be negative.

(2.) Next, for x, a_1, a_2, ... a_n, respectively put

$$r(\cos\varphi+\sqrt{-1}\sin\varphi),\ c_1(\cos\theta_1+\sqrt{-1}\sin\theta_1),$$

$$c_2(\cos\theta_2+\sqrt{-1}\sin\theta_2),\ \ldots\ldots c_n(\cos\theta_n+\sqrt{-1}\sin\theta_n).$$

The substitutions being effected there will result

$$\mathrm{U}=r_n\,.\cos n\varphi+c_1 r^{n-1}\cos(\overline{n-1}\,\varphi+\theta_1)+\ldots+c_n\cos\theta_n$$

Whence

$$\begin{aligned}\mathrm{U}^2=\ &r^{2n}.\cos^2 n\varphi+2c_1r^{2n-1}.\cos n\varphi.\cos(\overline{n-1}\,\varphi+\theta_1)\\ &+r^{2n-2}\{c_1{}^2\cos^2(\overline{n-1}\varphi+\theta_1)+2c_2\cos n\varphi.\cos(\overline{n-2}\varphi+\theta_2)\}\\ &+\ \ldots\ldots+c_n{}^2\cos^2\theta_n\end{aligned}$$

$$\begin{aligned}\text{and } \mathrm{V}^2=\ &r^{2n}\sin^2 n\varphi+2c_1r^{2n-1}.\sin n\varphi.\sin(\overline{n-1}\,\varphi+\theta_1)\\ &+r^{2n-2}\{c_1{}^2\sin^2(\overline{n-1}\varphi+\theta_1)+2c_2\sin n\varphi.\sin(\overline{n-2}\varphi+\theta_2)\}\\ &+\ \ldots\ldots+c_n{}^2\sin^2\theta_n\end{aligned}$$

$$\begin{aligned}\therefore \mathrm{U}^2+\mathrm{V}^2=\ &r^{2n}+2c_1r^{2n-1}\cos(\varphi-\theta_1)+r^{2n-2}\{c_1{}^2+2c_2.\mathrm{cas}(2\varphi-\theta_1)\}\\ &+\ \ldots\ldots+c_n{}^2.\end{aligned}$$

From this equation it appears that $\mathrm{U}^2+\mathrm{V}^2$ becomes $c_n{}^2$ when $r=0$, and that when r is continually increased, it approximates to r^{2n} in value, and accordingly is infinite when r is so.

But $r^2=r^2\cos^2\varphi+r^2\sin^2\varphi=u^2+v^2$; so that when $u=0$ and $v=0$ together, then also $r=0$, and $\mathrm{U}^2+\mathrm{V}^2=c_n{}^2$, a finite value; also when r^2 i. e. u^2+v^2 is increased indefinitely, i. e. when u or v or both are increased indefinitely, so also will $\mathrm{U}^2+\mathrm{V}^2$ be.

It is now ascertained that $\mathrm{U}^2+\mathrm{V}^2$ admits of a finite value, which may be indefinitely increased; and from its form it is incapable of a negative value. Put A for the least value it can have: let u_0, v_0 be the corresponding values of u, v. Also put in general $u=u_0+z\cos\Psi$, $v=v_0+z\sin\Psi$. So that

$$x=u_0+z\cos\Psi+(v_0+z\sin\Psi)\sqrt{-1}=u_0+v_0\sqrt{-1}+z(\cos\Psi+\sqrt{-1}.\sin\Psi).$$

Substitute this value for x in $x^n + a_1 x^{n-1} + \ldots\ldots + a_n$, and order the result by ascending powers of z, putting $b_0 (\cos\beta_0 + \sqrt{-1}\sin\beta_0)$, $b_1 (\cos\beta_1 + \sqrt{-1}\sin\beta_1)$ for the coefficients in their order: then will

$$\mathrm{U} = b_0 \cos\beta_0 + b_1 z\cos(\Psi+\beta_1) + b_2 z^2 \cos(2\Psi+\beta_2) + \&c.$$

Whence

$$\mathrm{U}^2 = b_0{}^2\cos^2\beta_0 + 2b_0\cos\beta_0\{b_1 z\cos(\Psi+\beta_1) + b_2 z^2\cos(2\Psi+\beta_2) + \&c\} \\ + \{b_1 z\cos(\Psi+\beta_1) + b_2 z^2\cos(2\Psi+\beta_2) + \&c.\}^2$$

and

$$\mathrm{V}^2 = b_0\sin^2\beta_0 + 2b_0\sin\beta_0\{b_1 z(\sin\Psi+\beta_1) + b_2 z^2\sin(2\Psi+\beta_2) + \&c.\} \\ + \{b_1 z\sin(\Psi+\beta_1) + b_2 z^2\sin(2\Psi+\beta_2) + \&c.\}^2$$

$$\therefore \mathrm{U}^2 + \mathrm{V}^2 = b_0{}^2 + 2b_0\{b_1 z\cos(\Psi+\beta_1-\beta_9) + b_2 z^2\cos(2\Psi+\beta_2-\beta_0) + \&c.\} \\ + b_1{}^2 z^2 + \&c.$$

Of the coefficients b_1, b_2, &c. every one cannot be zero, as in that case $\mathrm{U}^2 + \mathrm{V}^2$ would be reduced to the constant value $b_0{}^2$, contrary to what has been shewn. Some one of them, therefore, at least must have value: let the first which has value be b_m. The equation is then reduced to

$$\left\{\begin{array}{l} \mathrm{U}^2 + \mathrm{V}^2 = b_0{}^2 + 2b_0\{b_m z^m\cos(m\Psi + \beta_m - \beta_0) + \&c.\} \\ \qquad\qquad + \{b_m{}^2 + \&c.\}z^{2m} + \&c. \end{array}\right\}$$

Taking the values u_0, v_0 of u, v; i. e. making $\mathrm{U}^2 + \mathrm{V}^2 = \mathrm{A}^2$ this equation is reduced to $\mathrm{A} = b_0{}^2$, $\therefore b_0 = \mathrm{A}^{\frac{1}{2}}$.

Transposing $b_0{}^2$ and replacing b_0 by its value $\mathrm{A}^{\frac{1}{2}}$, there results

$$\left\{\begin{array}{l} \mathrm{U}^2 + \mathrm{V}^2 - \mathrm{A} = 2\mathrm{A}^{\frac{1}{2}}\{b_m z^m\cos(m\Psi + \beta_m - \beta_0) + \&c.\} \\ \qquad\qquad + \{b_m{}^2 + \&c.\}z^{2m} + \&c. \end{array}\right\}$$

Now A being the least value that $\mathrm{U}^2 + \mathrm{V}^2$ can have, the first member of the equation does not admit of a negative value, therefore the second member must also be such as not to admit of a negative value. Thus necessity requires that $\mathrm{A} = 0$, for otherwise the term which contains the lowest power of z will be $2\mathrm{A}^{\frac{1}{2}} b_m z^m \cos(m\Psi + \beta_m - \beta_0)$, and as z may be so assumed that this term shall be numerically greater than all the others taken together, the sign of the second member of the equation may be made to be the same with that of this term, which, from the double value of $\mathrm{A}^{\frac{1}{2}}$ can be made negative. Hence $\mathrm{A} = 0$ i. e. $\mathrm{U}^2 + \mathrm{V}^2 = 0$ when $x = u_0 + v_0\sqrt{-1}$.

Therefore, if x be assumed $= u_0 + v_0\sqrt{-1}$ then will (by part 1) $x_n + a_1 x^{n-1} + \ldots\ldots + a_n = 0$ as required; u_0 and v_0 being finite (by part 2). *Q. E. D.*

Recapitulation. $x^n + a_1 x^{n-1} + \dots + a_n = 0$, when $U^2 + V^2 = 0$. $U^2 + V^2$ admits of a minimum finite value. Let this value be A.

The corresponding values of u and v are finite. Let these be u_0 and v_0.

On investigation it appears that $A = 0$. Wherefore the values u_0 and v_0 being taken for u and v, give $U^2 + V^2 = 0$: that is to say $x = u_0 + v_0 \sqrt{-1}$ satisfies the equation $x_n + a_1 x^{n-1} + \dots + a_n = 0$.

ARTICLE XV.

On the Principle of least Action.

By the Rev. Edmund Sibson, *of Ashton in Makerfield, near Warrington.*

Proposition.

If any number of bodies, m, m', &c. connected together in any manner whatever, and urged by any forces, F, F', &c. describe the spaces, ds, ds', &c. in the direction of the respective forces, in the indefinitely small time dt, and if dz, dz', &c. be the spaces actually described by the bodies, with the velocities v, v', &c. it is required to find the relation between the forces, spaces, and velocities, all the bodies being supposed to move in the same plane.

The bodies being at liberty to move freely, they will obey that Law of Nature, by which she always performs her operations in the easiest and shortest way. But, that a body may move in the easiest way, the path that it describes must be such, that the body shall require the least motive force to carry it in that path; and, that a body may move in the shortest way, the path that it describes must be the shortest possible. Therefore, that a body may move in the easiest and shortest way, the sum of the products of its motive force, and the space it describes uniformly, must be a minimum.

Let $\int$ be the sign of integration, and δ that of variation.

Then, by the principle of least action,

$\int mv\,dz + \int m'v'\,dz' + \&c. = \text{minimum}$; and taking the variation

$$\int m\,dz\,\delta v + \int mv\,d\delta z + \int m'\,dz'\,\delta v' + \int m'v'\,d\delta z' + \&c. = 0 \quad \dots\dots (1)$$

Cor. 1. Because $vdv = \mathrm{F}ds$, by integration $v^2 = 2\int \mathrm{F}ds + c$. But, in the equation $vdv = \mathrm{F}ds$, the variable force F, is considered constant, during the time in which ds is described. Hence, in taking the variation, F may be considered constant, during the time of describing ds. But the time of describing ds equals the time in which the variation δs is made; therefore F may be considered constant, during the time in which the variation δs is made. Hence, taking the variation of $v^2 = 2\int \mathrm{F}ds + c$,

$$v\delta v = \int \mathrm{F}d\delta s = \int d(\mathrm{F}\delta s) = \mathrm{F}\delta s.$$

But $v = \frac{dz}{dt}$; hence, by substitution $\frac{dz}{dt}\delta v = \mathrm{F}\delta s$, and $\int mdz\delta v = \int mdt\,\mathrm{F}\delta s$.

Also, $\int mvd\delta z = -\int mdv\delta z$, by neglecting the definite part, Therefore by substitution in equation (1)

$$\int mdt\,\mathrm{F}\delta s - \int mdv\delta z + \int m'dt\,\mathrm{F}'\delta s - \int mdv'\delta z' + \&c. = 0. \quad \ldots\ldots(2)$$

Cor. 2. If m, m', &c. move in given curves, $\delta s = ds$, $\delta s' = ds'$ &c. and $\delta z = dz$, $\delta z' = dz'$, &c. Hence, by differentiating equation (2)

$mdt\mathrm{F}'ds + m'dt\mathrm{F}'ds' + \&c. = mdvdz + m'dv'dz' + \&c. = mdvdtv + m'dv'dtv' + \&c.$ or

$$m\mathrm{F}ds + m'\mathrm{F}'ds' + \&c. = mvdv + m'v'dv' + \&c. \ldots\ldots(3);$$

where the sign of ds, &c. is $+$, when ds and F increase together, and the sign is $-$, when the one increases as the other decreases. This is the mechanical theorem, on which there is an able Paper, Repository, Vol. III. Part 2, page 15, by Mr. Lowry.

Cor. 3. If x and y be the abscissa and ordinate, at right angles to each other, belonging to the curve z, and x' and y' be those belonging to z'; if that part of the force F, which acts in the direction of x, be represented by X, and that part of F, which acts in the direction of y, be represented by Y; and if F' be resolved in the same manner into X' and Y', &c. Then, because x and y are at right angles to each other, the force X is independent of the force Y, &c.

$$\frac{d^2x}{dt^2} = \mathrm{X}, \text{ and } \frac{d^2y}{dt^2} = \mathrm{Y}, \&c.$$

And, multiplying by dx and dy, $\frac{dx\,d^2x}{dt^2} = \mathrm{X}dx$, and $\frac{dy\,d^2y}{dt^2} = \mathrm{Y}dy$; and, by addition

$$\frac{dx\,d^2x + dy\,d^2y}{dt^2} = \mathrm{X}dx + \mathrm{Y}dy; \text{ and by integrating}$$

$\frac{dx^2 + dy^2}{dt^2} = 2\int(dx\mathrm{X} + \mathrm{Y}dx) + c$. But $\frac{dx^2 + dy^2}{dt^2} = \frac{dz^2}{dt^2} = v^2$;

therefore,
$v^2 = 2\int(\mathrm{X}dx+\mathrm{Y}dy)+c$. And, taking the variation as in Cor. 1.
$v\delta v = \mathrm{X}\delta x + \mathrm{Y}\delta y$; hence $\int m dz\delta v = \int m dt(\mathrm{X}\delta x + \mathrm{Y}\delta y)$.

Also, $dz^2 = dx^2 + dy^2$; and, taking the variation, $dzd\delta z = dxd\delta x + dyd\delta y$; or, by substituting for dz,

$$\int mvd\delta z = \pm\int m\frac{dx}{dt}d\delta x \pm \int m\frac{dy}{dt}d\delta y = \mp\int m\frac{d^2x}{dt}\delta x \mp \int m\frac{d^2y}{dt}\delta y,$$

by rejecting the definite parts, and making dt constant, and using the negative sign to dx, &c. when dx increases, as X &c. increases, and the positive sign, when dx &c. decreases, as X, &c. increases. Hence, by substituting in equation (1)

$$\int mdt(\mathrm{X}\delta x + \mathrm{Y}\delta y) \mp \int\frac{md^2x}{dt}\delta x \mp \int\frac{md^2y}{dt}\delta y + \int m'dt(\mathrm{X}'\delta x' + \mathrm{Y}'\delta y')$$

$$\mp\int\frac{m'd^2x'}{dt}\delta x' \mp \int\frac{m'd^2y'}{dt}\delta y' + \text{\&c.} = 0, \text{ or}$$

$$m\left(\mathrm{X}\mp\frac{d^2x}{dt^2}\right)\delta x + m\left(\mathrm{Y}\mp\frac{d^2y}{dt^2}\right)\delta y + m'\left(\mathrm{X}'\mp\frac{d^2x'}{dt^2}\right)\delta x\mathrm{Y}' + m'\left('\mp\frac{d^2y'}{dt^2}\right)$$

$$\delta y' + \text{\&c.} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

From this is derived the equation used by Mr. Francais, Repository, No. XVIII. page 77.

Cor. 4. When $\delta x, \delta y, \delta x', \delta y'$, &c. are independent of each other

$$\mathrm{X}\mp\frac{d^2x}{dt^2} = 0,\ \mathrm{Y}\mp\frac{d^2y}{dt^2} = 0,\ \mathrm{X}'\mp\frac{d^2x'}{dt^2} = 0,\ \mathrm{Y}'\mp\frac{d^2y'}{dt^2} = 0, \text{\&c.}$$

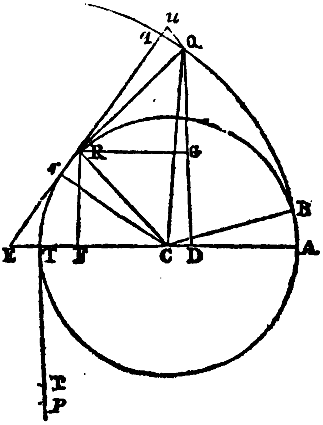
Hence, in a system of bodies connected together in any manner, there are as many equations as there are independent quantities, x, y, x', y', &c.

EXAMPLE 1.

Prize Question from the Ladies' Diary, 1828.

Suppose two given balls, P, Q, to be fastened to the end of a string, PTQ, of a given length, but void of thickness and weight, a given part of the string TB, being wound round the circumference, at the bottom, TRB, of a given cylinder, fixed on an indefinite smooth horizontal plane. Suppose the ball, P, to be impelled with a given velocity, in the direction TP of a tangent to the cylinder's base, at T; it is required to determine the circumstances of the motion of P and Q, and the position of the latter, at any time t, before the string has ceased to be in contact with the cylinder.

Let TB be the first position of that part of the string, which is in contact with the cylinder, then if P receive an impulse in the direction TP, the body, at B will begin to move in the same manner, as if it had received an impulse, in the direction of a tangent, at B; and Q will be continually drawn from that tangent by the action of P, in the direction of the string QR. Also, Q, by its circular motion, will acquire a centrifugal force, which will act upon it, in the direction RQ.



Let BQq be the curve, which Q will describe; and let PTRQ be the position of the bodies, and the string, at the time t; and let pTrq be the position of the bodies and the string at the time $t + dt$.

Let C be the centre of the base of the cylinder. Join CB, CR, and Cr, and draw Qu perpendicular to RQ, and meeting rq produced in u; then because the angle Qrq is very small, ruQ may be considered a right angle. Also, because QR and ur touch the circle at R, and r, CRr, and Cru are right angles; and, therefore the angle Qru = RCr.

Produce TC both ways, to meet the circumference in A, and to meet QR produced, in E; and draw QD, and RF perpendicular to TA. Join CQ, and draw RG perpendicular to QD.

Let RC $= 1$, arc TB $= a$, TR $= \varphi$, and consequently Rr $= -\, d\varphi$. Let the length of the string $= l$, RQ $= r$, TP $= p$, then TP $= l - \varphi - r$, $dp =$ Pp $= -\, qu = -(d\varphi + dr)$. Let CD $= x$, and QD $= y$. And, let $b =$ the velocity with which P begins to move, and therefore, the velocity with which Q begins to move, in the curve BQ, at B: $w =$ the velocity of Q, at Q, in the direction Qu.

Then, because the triangles RCr, Qru, are similar, CR : Rr :: rQ : Qu; or $1 : -d\varphi :: r + d\varphi : -\, r d\varphi$, nearly $=$ Qu; and therefore, Q$q = \sqrt{\text{Q}u^2 + uq^2} = \sqrt{r^2 d\varphi^2 + (d\varphi + dr)^2}$. Also because QRC, QDC, are right angled triangles, QD^2 + DC^2 = CR^2 + RQ^2; or $y^2 + x^2 = r^2 + 1$. And RQ^2 = RG^2 + QG^2 = FD^2 + (QD − RF)^2; or $r^2 = (x + \cos\varphi)^2 + (y - \sin\varphi)^2$; from which $x = \dfrac{y \sin\varphi - 1}{\cos\varphi}$. And, by substituting for x^2, in the equation

$y^2 + x^2 = r^2 + 1$; it is found, that $y = r\cos\phi + \sin\phi$; and, in the same manner, $x = r\sin\phi - \cos\phi$. Hence, also it is found, that $\sin\phi dx + \cos\phi dy = dr + d\phi$,

Also, the centrifugal force of Q $= \frac{w^2}{r} = \frac{r d\phi^2}{dt^2} = \text{F}$. And EQ : ED :: ER : EF :: RC : RF :: 1 : sin RT :: F : F $\sin\phi$ = X = accelerating force on Q, in the direction CD; and EQ : QD :: ER : RF :: RC : CF :: 1 : cos RT :: F : $\cos\phi$ = Y = accelerating force on Q, in the direction DQ.

Therefore, in equation (4), $\text{X} = \text{F}\sin\phi$, $\text{Y} = \text{F}\cos\phi$, $\text{X}' = 0$, $\text{Y}' = 0$, d^2x, and d^2y, are negative, because x and y increase as X, and Y, increase, $dx' = 0$, $dy' = dp = -(d\phi + dr)$, and $\delta y' = -(\delta\phi + \delta r) = -\sin\phi\delta x - \cos\phi\delta y$, dy' being negative, because the impulsive force, at the beginning of the motion, tended to increase y', $m = \text{Q}$, and $m' = \text{P}$; hence,

$$\text{Q}\left(\text{X} - \frac{d^2x}{dt^2}\right)\delta x + \text{Q}\left(\text{Y} - \frac{d^2y}{dt^2}\right)\delta y + \frac{\text{P}d^2p}{dt^2}\sin\phi\delta x + \frac{\text{P}d^2p}{dt^2}\cos\phi\delta y = 0.$$

But, δx is independent of δy; therefore,

$$\text{QX} + \frac{\text{P}d^2p}{dt^2}\sin\phi - \frac{\text{Q}d^2x}{dt^2} = 0, \text{ and } \text{QY} + \frac{\text{P}d^2p}{dt^2}\cos\phi - \frac{\text{Q}d^2y}{dt^2} = 0;$$

or, by substituting for X, and Y, and d^2p

$$-\frac{d^2x}{dt^2} = \left(\frac{\text{P}}{\text{Q}}\,\frac{d^2\phi + d^2r}{dt^2} - \frac{r d\phi^2}{dt^2}\right)\sin\phi = \psi\sin\phi. \;\ldots.(a)$$

$$\text{and } -\frac{d^2y}{dt^2} = \left(\frac{\text{P}}{\text{Q}}\,\frac{d^2\phi + d^2r}{dt^2} - \frac{r d\phi^2}{dt^2}\right)\cos\phi = \psi\cos\phi. \;\ldots.(b)$$

Multiply equation (a) by y, and (b) by x, and subtract them; and $\frac{xd^2y - yd^2x}{dt^2} = \psi(y\sin\phi - x\cos\phi) = \psi$; and by integration, dt being constant,

$$\frac{xdy - ydx}{dt} = f\psi + c, \; c \text{ being the correction.}$$

But, $d(\text{angle QCD}) = d\tan^{-1}\frac{y}{x} = \frac{xdy - ydx}{x^2 + y^2} = -d(\text{angle QCT}) = -d\phi - d\,\text{tang}^{-1}\,r = -d\phi - \frac{dr}{1 + r^2}$; therefore,

$xdy - ydx = -(r^2 + 1)d\phi - dr$. Hence $\frac{(r^2 + 1)d\phi + dr}{dt^2} = -f\psi - c$; and by substituting for ψ,

$$\frac{(r^2 + 1)d\phi + dr}{dt^2} = f\,\frac{r d\phi^2}{dt^2} - \frac{\text{P}}{\text{Q}}\,\frac{d\phi + dr}{dt^2} - c.$$

At the beginning of the motion, at B, r is infinitely small, and, therefore, the centrifugal force $\frac{rd\varphi^2}{dt^2}$ is infinitely small; hence $\frac{d\varphi + dr}{dt^2} = -\frac{P}{Q}\,\frac{d\varphi + dr}{dt^2} - c$. But at the beginning of the motion, $\frac{d\varphi + dr}{dt} = b$; therefore $c = -\frac{Q+P}{Q} + \frac{b}{dt}$.

$$\text{Hence} \frac{(r^2+1)\,d\varphi + dr}{dt^2} = f\frac{rd\varphi^2}{dt^2} - \frac{P}{Q}\,\frac{d\varphi + dr}{dt^2} + \frac{Q+P}{Q} + \frac{b}{dt};$$

$$\text{and } dt = \frac{Qr^2d\varphi}{b(Q+P)} + \frac{d\varphi + dr}{b} - \frac{Qfrd\varphi^2}{b(Q+P)}. \;\ldots\ldots(c)$$

Again, multiply equation (a) by dx, and (b) by dy, and add them, and

$$\frac{dxd^2x + dyd^2y}{dt^2} = -\psi\,(\sin\varphi dx + \cos\varphi dy) = -\psi\,(d\varphi + dr):$$

and, by integration, $\frac{dx^2 + dy^2}{2dt^2} = -f\psi\,(d\varphi + dr) + c'$; but $dx^2 + dy^2 = Qq^2 = r^2d\varphi^2 + (d\varphi + dr)^2$; therefore,

$$\frac{r^2d\varphi^2 + (d\varphi + dr)^2}{2dt^2} = f\frac{rd\varphi^2\,(d\varphi + dr)}{dt^2} - \frac{P}{Q}\,\frac{(d\varphi + dr)^2}{2dt^2} + c'.$$

At the origin, at B, for the reasons before given,

$$\frac{(d\varphi + dr)^2}{2dt^2} = -\frac{P}{Q}\,\frac{(d\varphi + dr)^2}{2dt^2} + c'; \text{ from which } c' = \frac{Q+P}{2Q} \times b^2.$$

Hence

$$\frac{r^2d\varphi^2 + (d\varphi + dr)^2}{2dt^2} = f\frac{rd\varphi^2\,(d\varphi + dr)}{dt^2} - \frac{P}{Q}\,\frac{(d\varphi + dr)^2}{2dt^2} + \frac{Q+P}{2Q} \times b^2,$$

from which

$$dt = \sqrt{\left(\frac{Qr^2d\varphi^2}{b^2(Q+P)} + \frac{(d\varphi + dr)^2}{b^2} - \frac{2Q}{b^2(Q+P)}frd\varphi^2(d\varphi + dr)\right)};$$

and from this and (c) the equation of solution is derived.

Cor. If the centrifugal force $= 0$, $\frac{rd\varphi^2}{dt^2} = 0$; hence

$$Qr^2d\varphi + (P + Q)(d\varphi + dr) = -Qcdt^2$$

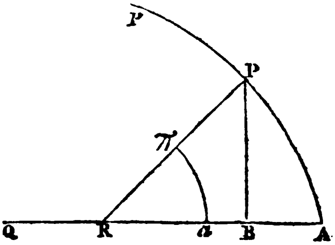
and $Q\,(r^2d\varphi^2 + (d\varphi + dr)^2) + P\,(d\varphi + dr)^2 = 2Qc'dt^2$, which are the equations used in the solution of this Problem, in the Ladies' Diary, by Petrarch.

EXAMPLE 2.

Landen's Memoirs, Vol. I. Mem. 7. Art 7. Ex. 2.
Leybourn's Repository, Vol. I. Quest. 105.
Wright's Solutions of the Cambridge Problems, Vol. II. Prop. 572.

Two bodies, whose weights are known, lying on a smooth horizontal plane, are connected by a flexible line, passing through a small ring fixed at a given point between them; in this position a given velocity is communicated to one of them, in a direction perpendicular to the line, that joins their centres, and the other body is made to move directly towards the ring: investigate the motion of the projected body, and find the angle described when the other body arrives at the ring.

Let R be the ring, QRP the position of the string and the bodies, P and Q, at the time t: A the point in the straight line ARQ, from which P begins to move at right angles to AR, with a velocity b, in the curve APp,



Let $a\pi = \varphi =$ arc of a circle, whose radius $R\pi = 1$; and draw PB perpendicular to AR. Let $PR = r$, $l =$ length of the string, then $RQ = x' = l - r$; and let $RB = x$ and $BP = y$.

Then, the body P, by its circular motion about R, acquires a centrifugal force, by which it is continually urged in the direction RP. And, $1 : d\varphi :: r : rd\varphi =$ circular arc described by P, in dt, and $\frac{rd\varphi}{dt} =$ velocity of P in that circular arc; therefore, $\frac{rd\varphi^2}{dt^2} = F =$ centrifugal force of P. And, by resolving the force F, $\frac{Fx}{r} = X =$ force on P, in the direction RB, and $\frac{Fy}{r} = Y =$ force on P, in the direction BP.

Therefore, in equation (4) $X = \frac{Fx}{r}$, $Y = \frac{Fy}{r}$, $X' = 0$, $Y' = 0$, $m = P$, $m' = Q$, d^2x and d^2y are both negative, because x and y increase, as X and Y increase, $y' = 0$, and $d^2x' = d^2r$ is positive, because the impulsive force at A, tends to diminish RQ, and

$\delta x' = -\delta r = -\delta\sqrt{x^2 + y^2} = -\frac{x\delta x}{r} - \frac{y\delta y}{r}$; hence,

$$\text{P}\left(\text{X} - \frac{d^2x}{dt^2}\right)\delta x + \text{P}\left(\text{Y} - \frac{d^2y}{dt^2}\right)\delta y - \frac{\text{Q}d^2r}{dt^2}\delta r = 0; \text{ or}$$

$$\left(\text{PX} - \frac{\text{Q}x}{r}\frac{d^2r}{dt^2} - \frac{\text{P}d^2x}{dt^2}\right)\delta x + \left(\text{PY} - \frac{\text{Q}y}{r}\frac{d^2r}{dt^2} - \frac{\text{P}d^2y}{dt^2}\right)\delta y = 0.$$

And, because δx and δy, are independent of each other,

$$\frac{xd\varphi^2}{dt^2} - \frac{\text{Q}}{\text{P}}\frac{x}{r} \times \frac{d^2r}{dt^2} = \frac{d^2x}{dt^2}, \text{ and}$$

$$\frac{yd\varphi^2}{dt^2} - \frac{\text{Q}}{\text{P}}\frac{y}{r} \times \frac{d^2r}{dt^2} = \frac{d^2y}{dt^2}.$$

Cor. If the centrifugal force $= 0$, $\frac{xd\varphi^2}{dt^2} = 0$ and $\frac{yd\varphi^2}{dt^2} = 0$;

$$\text{hence } \frac{d^2x}{dt^2} + \frac{\text{Q}}{\text{P}}\frac{x}{r}\frac{d^2r}{dt^2} = 0$$

$$\text{and } \frac{d^2y}{dt^2} + \frac{\text{Q}}{\text{P}}\frac{y}{r}\frac{d^2r}{dt^2} = 0,$$

which are Mr. Ivory's equations, in his solution, in Leybourn's Repository.

ARTICLE XVI.

On the Stereographic Projection.

By Mr. T. S. DAVIES.

(In a Letter to the Editor.)

MY DEAR SIR, *Bath, October,* 1829.

In my "Geometry of three dimensions" will be found a Theorem allied to this subject much more general than any that has yet been published; and which includes the projection of surfaces of the second order, as well as the more limited case of the sphere, as very particular examples. Yet since the subject is a very important one, and I have a few remarks to make both upon the history and demonstrations of the properties of this projection, which would be too episodal for that place, I shall trouble you with a short separate paper on the subject, the insertion of which in your forthcoming number will greatly oblige,

My dear Sir,
Your's very faithfully,
T. S. Davies.

I. (Historical.)

The three principal properties of this projection of the sphere are :—

1. That all circles are projected into circles :—
2. That the tangent of any arc, and its projection are equal :—
3. That the angle contained by any two circles is equal to its projection.

The first of these properties was partially known (that is in some of its cases) to Hipparchus, who flourished in the second century before our era: though Ptolemy, who lived nearly three centuries later, is commonly represented as the inventor of this method of projection.* This mistake, no doubt, arose from the loss of the writings of Hipparchus and the preservation of those of Ptolemy on the subject, by means of the Arabic version of Maslem. It is certain, however, that even to Ptolemy it was only known as a property of great circles and of less circles parallel to the plane of projection: and it is fair to conclude that this was the utmost generality to which the antients ever attained. As a universal truth it was first given in the "*Planisphærium Jordani*,"† published at Toulouse, in 1544, or soon after. It is remarkable in this production, as has also been noticed by Delambre, that he does not project upon a diamedral plane conjugate to the place of the eye, but upon a plane touching the sphere at the opposite pole. This practice has not been followed, I believe, by any other writer, though from its simplicity, I am convinced, it suggested, as well as furnished, a ready demonstration of the general property. There have been many other geometrical demonstrations given, but, as must be the case in an enquiry so elementary, they greatly resemble one another, being almost invariably founded upon the circular subcontrary sections of the cone. The chief objection to this method is, that it is incapable of extension to surfaces of the second order generally; at the same time that it is much less simple in principle than the more powerful method of Jordan. A very complete trigonometrical investigation of all the three properties, and their application to the construction of the planisphere is given by Delambre in the *Mémoires de l'Institut. tome 5. p.* 393—416; which so far as that method is concerned leaves nothing more to be done; and M. Gergonne has given an analytical investigation of the same properties, as simple as, perhaps, can be obtained by the method of co-ordinates, in the eleventh volume of his *Annales des Mathematiques*, pp. 153—164. The method of Jordan I will insert farther on, not from its rarity alone, but because I have made use of his method, slightly modified, in the demonstration of the more general property.

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* See Delambre, *Histoire d'Astron. Ancien.* tom. II. p. 433—57: and *Mémoires de l'Institut.* tom. V^{eme}. p. 393. "Ptolemée," says he in the last quoted place "dont le *Traité* est de plus ancien que nous avons sur le sujet, n'avoit sans doute qu'une idée imparfaite de cette propriété fondamentale, qu'il paroit supposée d'abord sans la démontrer ni même l'enoncer expressement." In the History of Antient Astronomy, he has discussed this question with great acuteness; but he has been less fortunate in his attempt to elucidate some points in another work of Ptolemy (the analemma) respecting the antique hour lines, as I shall shew on another occasion and in another place.

† *Sphæræ atque astrorum cœlestium ratio, natura & motus; ad totius mundi fabricationis cognitionem fundamenta*, M.D.XXXVI. *Valderus.*" As this collection of Tracts is become very scarce, and of course expensive, it may be worth while to put down the contents, to enable the mathematical antiquarian to form some notion before hand, whether the book contains any thing *likely* to be of interest with respect to the enquiries he may happen to be engaged in.

This property of circular projections has been *stated* by Legendre to hold in the ellipsoid of revolution, provided we

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The book is in reality an old quarto, though it bears much the appearance of a modern royal, or large paper octavo. Its contents are:—

1. Jacobus Zieglerus Landauus de Solidæ Sphæræ constructione
2. Proclus Diodachus Lycius de Sphæra sive globo cœlesti, scholiis ejusdem Ziegleri explicatus.
3. De canonica per Spæram operatione.
4. Hemicyclium Berosi.
5. Aratus Solensis de siderium natura et motu, simul in eundem cum commentariis Theonis Alexandrini Philosophi.
6. Planisphærium Claudii Ptolemæi.
7. Planisphærium Jordani.

Though it may seem somewhat irrelevant, it may not be improper to remark with respect to the fourth Tract in this list, that Berosus the Chaldean Astronomer, is to be distinguished from the Berosus who is put down in the common Classical Dictionaries, having probably lived as much as eleven or twelve centuries earlier. This is the only way by which the inconsistencies can be removed from the history of Berosus. See Weidler Historia Astronomia, p. 36, and after him, Bailly Astron. Ancien. p. 386.

take the equator and its pole for the plane of projection and place of the eye.* For the prolate ellipsoid a demonstration, rather complex for the subject, was given by M. Fresnel, and which with a little address he might have extended to the oblate.† M. Hachette extended the theorem to all cases of surfaces of the second order where the plane of projection was a circular diamedral section, and the place of the eye the extremity of the diameter conjugate to that section. The demonstration of M. Daviel is analytical and tolerably concise, when we consider the method he adopted.‡ This is the most general case of *circular* sections by the stereographic projections of surfaces of the second order.

The property then passed through the hands of Monge, and, as was the case with every thing he touched, it came out improved. It was elegantly demonstrated in all its generality by M. Chasles.§ I shall annex his demonstration to my own; and have only to remark concerning it that the method of limits which he has employed to find the plane of projection, though perfectly accurate, has not been fairly treated by any author whose works I have met with. This, it may also be noticed, is less elementary than either of those given for the sphere already referred to, The same author has also given a very perfect analytical solution in the notes of Hachette's " Géométre à Trois demensions," p. 270, ad fin.: and again in tome XVIII. of the *Annales des Mathèmatiques*, p. 305, he has resumed the subject, and treated it by the method of poles, which at the same time that it is more complicated offers nothing new in result which might not be readily obtained by simpler means. I ought also to mention an extract contained in tom. XVI. of the *Annales*, from a memoire of M. Dandelin, of Liege, in which there is a slight approximation towards the method of Jordan; though he almost immediately forsakes it for the system of cones.

My own attention was called to this enquiry some time ago, when studying a collateral subject (the temporary hour lines upon the antique dials,) which was before I had noticed any modern works where the planisphere was treated except those of Emerson and Bishop Horsley: and the principal object of this historical introduction is to save other enquirers the trouble of searching through voluminous works, often to no purpose, and always at great cost of time, should they feel disposed to pursue any such enquiry as this. I have not alluded to the common elementary writers on this subject: who are, in general but copyists of one another, or at best of Clavius, Stoflérinus, Bion, Aguillon, and other early mathematicians.

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* *Cosresp. sur l'E'c. Polyt.* tome Ier. p 76.

† Ib. p. 78.

‡ Ib. p. 80—2.

§ Ib. tome IIIème. p. 15, 16.

With respect to the second property, as well as the third, they follow at once from the method of Jordan: and we readily perceive by it that they belong exclusively to the sphere. These were altogether unknown to the ancients; and were, like the generalization of the first property, the immediate consequence of this elegant method. The third has, indeed, strangely enough been attributed by M. Gergonne to M. Puissant (Ann. tom. XI. p. 161.) and afterwards to Ptolemy himself, (Ann. tom. XVI. p. 322.). The researches of a higher authority, that of Delambre, have shewn, however, that it is due to neither of these names. There is not the slightest trace of it, nor any thing which can warrant the belief that he had the slightest knowledge of it to be found in the "Planispæhrium Ptolemœi" as I can affirm from a very careful examination of that work; and Delambre could not find in the writings of Synesius or Theon or any of the antient writers the slightest indication of the suspicion of such a property. "It is announced for the first time," says he, "in the Dictionary of Savérien, from which the article Stereographic Projection has been copied, word for word, into the *Encyclopédie.*" Mem de l'Inst. ub. sup. p. 393. The passage referred to runs thus:—"Dans la Projection Stéréographique, les angles que font les cercles sur la surface de la sphere sont égaux aux angles que les lignes de leurs projections respective, font entr' elles sur la plane de projection." Encycl. Ed. Neufch. 1765, Art *Projection Stéréographique.*

With respect to the remaining property, it is so intimately connected with that just quoted, that a knowledge of one could scarcely subsist without the other; and I should think this would result immediately to him who first demonstrated the theorem whose history we have considered so much at length.

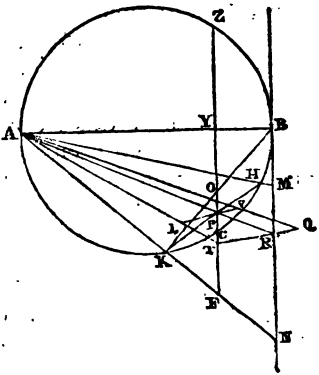
II. (GEOMETRICAL.)

(1.)

The Solution of Jordan.

After proving the particular cases of great circles and parallel less circles, he proceeds to shew that *every circle* is projected into a circle.

Let the less circle not be parallel to the equinoxial,* and let HR be its diameter. Draw the parallel circle whose diameter is ZC cutting it any manner, and let the common intersection of the circles be denoted by the line LVP; and AZBK be any circle passing by the poles and cutting the fore-named circles.



Draw the lines AKN, AHM, and KB. Prolong ZC to F. Then draw ALT, APR, AVQ to meet the plane of projection in T, R, Q, which (since these lines issuing from A to points in the same line KPV, are in the same plane) will be in the same straight line.

Then because AKB and FYA are right angles, and FYA common to the two triangles, the angles AFY, KBA are equal. But (20. III. Euc.) KBA = KHA; and, therefore, the triangles KFP, OHR are similar, O being the intersection of AH, YP. Hence

KP : OP :: FP : PH ; and therefore

KP . PH = OP . PF

= LP . PV, by the circle AVLR.

Hence, obviously by similar triangles NR, RM = TR . RQ : and the points T, Q are in the circle upon the diameter MN.

Q. E. D.

The above being in some respects, according to my view, an improvement upon the actual details of Jordan's demonstration, I annex as *literal* a translation of his text as his barbarous Latinity would permit. See page 284, 5. Fig. upon p. 283.

"Also, let there be one of the lesser circles not parallel to the equinoxial, of which HK is the diameter, cutting the former in any manner whatever: let their common section [differentia] be denoted by the line LPV, which shall cut the superficies of the circle AZBR passing through the poles, at right angles; and being equally inclined to each side, VP will be equal to LP. Then let the lines AKN, AHM and KB be drawn, and the line ZC be extended to F. Also from the point T let TQ be drawn parallel to LPV and representing it in the plane

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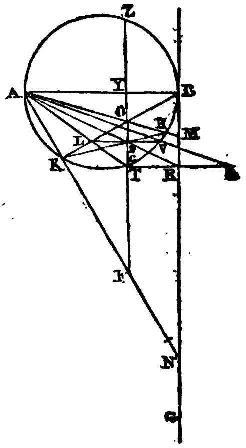
* Jordan, like Ptolemy, projects from the pole of the world in his planisphere.

[of projection]; and the lines ALT, AHR, AVQ. Since, therefore the angles AKB, and FYA are right angles, and the angle FAY is common to both the triangles, the angle AFY will be equal to the angle KBA; but the angle KBA, by Euc. III. 20., is equal to the angle KHA, and therefore the two triangles KFP, OHP will be similar, O being the intersection of AH, YP. Therefore as KP is to OP so is FP to PH. Wherefore, the rectangle under KP and PH is equal to that under FP and PO; but the rectangle under KP and PH is equal (because they cut each other in the same circle) to that under LP and PV. Hence the rectangle under NR [and RM] is equal to that under TR and RQ by reason of the parallelism of the lines. Therefore, the circumference whose diameter is KH, if represented on a plane [touching the sphere at B] will pass through the points M, T, N, Q.



Q. E. D.

I have little to say concerning the variation I have made from his demonstration. It is comprised in the position of the circle AZBK, and in the manner of determining the line TRQ. The circle when drawn orthogonally gives, as he determines it, the line MN for a diameter of the projection: which is an unnecessary restriction in demonstrating that the projection is a circle. The like remark may be made respecting the method of drawing the diamedral plane AZBK in the general solution following: if that plane be conjugate to the two planes HK, ZC, then the line HN will be a *diameter* of the projection. Those who require such a restricted method of demonstration can thus alter it to their own taste.

In the translation there are four bracketted phrases: the second and fourth are explicatory and added to prevent ambiguity: the first is merely the word in the text "*differentia*," which is evidently should be "*intersectio*:" and the third is the supply of a typographical omission. I have been compelled to depart very widely from the punctuation: and indeed the whole tract has much the appearance of never having been read in proof by the author, or indeed, by any competent Latinist.

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* The figure to Jordan is *half* the linear size of the original.

(2.)

My demonstration of the property in reference to surfaces of the second order.

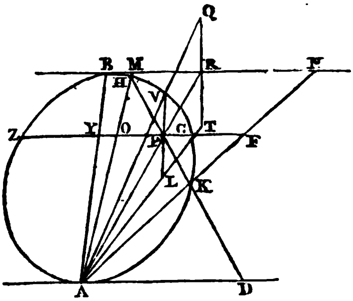
Note. The plane of projection is conjugate to the diameter in the extremity of which the eye is placed.

The following demonstration may be conveniently divided into two parts, for the purpose of distinctly enunciating the former, which is capable of very important uses in investigating the properties of conics, and surfaces of the second order.

(A). Let any transversal ZC be drawn conjugate to one of the angles A of a triangle ABC inscribed in any conic section to cut the sides and curve in Z, O, P, C, F as in the figure: then will

$$ZP . PC = OP . PF.$$

Draw the tangent AD and prolong HK to meet it.



Then by parallels $\left\{\begin{array}{l} HD : HP :: AD : OP, \\ DK : KP :: AD : PF \end{array}\right\}$ and therefore,

$$HD : DK : HP . PK :: AD^2 : ZP . PC.$$

But by the figure, $HD . DK : HP . PK :: AD^2 : OP . PF.$

Hence ex. eq. $OP . PF = CP . PZ.$ *Q. E. D.*

(B.) Let ABHK be any section in a surface of the second order cutting the given section in HR, and passing through the poles A, B. Let any other section conjugate to AB, viz. ZVCL, be drawn cutting the given section in VL, and let it cut HK in P. Draw AHM, AVQ, APR, ALT, and AKN to meet the plane of projection in M, Q, R, T, N. As in my version of Jordan, Q, R, T are in one line parallel to VPL; and MN is parallel to ZC.

Then $ZP . PC : VP . PL :: OP . PF : VP . PL$, by (A), and

$:: MR . RN : QR . RT$, by sim. triangles.

Hence the ordinates VL, QT having the same inclination to the axes ZC, MN, and the rectangles of their segments having in both figures the same ratio, the figures ZVCL and MQNT are similar. *Q. E. D.*

Cor. The theorems of Legendre and Hachette are evidently comprised in this.

(3.)

The Demonstration of M. Chasles, Corresp.Polyt. tom. 3. p. 16.

This demonstration presupposes that we can describe a cone to envelope any two sections of a surface of the second order: for the truth of which the English reader may consult my "Elementary Demonstrations," in the present Number of the Repository, this admitted, M. Chasles proceeds;—

"Any plane curve being traced upon a surface of the second degree, and projected upon a plane by a cone having its summit in a point of the surface for which the tangent plane is parallel to the plane of projection; then the projection of this curve will be a curve similar to that which we obtain by cutting the surface by a plane parallel to that of projection.

"In fact, let C, C′ be the two curves obtained by cutting the surface by two planes parallel to that of projection. By the curve C′ and the plane curve A traced upon the surface, describe a cone, it will meet the plane of projection in a curve B similar to C′, and therefore to C. But the cutting plane which gives the curve C′ being moved parallel to itself in approaching indefinitely near to the tangent plane will vary the curve, which is nevertheless always similar to C. Then, at the limit, that is, when the secant plane C′ becomes confounded with the tangent, the curve B is still similar to the curve C."

(4.)

The Analytical Demonstration of the general property, by M. Chasles. (Supp. Hachette, p. 271.)

Theorem. If by the plane curves S, S′....traced upon a surface of the second degree, we describe cones which have their summits in the same point P, and then cut them all by a plane parallel to that which touches the surface in P, the sections are curves similar to one another and similarly situated. They will also have their centre upon the straight lines which join the point P and the summits of the cones which respectively circumscribe the surface in the given curves S, S′.....

Demonstration. Let $Ax^2 + A'y^2 + A''z^2 - K = 0$ ....(1). both equation of the surface of the second degree referred to conjugate diameters.

Let f, g, h be the co-ordinates of the summit of the cone which touches the surface in the curve S; the equation of the plane of this curve of contact is

$$Afx + A'gy + A''hz - K = 0 \ldots\ldots\ldots\ldots\ldots (2).$$

Let us consider the same curve S as the base of a second cone (S, P) which has its summit at the extremity P of the diameter of the surface directed along the axis z, then we shall have for the co-ordinates of the summit P,

$$x = 0,\ y = 0,\ z = \sqrt{\frac{K}{A''}},\ \text{or for brevity } z = c.$$

The equation of the tangent plane to the surface of the second degree in the same point P will be

$$z = \sqrt{\frac{K}{A''}}$$

and the equation of the diamedral section parallel to this plane will be

$$Ax^2 + A'y^2 - K = 0 \ \dots\dots\dots\dots\dots\dots (3).$$

We proceed now to shew that the cone (S, P) being cut by the diamedral plane parallel to the tangent plane at the point P, the curve which results and the diamedral section of the given surface situated in the same plane, are curves similar and similarly situated.

Let us seek in the first place the equation of the conic surface (S, P). The equations of the generating straight lines are

$$x = L\,(z - c), \ \dots\dots\dots\dots\dots\dots\dots\dots (4)$$

$$y = M\,(z - c), \ \dots\dots\dots\dots\dots\dots\dots\dots (5)$$

Substitute these values of x and y in the equations (1) and (3), and recollect that $A''c^2 - K = 0$, because $c^2 = \frac{K}{A''}$, we have, dividing by $z - c$,

$$(z - c)\,(AL^2 + A'M^2 + A'') + 2A''c = 0, \text{ and}$$

$$(z - c)\,(ALf + A'Mg + A''h) + A''ch - K = 0;$$

and eliminating $z - c$ between these two equations, we shall have

$$\left\{ \begin{array}{l} (K - A''ch)\,(AL^2 + A'M^2 + A'') + \\ + 2A''c\,(ALf + A'Mg + A''h) = 0 \ \dots\dots\dots\dots (6). \end{array} \right.$$

This equation expresses the relation that exists between the quantities L and M, and shews that if one of them take all the values which correspond to the divers position of the rectilinear generatrix of the cone, then the values of the other for the same positions will be determined.

It hence follows that the system of the three equations (4), (5), (6) represents the conic surface (S, P).

Eliminating L and M by means of these equations, or substituting in equation (6) the values $\frac{x}{z - c}$, $\frac{y}{z - c}$ of L and M deduced from (4) and (5) we shall have for the equation of the conic surface (S, P):

$$\left\{\begin{array}{l}(\mathrm{K}-\mathrm{A}''ch)\left(\mathrm{A}x^2+\mathrm{A}'y^2+\mathrm{A}''(z-c)^2\right)+\\ +2\mathrm{A}''c\left(\mathrm{A}fx+\mathrm{A}'gy^2+\mathrm{A}'h(z-c)\right)(z-c)=0\ \ldots\ldots\ (7).\end{array}\right.$$

We perceive that in making $z=0$, the diamedral plane which contains the curve of the surface of the second degree which is represented by the equation (3) cuts the cone (S, P) in the line whose equation is:

$$\mathrm{A}x^2+\mathrm{A}'y^2-\frac{2\mathrm{A}''c^2(\mathrm{A}fx+\mathrm{A}'gy)}{\mathrm{K}-\mathrm{A}''ch}+\frac{\mathrm{K}(\mathrm{K}+\mathrm{A}''ch)}{\mathrm{K}-\mathrm{A}''ch}=0\ldots\ldots(8)$$

which may be put under the form:

$$\mathrm{A}\left(x-\frac{cf}{c-h}\right)^2+\mathrm{A}'\left(y-\frac{cg}{c-h}\right)^2+\text{const.}=0\ldots\ldots(8').$$

Since this does not differ from equation (3) except by linear terms and constants, it follows that these two sections are similar and similarly situated.

The centre of the section represented by equation (8) is upon the straight line which joins the point P to the summit (f, g, h) of the cone circumscribing the surface of the second degree in the curve S, and of which the equations are

$$x=\frac{f(c-z)}{c-h},\ y=\frac{g(c-z)}{c-h}\ \ldots\ldots\ldots\ldots\ldots(9).$$

In fact, if we cut the cone (S, P) by the plane whose equation is $z=h$, which passes by the summit (f, g, h) we have the equation of this section; and by putting in equation (7) for z its value h, it becomes,

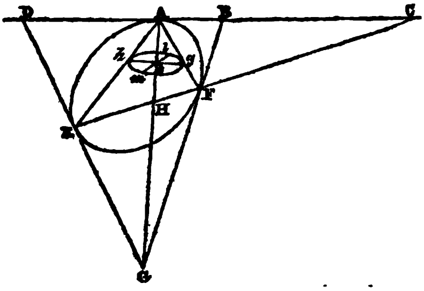
$$\mathrm{A}(x-f)^2+\mathrm{A}'(y-g)^2-(\mathrm{A}f^2+\mathrm{A}'g^2+\mathrm{A}''h^2-\mathrm{K})=0.$$

The centre of this section is the summit (f, g, h) of the circumscribing cone. But all the parallel sections of a cone have their centres upon the same straight line which passes by the summit; and, therefore, the centre of the section (8) is the point of intersection of the diamedral plane $z=0$, and of the straight line (9) which joins the point P and the summit of the circumscribing cone: a proposition which is again demonstrated by equation (8) in which the quantities $\frac{cf}{c-h}$, $\frac{cg}{c-h}$ are the co-ordinates of the point where the straight line represented by equation (9) cuts the plane of xy. The theorem, then, is proved.

(5.)

The beautiful property annexed by M. Chasles to the Ptolemaic one, deserves a geometrical consideration, and is found thereby to admit of an elegant proof, thus:

Let any plane be drawn through G the summit of the cone and the place of the eye, cutting the tangent plane in BD, the plane of contact in FE, and the sides of the cone in GF, GE. Also let it cut one of the planes conjugate to A in *gh*.



Then it is a well known property of the figure that

EH : HF .: EC : CF,

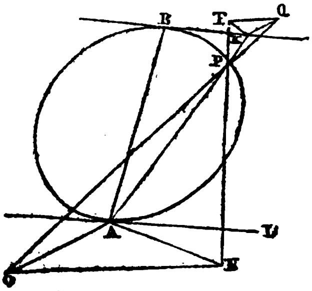
and ∴ by Leslies' Elem. Geom. VI. Pr. 6. we learn that GH is bisected in *k*.

In like manner all the other lines produced in the same manner as *gh* such as *lm* are bisected by GA in *k*. Hence *k* is the centre of the section *glhm*. *Q. E. D.*

(6.)

With respect to the demonstration of the second and third properties, I propose the following method.

Draw the tangent planes at A, B, P, intersecting in OQ, TN.



Let the traces of the planes of the two intersecting sections of the given surface upon the tangent plane at P be OQ, TN: and draw APB, TR, QR, AO, AN. Then TRQ = OAQ = projection of the angle made by the tangents to the two sections at the point of contact.

Now, in the sphere, because OPN, OAN are tangent planes at P and A, we have

NA = NP and OA = GP: and hence by similar figures
RT = TP and RQ = QP. Which is the second property.

Also because TP and PQ are equal to TR and RQ, and TQ common, we have the angles TPQ, TZQ equal. Which is the third property.

But to ascertain whether these properties are peculiar to the sphere, we remark that by parallels,

PQ : QR :: PO : OA, and PT : RT :: PN : NA.

Hence it depends upon the linear tangents of the several sections of the surface of the second order. But, generally, the properties do not inhere in other than spherical bodies. However there are in all the surfaces specific points for which they take place.

Let A be the projecting point: then if a tangent plane be described at A, it is evident that if in the diamedral section through A, we draw lines parallel to the principal axes, we shall have an inscribed rectangle, from either of the three remaining angles of which tangents being drawn to meet that at A, the parts mutually intercepted shall, in pairs, be equal to one another. If moreover, we draw tangent planes to the surface at these points, these tangent planes will be related to one another as the tangent planes of the sphere already investigated. In short, that for any projecting point, a certain number of other points may be found, and that any sections having those points for their conjugates shall have their tangents, and the angles contained by the tangents projected into equal tangents and equal angles.

Should the surface, however, be one of revolution, then there are not only four, but an infinite number of points which have this property: viz. all those contained in the sections vertical to the axis and passing through the four points.

The point opposite to the projecting point has for a tangent plane, a plane *parallel* to that at A: and, therefore, they limit each other only at an infinite distance. In this case the tangents and their projections coincide, and on that account are equal. The same may be said of the angles.

In the other cases it is evident that the points of contact of the two tangent planes with the general surface may be made the points of contact with a sphere: and hence we have another consideration from which the truth of the proposition may be derived.

It may not be altogether unnecessary to remark that a little inaccuracy of expression has occasionally been employed on this subject, which has given rise to some misconception. The inclination of the circles on the sphere is the same as the inclination of the planes; but this is not the case with less circles: but regardless of this difference, the inclination of the tangents to the circles has sometimes been expressed as that of the inclined planes, and the dihedral angle has been said be projected into one equal to itself. The supposition itself is absurd, and therefore the operation is impossible. If we understand, as is done by all good writers, by the phrase "inclination of two circles," or "inclination of two sections" of any given surface, the *angle contained by the tangents at the point of intersection*, all ambiguity will be avoided. In great circles this becomes the same as the spherical angle of our common trigonometry. On a plane, the angle of the tangents is the same as the angle of the curves themselves: and, in short, in all cases the angle made by any two sections of a surface is the same as that made by the traces of the sectional planes upon the tangent plane at the point of intersection.

T. S. D.

ARTICLE XVII.

Researches in the Geometry of Three Dimensions.

By Mr. T. S. Davies, *Bath.*

(Continued from page 121.)

The surfaces of the second order, whether considered in reference to their own curious and diversified properties, or as a key to the ultimate structure of surfaces in general, are the most interesting as well as the most important in the whole range of Solid Geometry. The properties, indeed, of most other bodies become so complicated, that whether investigated by means of lines actually traced in them, or by means of their representative equations, they alike defy the scrutiny of the most ardent and the most sagacious geometer: but the properties of surfaces of the second order are so completely under the control of both methods that they offer scarcely greater difficulties than the lines of the same order; whilst in many instances they have furnished properties of the plane sections, which before that circumstance might have lain concealed for ages yet to come. When, however, we reflect that the curvature at each point of a surface being known; and moreover that the curvature of any surface at any point is the same with that of a surface of the second order, which we can generally determine, osculating it at the point in question:*—we shall see at once the great value of these surfaces in all enquiries of a more recondite character in solid geometry, and a reason why they have been so assiduously investigated by the mathematicians of the continent.

The majority of the properties of *lines* of the second order depends upon the conjugate diameters: and the next in point of number and importance upon the doctrine of transversals, modified to suit the case. So likewise, in *surfaces* of the second order. the same circumstances will be found to hold good. It may, indeed, be said that in both cases, the doctrine of conjugates might be easily derived from that of transversals, and therefore it would be more systematic to profess to derive the whole course of the investigation at once from the transversals: and this, in truth is, what the illustrious Pascal seems to have done,† and in imitation of whom several distinguished Foreign Geometers have more recently adopted as the fundamental principle of their enquiries. Yet the most systematic is not always the most simple or the most intelligible process; and it has, therefore, by many been deemed better to sacrifice the doubtful advantages of such systems to the paramount considerations of brevity and perspicuity. Such is the plan I adopt in these miscellaneous speculations: viz. to employ in all cases the method which most readily brings me to the object before me, and rest fully satisfied by establishing the properties I bring forward without adhering exclusively to one or other method of effecting it.

One difficulty presses on me in the outset—that of writing in a language which does not boast a single work containing the leading propositions which I am obliged to employ. I am, therefore, unable to quote specific authorities, so as to render the propositions whch I may give, dependant upon any course of reading in which the mere English Geometer may be supposed to have indulged. I shall, in consequence, be obliged to quote the properties I may need at length, but shall generally do it without demonstration. I had, indeed, originally intended to trace the progress of this branch of mathematical science from its first buddings; and to examine at length the relative advantages of the different methods of discussion: but I find the plan will lead me into details inconsistent with the limits necessarily prescribed for such subjects in a work like the Repository; I, therefore, abandon that project and shall reserve the notes I had made for some other purpose—probably as an appendix to a separate work specially devoted to Solids and Descriptive Geometry.

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* This remarkable property was discovered by M. le Baron Dupin, and is made by him the basis of the most profound and systematic investigations into the doctrine of surfaces which has yet appeared. See his *Développements de Géométrie*, pp. 12, 13.

† Vide the letter of Leibnitz to M. Perrier; *Ouvrages Père Pascal*; tom. Veme, fin.

SECTION II.

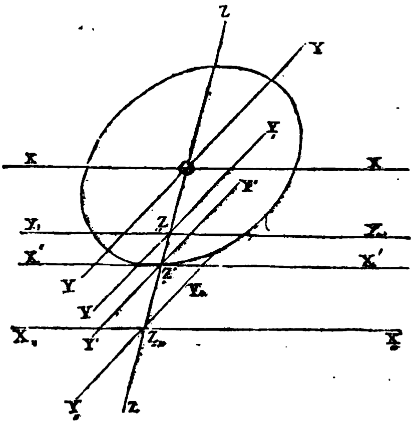
The conjugates (diameters and diamedral planes) of the second order.

I.

DEFINITIONS. 1. Lines given in position are conveniently represented by single letters taken as their symbols, such as X, Y, Z. If specific lengths measured from specified points be mentioned they are denoted by X′, Y′, Z′, and X″, Y″, Z″, &c.

2. A plane is denoted by conjoining the symbols of two lines situated in that plane, as XY, YZ, ZX, &c. which denote the plane in which are situated the lines X and Y, Y and Z, X and Z, &c.

3. The intersection of two planes is represented by a vertical line between their symbols, as XY | VS, indicates the line in which the plane XY cuts VZ.



4. The intersection of a line and plane being a point, the symbol of that point is a dot between their respective symbols, situated at the *upper* part the line (in the manner that Dr. Hutton places his decimal point, and nearly for the same reason, to distinguish it from the sign of multiplication): thus

X · ZY

signifies the point in which the line X meets the plane ZY.

5. On the same principle the intersection of the two lines X and Y is denoted by

X · Y.

6. The intersection of two lines each of which is the intersection of two planes will have for its symbol

(ST | QP) · (XY | VZ).

or more simply

ST | PQ · XY | VZ.

7. Three intersections of three several pairs of planes may be thus represented

ST | QP · XY | VZ · OV | RN

but if we wish to express that the three points of intersection are coincident we should enclose them in parenthesis, thus:—

{ST | ....}.

8. The same with respect to planes, the symbol

| ST | QP | VR | NZ |

will signify that four planes mutually intersect one another, but enveloped in parentheses they will meet in the same line, and if in double parentheses that they meet in a point.

Thus {(| ST | QP | VR | NZ |)}.

This system of symbols might obviously be carried to any desired extent and is remarkably simple and systematic. Though the same in principle with that employed in the *Géométrie de Position*, it has the advantage of being less liable to mistakes, from drawing a line a little too long or too short—a fault to which that of Carnot is very liable, as my own experience has often shewn me.

II.

PRINCIPLES. 1. *Conjugate diamedral planes*, or simply *conjugate diamedrals*, are in systems of threes: their symbols are

XY, XZ, YZ.

2. *Conjugate diamedral lines.* or simply, *conjugate diameters*, are also in systems of threes: their symbols are

X, Y, Z; or XY | YZ, XY | XZ, ZY | ZX

according as we consider them to be the original lines upon which we found our considerations, or the intersection of the diamedral planes mentioned in the last paragraph.

3. When one diamedral as XY, is given; the intersection of the other two is given also viz. XZ | ZY, or Z is given also.

4. When one *diameter* as Z, is given, the plane of the other two, XY is given also.

5. The *diameter* and *diamedral* so related are said to be conjugate to one another, or simply to be *conjugates*.

6. When two *diameters* are given, the *diamedral* is also given, and hence the *third diameter* is also given.

7. When two *diamedrals* are given, their *intersection* is given, and hence the *third diamedral* is given also.

III.

Principles continued 1. Any diameter Z being drawn in a surface (S) of the second order, and a plane parallel to its conjugate diamedral, *this plane is said to be conjugate to* Z, and is denoted by $X_{\prime}Y_{\prime}$ to distinguish from XY the conjugate diamedral. The two Z, $X_{\prime}Y_{\prime}$ are called conjugates, and their symbol is

$$Z,\ X_{\prime}Y_{\prime}.$$

2. Measuring upon Z the distance $Z_{\prime}$ and $Z_{\prime\prime}$, and putting Z' the value of the line Z intercepted by the surface, if we make this relation amongst the three quantities

$$Z_{\prime} \cdot Z_{\prime\prime} = Z'^2$$

then $Z_{\prime}$ and $Z_{\prime\prime}$ are the *reciprocals of each other:* and the corresponding conjugates $X_{\prime}Y_{\prime}$ and $X_{\prime\prime}Y_{\prime\prime}$ (parallel to XY) are called *reciprocal conjugates.*

3. The points were the *reciprocal planes* meet Z in $Z_{\prime}$ and $Z_{\prime\prime}$ are called the *reciprocal poles* $Z_{\prime}$ and $Z_{\prime\prime}$; and the *reciprocal planes* themselves are called *reciprocal polar planes.* The lines $X_{\prime}$ and $Y_{\prime\prime}$ or $X_{\prime\prime}$ and $Y_{\prime}$ are denominated *reciprocal polars.* Sometimes, indeed, the *pole* and its *reciprocal polar plane* are also collectively termed *reciprocal polars,* also. This nomenclature of poles, polars, and polar planes are good in respect to the important properties of these surfaces discovered by Monge, and presently to be more particularly referred to: but as we lose sight of either the *conjugation* or the *reciprocity* of these planes by that system of terms, I have preferred, for the present at all events, to adopt the terms *reciprocal conjugates,* when mentioning either $X_{\prime}$, $Y_{\prime\prime}$. or $Z_{\prime}$, $X_{\prime\prime}Y_{\prime\prime}$. The course of my enquiries is more dependant upon reciprocal conjugates than upon polars.

4. When we wish to distinguish between the pairs of reciprocal conjugates, we may call $X_{\prime}$, $Y_{\prime\prime}$ or $X_{\prime\prime}$, $Y_{\prime}$ the *mean reciprocal conjugates.* and $Z_{\prime}$, $X_{\prime\prime}Y_{\prime\prime}$ or $Z_{\prime\prime}$ $X_{\prime}Y_{\prime}$ the *extreme reciprocal conjugates.*

5. The planes $X_{\prime}Y_{\prime}$ and $X_{\prime\prime}Y_{\prime\prime}$ being so related that one always cuts the surface and the other never, they are conveniently distiguished into *interior* and *exterior reciprocal conjugate planes.*

IV.

MONGE'S THEOREM, *Géom. Descript. section* II. *art.* 38—41.

If a cone (C) having its centre *c* in one of the reciprocal conjugates, envelope a surface (*s*), then the plane of contact (P) shall envelope the other reciprocal conjugate of (*s*).

Conversely. If a plane (P) envelope one of the reciprocal conjugates of the surface (*s*), the centre of C of the cone shall be enveloped by the other reciprocal conjugate.

Thus, in reference to our notation, if a cone have **its centre** in $\mathrm{Y}_{,}$ its plane of contact shall pass through $\mathrm{X}_{,,}$; and if a cone have its centre in $\mathrm{X}_{,,}\mathrm{Y}_{,,}$ then its plane of contact will always pass through $\mathrm{Z}_{,}$.

The same will, conversely, take place by interchanging $\mathrm{X}_{,}$, $\mathrm{Y}_{,}$, $\mathrm{Z}_{,}$ and $\mathrm{X}_{,,}$, $\mathrm{Y}_{,,}$, $\mathrm{Z}_{,,}$.

If, however, we denominate in the cone, the centre and plane of contact, reciprocal conjugates, we shall have the theorem in this form:—

If one pair of reciprocal conjugates of the cone and surface coalesce, then the other pair will also coalesce.

Without doubt this beautiful theorem is due to Monge, though the corresponding plane one had been known long before. There are some traces of it to be found in the writings of Desarguis and of Gregory St. Vincent: and it is very completely stated and proved by Lahire, in his *Sectiones Conicæ* (1685), as well as several other theorems which are in this country referred to a very recent period. The first traces that I have been able to find of any English writer having attended to this property, is in the writings of Dr. Matthew Stewart, who had examined the circle and determined many of the more obvious and more elegant properties of these reciprocal conjugates. They have been variously treated by piecemeal in our periodical works; but as no specific series of results has been drawn from them, further notice of the works in which they occur is needless on the present occasion.

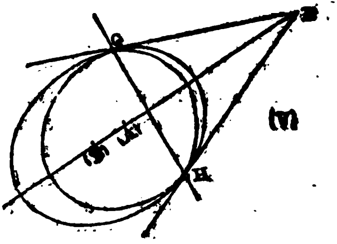
There is, however, a very complete series of these properties to be found in a paper by Dr. Hey, in the Philosophical Transactions for 1814, where they are used as preliminary to the solution of some problems concerning the perspective representation of circles. As this paper will come before us again in another shape, I shall forbear any remark upon it here.

It is to foreign mathematicians, however, that we must look for the most simple demonstrations of this series of propositions and the kindred ones in surfaces of the second order: and it is to their discussions of these that we owe the developement of that beautiful system of poles which enters more or less into almost every geometrical enquiry concerning these surfaces, and which has led to many of the improvements in their system of transversals.

V.

Having stated the theorem of Monge in terms as brief and comprehensive as the subject seems to admit, let us enquire whether the envelope surface is necessarily a cone, or whether any other concentric surface might not be endowed with the same properties. We shall thus find that no other surface than the cone has that quality: and the consideration by which we arrive at that conclusion is extremely simple.

1. Let (A) be a fixed conic section, and B the centre of a series of conic sections each of which is to touch (A) in two points, as G, H. Then because the section (B) touches (A) at G, H, they will have common tangents, HF, GF at these points. Join GH, FA, FB. Then because FA is drawn from the intersection of the tangents to (A) to the centre, it bisects GH the chord of contact: and in the same way we find that FB bisects GH. Hence we infer that F is always in a given line AB, and that HG is always parallel to itself.



2. It is easy to see also that if any surface of the second order whose centre is B, have a plane contact with a given surface (A), the cones of contact will have their centres in the line AB; and that the planes of contact are parallel to one another. For the several diamedral sections of the cone and surface will produce the case we have already considered *in plano*.

Now, as these sections are parallel they do not pass through any specific point or line reciprocally conjugate to that in which B is situated: and there is not, consequently, any such a property belonging to the second order in general, as that which Monge has given with respect to the cone. There are, however, some curious properties belonging to the figure which we shall notice further on.

VI.

Let now two planes (P), (P′) be drawn through F, cutting the surfaces in (s), (s'): then if in (s), (s') diameters be drawn parallel to $X_{,,}$, these shall be conjugates to

$$(P) \mid YZ \text{ and } (P') \mid YZ.$$

For draw the *diamedrals* parallel to (s), (s') and these will intersect in X: but X is parallel to $X_{,,}$, and therefore YZ is conjugate to the diameters, in (s), (s'). Hence these diameters are conjugate to

$$(P) \mid YZ \text{ and } (P') \mid YZ.$$

Hence we derive a test of the capability of two lines of the second order (s), (s') situated in different planes to be inscribed in the same surface of the second order.

Produce the planes to meet in $X_{,,}$: draw diameters in (s), (s) parallel to $X_{,,}$ and in the same sections, conjugates to these diameters: if these intersect $X_{,,}$ in the same point, the sections are inscriptible, otherwise they are not.

VII.

These conditions are necessary, and they are also sufficient. But they do not determine the individual surface in which these sections are restricted to cohere: nor is there such a surface determinable till we have fixed upon some other condition. The centre must be somewhere in the plane YZ to fulfil the forenamed conditions; but it may fulfil them wherever in that plane it be posited. There will hence be innumerable positions for that centre, and therefore innumerable surfaces in which the two sections may cohere. We learn from this, that

If two lines of the second order be inscriptible in one surface of the second order, they are capable of innumerable inscriptions.

VIII.

If two planes be drawn to rest upon two such sections, their line of intersection will cut the diamedral YZ in a point, which will be the centre of their enveloping cone: and this cone will have its vertex inward or outward as the planes meet between the given conic sections or beyond the less. Hence a very particular case of this proposition, but a useful one is, that, *if two lines of the second order are inscriptible one way in a cone, they are capable of the other inscription.*

IX.

From the same principles it will be easy to demonstrate the two following beautiful propositions, the first of which is due to Brianchon and the second to Monge.

If a cone pierce a surface of the second order making two isolated branches, one of which is plane, then the other also is plane. Journ. de l'ec. Polyt. Cah. XIII. p. 304.

If two surfaces of the second order are tangential (in a plane curve) to a third surface, then these two surfaces will have a plane intersection. Corresp. de l'ec. Polyt. tom. II. p. 321.

Nor will it be difficult to prove the following more general ones:—

If two surfaces of the second order intersect one another in two isolated branches, and the one be plane, then the other also is plane.

If each of two surfaces of the second order intersects a third surface of the second order in a plane curve they will (if they intersect at all) intersect one another in a plane curve.

These propositions will again appear as corollaries to some of our subsequent speculations, on which account I shall dismiss them at present with a bare enunciation.

X.

It has been ascertained in VI. and VII. that when any two plane sections are inscriptible, they are inscriptible, in innumerable surfaces of the second order: the only condition to which they are subject being that their centres are in the diamedral conjugate to both the sections. In like manner if we conceive a third section (s''') which is inscriptible separately with each of the others (s'), (s'') there are innumerable surfaces which may envelope (s'''), (s') and innumerable others which may envelope (s'''), (s''); the only conditions to which these are subject being that they shall have their centres in the two corresponding conjugate diamedrals.

Moreover, if any (c'), (c'') be the conjugates respectively of (s''), (s''') and (s'), (s''') then any point in (c'), (c'') may be the centre of two surfaces which will envelope (s''), (s''') and (s'), (s'''): and this property belongs to no point out of the line $(c') \mid (c'')$. The statement being made concerning each of the other pairs of planes, we are led ultimately to remark that the common intersection of the three planes is a point from which a surface may be described to envelope all three of the sections.

Hence, *if three sections taken two and two are inscriptible they are all three inscriptible in the same surfaces of the second order.*

It is, moreover, plain that they admit of inscription, generally, in only *one* such surface.

Other tests of inscriptibility will hereafter be laid down: but it will in the sequel appear that they are but modifications of this, or at most very proximate consequences of it.

XI.

The property in (X) is perhaps one of the most fecund in this method of enquiry that can be proposed; and we shall in future parts of this *olla podrida*, have frequent occasion to employ it. I shall therefore only deduce from it one single proposition, and that solely on account of its vast importance in the doctrine of curvatures.

If any three sections of the second order be about the same diameter, they can be inscribed in a surface of the second order. Dupin, developp. de Géom. p. 60.

For from the middle of the line draw diameters conjugate to that line in each section: these are in the diamedral conjugate to those two sections, and those two diameters meet the given line in the same point: and therefore the two sections are inscriptible. In like manner each pair is inscriptible, and therefore all three are inscriptible.

XII.

Let two surfaces $(A)_1$ (B) be given. Join these centres by the line AB or X. Conjugate to this line draw diamedrals, $(YZ)_1$ and $(YZ)_2$* meeting in Y_1. Then any cone whose centre is in $(YZ)_1$ will have its reciprocal conjugate (or plane of contact parallel to X, as will likewise any cone having its centre in $(YZ)_2$. Their intersection will then be parallel to X.

Now when the centres coalesce with Y_1 then their reciprocal conjugates coalesce with $\overset{1}{X}_{//}$, $\overset{2}{X}_{//}$: and conversely when the planes of contact coalesce with $\overset{1}{X}_{//}$, $\overset{2}{X}_{//}$ the centres of the cones are in Y_1.

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* This notation merely refers to two surfaces *mutually* related to one another, and *symmetrically* to any other magnitude. Every method which can be devised by means of notation for presenting to the eye at once the symmetry and relations of the objects of our contemplation contributes materially to the ready comprehension of their properties by the understanding, and thereby greatly facilitates both acquisition and discovery. The Geometer will find much advantage, in this respect, in the adoption of some of the happy devices of the analyst.

The functions of the surfaces marked with the subscribed numbers in the sinister-base (borrowing a term from the vocabulary of the Herald's College) shew that they belong to the surfaces which are numbered in accordance with them. The dot beneath the accented $\acute{z}$ is used to express the portion of that axis which constitutes the semidiameter of the surface. It is sometimes convenient to put the numbers *over* the symbols.

If, moreover, the cones have a common centre in Y, the planes of contact will then be parallel to one another: and, conversely, if the planes of contact be parallel the centres will be coincident.

Lastly, if the planes of contact become coalescent then there is one position of coalescence of the centres corresponding to it, and *only one*. Hence *there is always one point, from which tangent cones being drawn to two given surfaces, the planes of contact shall coalesce:* and it is evident that there is only one such point.

XIII.

The same conclusion might have been derived analytically: thus, Let the surfaces be denoted by the equations

$$1.\quad \left.\begin{aligned} &Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy + 2Gx \\ &+ 2Hy + 2Jz = K \end{aligned}\right\}$$

$$2.\quad \left.\begin{aligned} &A'x^2 + B'y^2 + C'z^2 + 2D'yz + 2E'xz + 2F'xy + 2G'x \\ &+ 2H'y + 2J'z = K'. \end{aligned}\right\}$$

Then the planes of contact have for equations (x', y', z' being co-ordinates of the centre)

$$1'.\quad \left.\begin{aligned} &x(Ax' + Fy' + Ez' + G) + y(Fx' + By' + Dz' + H) \\ &+ z(Ex' + Dy' + Cz' + J) + (Gx' + Hy' + Jz' + K) = 0 \end{aligned}\right\}$$

$$2'.\quad \left.\begin{aligned} &x(A'x' + F'y' + E'z' + G') + y(F'x' + B'y' + D'z' + H') \\ &+ z(E'x' + D'y' + C'z' + J') + (G'x' + H'y' + J'z') + K') = 0. \end{aligned}\right\}$$

(See Monge Analyse Applique, p. 14.)

Now if we multiply the first of these quantities by an indeterminate constant p, the equation will continue to represent the same plane: and in order that the two equations $x \times 1'$ and $2'$ may represent the same plane it is necessary that the coefficients of the variables x, y, z, in each equation should be equal each to each. This gives the four equations

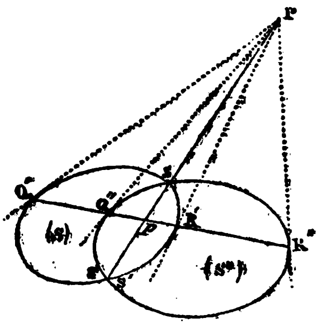
$$p \cdot (Ax' + Fy' + Ez' + G) = A'x' + F'y' + E'z' + G'$$
$$p \cdot (Fx' + By' + Dz' + H) = F'x' + B'y' + D'z' + H'$$
$$p \cdot (Ex' + Dy' + Cz' + J) = E'x' + D'y' + C'z' + J'$$
$$p \cdot (Gx' + Hy' + Jz' + K) = G'x' + H'y' + J'z' + K'.$$

From these equations we find the four quantities p, x', y', z': and thence the position of the summit of the cone. It is likewise obvious from the equation being of the first degree in x', y', z', that there is but one point from which it can be placed.

XIV.

From this property we may derive several curious results: one of which shall suffice for the present, viz. that proposed as Question 14. Gent. Diary of the present year.

Let P be the point from which the cones of contact being drawn, its reciprocal conjugates (the planes of contact) shall coincide. Let any line drawn from P through a point s in the curve of penetration of the surfaces (s'), (s'') meeting the common plane of contact in p, and the two surfaces again in s', s''. Now by well known properties of the figure, the line Pp will be harmonically divided in s, s' by the surface (s'), and in s, s'' by the surface (s''). Hence three of the points P, s, p being identical in both cases, the fourth s', s'' must also coincide: or the line Pp again passes through the curve of mutual penetration. The same happens to all the lines from P through the points of the said curve, and hence that *curve is wholly situated upon the cone formed by those lines.*



Note. The annexed figure represents two sections made in the surfaces (s'), (s'') by *any plane* through the line Ps; and the line Q′R″ section of the common reciprocal conjugate of P. We have proved that s', s'' coincide.

XV.

By a method very similar to that employed in XIII. we might have deduced this proposition, without the intervention of the property there established and passing through the route sketched out in XIV.: but it is unnecessary, and therefore as the theorem is only a case of a much more general one, which may be demonstrated in the same way, I shall give it in preference.

If two surfaces of the second order intersect, there are innumerable surfaces of the second order which have a common section with them.

Let the two surfaces be denoted by the equations employed in XIII. Multiply each by the indeterminate constants m, n: then adding the resulting equations together we shall have

$$\left.\begin{array}{l}(m\text{A}+n\text{A}')x^2+(m\text{B}+n\text{B}')y^2+(m\text{C}+n\text{C}')z^2\\ +2(m\text{D}+n\text{D}')yz+(2m\text{E}+n\text{E}')xz+2(m\text{F}+n\text{F}')xy\\ +2(m\text{G}+n\text{G}')x+2(m\text{H}+n\text{H}')y+2(m\text{J}+n\text{J}')z\end{array}\right\}=m\text{K}+n\text{K}'.$$

Though for the sake of symmetry *two* indeterminates have been introduced into this equation, it is obvious that *one* would have sufficed; and moreover that there is in reality but one *independent* intermediate employed. We have then to remark that if we refer this equation to the centre of the surface we shall eliminate the terms which contain the first powers of x, y, z: and we shall have a new equation of this form

$$A''x^2 + B''y^2 + C''z^2 + D''yz + E''zx + F''xy = K''$$

Now this equation involves the arbitrary quantity $\frac{m}{n}$, and hence will vary with that quantity, always, however, being referred to the centre as the origin; and hence innumerable surfaces are represented by this equation all of which have a common intersection with the two given ones. *The cone in* XIII. *is a particular case of this.*

The geometrical demonstration of this property is not difficult but depends upon other properties which are rather remote from those already laid down: and, therefore, on account of its similarity in the enunciation, I have put it here though obliged to employ the algebraical geometry to establish it. I shall now conclude with a notice of some propositions respecting cones of the second order in different planes, analogous to those in my former paper concerning circles on the same plane.

XVI.

The properties established concerning circles are true concerning similar conic sections similarly placed, *on the same plane*, as is obvious from (XVI.) of that series. For such would be the result of cutting the cylinders and planes there mentioned, by a plane oblique to the axis. M. Chasles has employed the stereographic projection of surfaces of the second order for establishing the properties of centres and axes of similitude, and other properties of this kind (Ann. des Math. tom. XVIII. p. 305—20.) This method though unnecessary for simple establishment of that property is yet very ingenious when applied to the demonstration of the corresponding properties of plane lines traced upon a surface of the second order.

XVII.

If two sections (s), (s') of a surface of the second order (s) be given: and any two other sections (s'), (s'') be described tangential to these, then the cones enveloping these latter sections and having their vertices internally situated, these vertices will be in the same straight line.

For produce the planes of (s), (s') to meet in (s) | (s'). Then the cones in question will touch in all their length these two planes; and will hence have their vertices in the line (s) | (s).

XVIII.

The *internal* cones will have their vertices all in one plane: viz. the reciprocal conjugate of the cone which envelopes (s), (s'): as is too obvious to need a detailed proof.

(To be continued.)

THE END OF THE SECOND PART OF THE FIFTH VOLUME.

THE
MATHEMATICAL REPOSITORY.

VOL. V. PART III.

MATHEMATICAL MEMOIRS, EXTRACTED FROM WORKS OF EMINENCE.

ARTICLE I.

On the determination of the attraction which a planet would exert upon any point given by position, if its mass were distributed along the whole orbit,—uniformly in relation to the time in which each part of the orbit is described.

By CHARLES FRED. GAUSS, *presented to the Royal Society of Sciences 17th January,* 1818.

The secular variations which the elements of the orbit of a planet suffer from the disturbing force of another planet, are independent of the position of the latter in its orbit; and will be the same whether the disturbing body move in an elliptic orbit according to the laws of Kepler, or its mass be conceived to be distributed along its orbit, in such a manner that equal portions of the mass shall be every where assigned to the parts of the orbit, which the planet, if in motion, would describe in equal times; provided that the times of revolution of the disturbing and of the disturbed planet are not commensurable. This elegant theorem, if it has not hitherto been formally stated by any one, is at any rate very easily demonstrated from the principles of physical astronomy. A problem therefore presents itself which, both on its own account, and for many artifices required in its solution, is highly worthy of attention;—namely,—To determine exactly the attraction exerted by a planetary orbit upon any point given by position;—or,—as it may be otherwise expressed,—by any elliptic ring, whose thickness is infinitely small, and varies according to the law above specified.

2. Denoting the excentricity of the orbit by e, and the excentric anomaly of any point in the orbit by E, the corresponding elements of the excentric anomaly and of the mean anomaly will be $d\mathrm{E}$ and $(1 - e\cos \mathrm{E})\, d\mathrm{E}$; wherefore the element of the mass to be assigned to the indefinitely small portion of the orbit to which these elements correspond, will be to the whole mass, which we shall assume as the unit, as $(1-e\cos \mathrm{E})\, d\mathrm{E}$ to 2π, π expressing the semicircumference of a circle whose radius is unity. Putting therefore the distance of the point attracted from any point in the orbit $= \varrho$, the attraction exerted by the element of the orbit will be

$$= \frac{(1 - e\cos \mathrm{E})\, d\mathrm{E}}{2\pi\varrho^2}.$$

Let us denote the greater semiaxis by a and the less by b; and let us assume the former as the line of the abscissae, the centre of the ellipse being considered as their origin. Then we have $a^2 - b^2 = a^2e^2$, the abscissa of any point of the orbit $= a\cos \mathrm{E}$ and its ordinate $= b\sin \mathrm{E}$. Lastly, let us denote the distance of the attracted point from the plane of the orbit by C, and the remaining co-ordinates drawn from the attracted point parallel to the greater and less axes by A and B. These things being premised, the attractive force exerted by the element of the curve, when resolved into two parallel to the greater and less axes, and a third perpendicular to the plane of the orbit, gives the three following expressions which we shall put equal to $d\xi$, $d\eta$, $d\zeta$,

$$\frac{(\mathrm{A} - a\cos \mathrm{E})(1 - e\cos \mathrm{E})\, d\mathrm{E}}{2\pi\varrho^3} = d\xi$$

$$\frac{(\mathrm{B} - b\sin \mathrm{E})(1 - e\cos \mathrm{E})\, d\mathrm{E}}{2\pi\varrho^3} = d\eta$$

$$\frac{\mathrm{C}(1 - e\cos \mathrm{E})\, d\mathrm{E}}{2\pi\varrho^3} = d\zeta$$

where $\varrho = \sqrt{((\mathrm{A} - a\cos \mathrm{E})^2 + (\mathrm{B} - b\sin \mathrm{E})^2 + \mathrm{CC})}$.

These differentials being integrated between the limits $\mathrm{E} = 0$

and $E = 360°$ will give the partial attractions ζ, η, ξ, exerted in directions opposite to the directions of the co-ordinates. The whole attraction is compounded of the partial attractions ξ, η, ζ, which may, by a known method, be referred to any direction whatever.

3. The whole matter now depends upon introducing another variable instead of E, so that the radical quantity may be reduced to a more simple form.

For this purpose we shall put

$$\cos E = \frac{\alpha + \alpha' \cos T + \alpha'' \sin T}{\gamma + \gamma' \cos T + \gamma'' \sin T}$$

$$\sin E = \frac{\beta + \beta' \cos T + \beta'' \sin T}{\gamma + \gamma' \cos T + \gamma'' \sin T}$$

where the nine coefficients α, α', α'', &c. are manifestly not altogether arbitrary, but ought to satisfy certain conditions, which it is proper first of all to examine. We observe first, that the substitution would remain the same, although all the coefficients were multiplied by the same factor, so that without rendering the result less general we may suppose one of them to have a determinate value; we may put, for example $\gamma = 1$: for the sake of neatness, however, we shall in the mean time leave all the nine coefficients indefinite. We remark besides, that such values are to be excluded as would give α, α', α'', or β, β'. β'' proportional to γ, γ', γ'' respectively: for E would then no longer remain indeterminate. The differences, therefore, $\gamma'\alpha'' - \gamma''\alpha'$, $\gamma''\alpha - \gamma\alpha''$, $\gamma\alpha' - \gamma'\alpha$ cannot vanish at the same time.

The coefficients α, α', α'', &c. must evidently be compared in such a manner, that independently of any particular values which may be assigned to them, we may have,

$$\left.\begin{array}{l} (\alpha + \alpha' \cos T + \alpha'' \sin T)^2 \\ + (\beta + \beta' \cos T + \beta'' \sin T)^2 \\ - (\gamma + \gamma' \cos T + \gamma'' \sin T)^2 \end{array}\right\} = 0$$

this function must therefore necessarily be of the form

$$k(\cos T^2 + \sin T^2 - 1).$$

Hence we obtain these six conditional equations,

$$\left.\begin{array}{l} -\alpha\alpha - \beta\beta + \gamma\gamma = k \\ -\alpha'\alpha' - \beta'\beta' + \gamma'\gamma' = -k \\ -\alpha''\alpha'' - \beta''\beta'' + \gamma''\gamma'' = -k \\ -\alpha'\alpha'' - \beta'\beta'' + \gamma'\gamma'' = 0 \\ -\alpha''\alpha - \beta''\beta + \gamma''\gamma = 0 \\ -\alpha\alpha' - \beta\beta' + \gamma\gamma' = 0 \end{array}\right\} \qquad \text{(I)}$$

Upon these equations depend many others, which it will be worth while to examine. Putting for the sake of brevity,

$$\left.\begin{array}{r} \alpha\beta'\gamma'' + \alpha'\beta''\gamma + \alpha''\beta\gamma' \\ -\alpha\beta''\gamma' - \alpha'\beta\gamma'' - \alpha''\beta'\gamma \end{array}\right\} = \varepsilon \ldots\ldots \text{(II)}$$

by combining equations (I), the nine following are easily derived;

$$\left.\begin{array}{lll} \varepsilon\alpha & = -k(\beta'\gamma'' & - \gamma'\beta'') \\ \varepsilon\beta & = -k(\gamma'\alpha'' & - \alpha'\gamma'') \\ \varepsilon\gamma & = +k(\alpha'\beta'' & - \beta'\alpha'') \\ \varepsilon\alpha' & = +k(\beta''\gamma & - \gamma''\beta) \\ \varepsilon\beta' & = +k(\gamma''\alpha & - \alpha''\gamma) \\ \varepsilon\gamma' & = -k(\alpha''\beta & - \beta''\alpha) \\ \varepsilon\alpha'' & = +k(\beta\gamma' & - \gamma\beta') \\ \varepsilon\beta'' & = +k(\gamma\alpha' & - \alpha\gamma') \\ \varepsilon\gamma'' & = -k(\alpha\beta' & - \beta\alpha') \end{array}\right\} \qquad \text{(III)}$$

From the three first of these we again easily obtain this equation:

$$\varepsilon\alpha(\beta'\gamma'' - \gamma'\beta'') + \varepsilon\alpha(\gamma'\alpha'' - \alpha'\gamma'') + \varepsilon\gamma(\alpha'\beta'' - \beta'\alpha'') =$$

$$-k(\beta'\gamma'' - \gamma'\beta'')^2 - k(\gamma'\alpha'' - \alpha'\gamma'')^2 + k(\alpha'\beta'' - \beta'\alpha'')^2$$

to which the following is equivalent:

$$\varepsilon\varepsilon = k\,(-\alpha'\alpha' - \beta'\beta' + \gamma'\gamma')\,(-\alpha''\alpha'' - \beta''\beta'' + \gamma''\gamma'')$$
$$- k\,(-\alpha'\alpha'' - \beta'\beta'' + \gamma'\gamma'')^2$$

which, by the assistance of equations 2, 3, 4 in (I) is reduced to

$$\varepsilon\varepsilon = k^3 \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathrm{IV})$$

With equal facility, from equations (I) are derived the following:

$$\left.\begin{aligned}
(\beta'\gamma'' - \gamma'\beta'')^2 &= -k\,(k - \alpha'\alpha' - \alpha''\alpha'')\\
(\gamma'\alpha'' - \alpha'\gamma'')^2 &= -k\,(k - \beta'\beta' - \beta''\beta'')\\
(\alpha'\beta'' - \beta'\alpha'')^2 &= +k\,(k + \gamma'\gamma' + \gamma''\gamma'')\\
(\beta''\gamma - \gamma''\beta)^2 &= +k\,(k + \alpha\alpha - \alpha''\alpha'')\\
(\gamma''\alpha - \alpha''\gamma)^2 &= +k\,(k + \beta\beta - \beta''\beta'')\\
(\alpha''\beta - \beta''\alpha)^2 &= -k\,(k - \gamma\gamma + \gamma''\gamma'')\\
(\beta\gamma' - \gamma\beta')^2 &= +k\,(k + \alpha\alpha - \alpha'\alpha')\\
(\gamma\alpha' - \alpha\gamma')^2 &= +k\,(k + \beta\beta - \beta'\beta')\\
(\alpha\beta' - \beta\alpha')^2 &= -k\,(k - \gamma\gamma + \gamma'\gamma')
\end{aligned}\right\} \quad (\mathrm{V}$$

For the sake of illustration we shall indicate how the first of these equations is obtained: the rest are easily formed in the same manner. Equations 4, 2, 3 in (I) give,

$$(\gamma'\gamma'' - \beta'\beta'')^2 - (\gamma'\gamma' - \beta'\beta')\,(\gamma''\gamma'' - \beta''\beta'') = \alpha'\alpha'\alpha''\alpha''$$
$$- (\alpha'\alpha' - k)\,(\alpha''\alpha'' - k)$$

which equation when reduced gives immediately the first in (V).

From these equations (V) we conclude that the value, $k = 0$, is not admissible in our present inquiry; for if $k = 0$, all the nine quantities $\beta'\gamma'' - \gamma'\beta''$, &c. would necessarily vanish; that is the coefficients α, α', α'' would become proportional to β, β', β'', as also to γ, γ', γ''. Hence likewise, it appears from equation (IV) that the quantity ε can never vanish; wherefore the quantity k must necessarily be positive,—if, so be that, all the coefficients $\alpha, \alpha', \alpha''$, &c. are real quantities. By combining the first three equations in (III) with the first three in (V), there arise these new equations, which evidently depend upon the value of k not becoming evanescent:

$$\left.\begin{aligned} \alpha\alpha - \alpha'\alpha' - \alpha''\alpha'' &= -k \\ \beta\beta - \beta'\beta' - \beta''\beta'' &= -k \\ \gamma\gamma - \gamma'\gamma' - \gamma''\gamma'' &= +k \end{aligned}\right\} \quad \text{(VI)}$$

The combination of the rest would produce the same equations. To these we finally annex the three following:

$$\left.\begin{aligned} \beta\gamma - \beta'\gamma' - \beta''\gamma'' &= 0 \\ \gamma\alpha - \gamma'\alpha' - \gamma''\alpha'' &= 0 \\ \alpha\beta - \alpha'\beta' - \alpha''\beta'' &= 0 \end{aligned}\right\} \quad \text{(VII)}$$

which are easily derived from equations (III);—for example,—the second, fifth, and eighth give

$$\varepsilon\beta\gamma - \varepsilon\beta'\gamma' - \varepsilon\beta''\gamma'' = k\gamma(\gamma'\alpha'' - \alpha'\gamma'') - k\gamma'(\gamma''\alpha - \alpha''\gamma)$$
$$- k\gamma''(\gamma\alpha' - \alpha\gamma') = 0.$$

These equations also evidently depend upon the value $k = 0$ being excluded *.

Since, as we have shown above, all the coefficients may be multiplied by the same factor, the value of k will in that case become multiplied by the square of that factor; we shall now therefore always suppose

$$k = 1$$

so that we shall also necessarily have $e = +1$, or $e = -1$. It appears therefore, that the nine coefficients α, α', α'', &c. among which subsist six conditional equations ought to be reducible to three quantities independent of one another; which

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* It may not perhaps be superfluous to remark, that we made choice of the preceding analysis after due consideration; and employed it in preference to another method of deriving the relations III.—VII., because the latter though it appeared somewhat more elegant, was found, upon being accurately examined, to be liable to certain ambiguities, from which it could not be freed without a circuitous mode of proceeding.

is effected most conveniently indeed by means of three angles, in the following manner:

$$\alpha = \cos \mathrm{L} \tan \mathrm{N}$$

$$\beta = \sin \mathrm{L} \tan \mathrm{N}$$

$$\gamma = \sec \mathrm{N}$$

$$\alpha' = \cos \mathrm{L} \cos \mathrm{M} \sec \mathrm{N} \pm \sin \mathrm{L} \sin \mathrm{M}$$

$$\beta' = \sin \mathrm{L} \cos \mathrm{M} \sec \mathrm{N} \mp \cos \mathrm{L} \sin \mathrm{M}$$

$$\gamma' = \cos \mathrm{M} \tan \mathrm{N}$$

$$\alpha'' = \cos \mathrm{L} \sin \mathrm{M} \sec \mathrm{N} \mp \sin \mathrm{L} \cos \mathrm{M}$$

$$\beta'' = \sin \mathrm{L} \sin \mathrm{M} \sec \mathrm{N} \pm \cos \mathrm{L} \cos \mathrm{M}$$

$$\gamma'' = \sin \mathrm{M} \tan \mathrm{N}$$

Where the upper of the ambiguous signs belongs to the case $e = +1$, and the lower to the case $e = -1$. Analytic investigation proceeds, however, for the most part, more elegantly, without the use of these angles. But it would not be difficult to assign the geometrical signification both of the angles and of the other auxiliary quantities which occur in this investigation; but this, which is not necessary for our present purpose, we shall leave to be followed out by our skilful reader.

4. If the values assumed above for cos E and sin E are now substituted in the expression for the distance ϱ, it becomes of the form $\varrho =$

$$\frac{\sqrt{(\mathrm{G} + \mathrm{G}' \cos \mathrm{T}^2 + \mathrm{G}'' \sin \mathrm{T}^2 + 2\mathrm{H} \cos \mathrm{T} \sin \mathrm{T} + 2\mathrm{H}' \sin \mathrm{T} + 2\mathrm{H}'' \cos \mathrm{T})}}{\gamma + \gamma' \cos \mathrm{T} + \gamma'' \sin \mathrm{T}}$$

where the coefficients $\alpha, \alpha', \alpha''$, &c. are to be so determined that the six conditional equations being satisfied, namely,

$$\left.\begin{array}{rl} -\alpha\alpha - \beta\beta + \gamma\gamma &= 1 \\ -\alpha'\alpha' - \beta'\beta' + \gamma'\gamma' &= -1 \\ -\alpha''\alpha'' - \beta''\beta'' + \gamma''\gamma'' &= -1 \\ -\alpha'\alpha'' - \beta'\beta'' + \gamma'\gamma'' &= 0 \\ -\alpha''\alpha - \beta''\beta + \gamma''\gamma &= 0 \\ -\alpha\alpha' - \beta\beta' + \gamma\gamma' &= 0 \end{array}\right\} \quad [1]$$

and consequently the other equations also which arise from them, we may have

$$\mathrm{H} = 0, \qquad \mathrm{H}' = 0, \qquad \mathrm{H}'' = 0;$$

in which manner the problem, generally speaking, will be rendered determinate.

If therefore we denote the denominator of ϱ by t, the following function of the three quantities t, $t \cos \mathrm{E}$, $t \sin \mathrm{E}$,

$$(\mathrm{AA} + \mathrm{BB} + \mathrm{CC})\, tt + aa\,(t \cos \mathrm{E})^2 + bb\,(t \sin \mathrm{E})^2$$
$$- 2a\mathrm{A}t \cdot t \cos \mathrm{E} - 2b\mathrm{B}t \cdot t \sin \mathrm{E}.$$

ought by the substitution,

$$t \cos \mathrm{E} = \alpha + \alpha' \cos \mathrm{T} + \alpha'' \sin \mathrm{T}$$
$$t \sin \mathrm{E} = \beta + \beta' \cos \mathrm{T} + \beta'' \sin \mathrm{T}$$
$$t = \gamma + \gamma' \cos \mathrm{T} + \gamma'' \sin \mathrm{T}$$

to become

$$\mathrm{G} + \mathrm{G}' \cos \mathrm{T}^2 + \mathrm{G}'' \sin \mathrm{T}^2.$$

This is evidently the same thing as to say that the function of the three indeterminate quantities x, y and z,

(W)

$$aaxx + bbyy + (\mathrm{AA} + \mathrm{BB} + \mathrm{CC})zz - 2a\mathrm{A}xz - 2b\mathrm{B}yz,$$

ought by the substitution

$$x = \alpha u + \alpha' u' + \alpha'' u''$$
$$y = \beta u + \beta' u' + \beta'' u''$$
$$z = \gamma u + \gamma' u' + \gamma'' u''$$

to pass into this function of the indeterminate quantities u, u', u'',

$$\mathrm{G}uu + \mathrm{G}'u'u' + \mathrm{G}''u''u''$$

But seeing that from these formulas, joined with the equations [1], we easily obtain

$$u = -\alpha x - \beta y + \gamma z$$
$$u' = \alpha' x - \beta' y - \gamma' z$$
$$u'' = \alpha'' x - \beta'' y - \gamma'' z$$

it is evident that the function (w) ought to be identical with the following :

$$G(-\alpha x-\beta y+\gamma z)^2+G'(\alpha' x+\beta' y-\gamma' z)^2+G''(\alpha'' x+\beta'' y-\gamma'' z)^2;$$

whence we have six equations

$$\left.\begin{aligned} aa &= G\alpha\alpha + G'\alpha'\alpha' + G''\alpha''\alpha'' \\ bb &= G\beta\beta + G'\beta'\beta' + G''\beta''\beta'' \\ AA+BB+CC &= G\gamma\gamma + G'\gamma'\gamma' + G''\gamma''\gamma'' \\ bB &= G\beta\gamma + G'\beta'\gamma' + G''\beta''\gamma'' \\ aA &= G\gamma\alpha + G'\gamma'\alpha' + G''\gamma''\alpha'' \\ 0 &= G\alpha\beta + G'\alpha'\beta' + G''\alpha''\beta'' \end{aligned}\right\} \qquad [2]$$

From these twelve equations [1] and [2] our unknown quantities G, G', G'', α, α', α'', &c. must be determined.

5. From the combination of equations [1] and [2], the following are easily derived

$$-\alpha aa+\gamma aA=\alpha G$$

$$-\beta bb+\gamma bB=\beta G$$

$$\gamma(AA+BB+CC)-\alpha aA-\beta bB=\gamma G;$$

whence arise besides

$$\alpha=\frac{\gamma aA}{aa+G} \qquad [3]$$

$$\beta=\frac{\gamma bB}{bb+G} \qquad [4]$$

$$AA+BB+CC-\frac{aaAA}{aa+G}-\frac{bbBB}{bb+G}=G.$$

The last we can also put under this form

$$\frac{AA}{aa+G}+\frac{BB}{bb+G}+\frac{CC}{G}=1 \;\ldots\ldots\ldots [5]$$

In like manner, from the combination of equations [1] and [2] we deduce

$$\alpha' aa - \gamma' a\mathrm{A} = \alpha' \mathrm{G}'$$

$$\beta' bb - \gamma' b\mathrm{B} = \beta' \mathrm{G}'$$

$$-\gamma' (\mathrm{AA} + \mathrm{BB} + \mathrm{CC}) - \alpha' a\mathrm{A} - \beta' b\mathrm{B} = \gamma' \mathrm{G}',$$

and hence we find

$$\alpha' = \frac{\gamma' a\mathrm{A}}{aa - \mathrm{G}'} \ \ldots\ldots\ldots\ldots\ldots\ldots [6]$$

$$\beta' = \frac{\gamma' b\mathrm{B}}{bb - \mathrm{G}'} \ \ldots\ldots\ldots\ldots\ldots\ldots [7]$$

$$\frac{\mathrm{AA}}{aa - \mathrm{G}'} + \frac{\mathrm{BB}}{bb - \mathrm{G}'} - \frac{\mathrm{CC}}{\mathrm{G}'} = 1 \ \ldots\ldots\ldots\ldots [8]$$

and again

$$\alpha'' = \frac{\gamma'' a\mathrm{A}}{aa - \mathrm{G}''} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [9]$$

$$\beta'' = \frac{\gamma'' b\mathrm{B}}{bb - \mathrm{G}''} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [10]$$

$$\frac{\mathrm{AA}}{aa - \mathrm{G}''} + \frac{\mathrm{BB}}{bb - \mathrm{G}''} - \frac{\mathrm{CC}}{\mathrm{G}''} = 1 \ \ldots\ldots\ldots [11]$$

It appears therefore that G, $-\mathrm{G}'$, $-\mathrm{G}''$ are the roots of the equation

$$\frac{\mathrm{AA}}{aa + x} + \frac{\mathrm{BB}}{bb + x} + \frac{\mathrm{CC}}{x} = 1 \ \ldots\ldots\ldots\ldots [12]$$

which being properly reduced becomes

$$x^3 - (\mathrm{AA} + \mathrm{BB} + \mathrm{CC} - aa - bb)\, xx$$

$$+ (aabb - aa\mathrm{BB} - aa\mathrm{CC} - bb\mathrm{AA} - bb\mathrm{CC})\, x$$

$$- aabb\mathrm{CC} = 0 \ \ldots\ldots\ldots\ldots [13]$$

6. Now, concerning the nature of this cubic equation; the following things are to be remarked.

I. From the last term $-aabbCC$, it appears that the equation has at least one real root which is either positive, or vanishes when $C = 0$. Let us denote this real root, which is not negative by g.

II. By subtracting from equation 12 put under this form

$$x = \frac{AAx}{aa+x} + \frac{BBx}{bb+x} + CC;$$

this other

$$g = \frac{AAg}{aa+g} + \frac{BBg}{bb+g} + CC;$$

and dividing by $x-g$, there arises a new equation, containing the two remaining roots, namely,

$$1 = \frac{aaAA}{(aa+x)(aa+g)} + \frac{bbBB}{(bb+x)(bb+g)}$$

which being properly arranged and resolved gives

$$2x = \frac{aaAA}{aa+g} + \frac{bbBB}{bb+g} - aa - bb$$

$$\pm\sqrt{\left(\left(aa-bb-\frac{aaAA}{aa+g}+\frac{bbBB}{bb+g}\right)^2+\frac{4aabbAABB}{(aa+g)(bb+g)}\right)}$$

This expression, seeing the quantity under the radical sign is from its nature positive, or at least not negative, shews that the other two roots are also always real.

III. But, on the other hand, by subtracting those equations put under this form

$$gx = \frac{AAgx}{aa+x} + \frac{BBgx}{bb+x} + gCC;$$

$$gx = \frac{AAgx}{aa+g} + \frac{BBgx}{bb+g} + xCC;$$

and dividing by $g-x$, we obtain an equation containing the two remaining roots of the form

$$0 = \frac{AAgx}{(aa+g)(aa+x)} + \frac{BBgx}{(bb+g)(bb+x)} + CC$$

which, if g is a positive quantity, is evidently incapable of being satisfied by a positive value of x. Whence we conclude that our cubic equation cannot have more than one positive root.

IV. As often therefore as 0 is not among the roots of our equation there are necessarily one positive root and two negative. But as often as $C = 0$, and therefore 0 is one of the roots, the other two are contained in the equation.

$$xx - (AA + BB - aa - bb)x + aabb - aaBB - bbAA = 0;$$

whence these roots will be expressed by

$$\tfrac{1}{2}(AA + BB - aa - bb)$$

$$\pm \tfrac{1}{2}\sqrt{((AA - BB - aa + bb)^2 + 4AABB)}$$

Here again it will be proper to distinguish three cases.

First, if the last term $aabb - aaBB - bbAA$ is positive, (that is, if the point attracted lie in the plane of the attracting ellipse *within* the curve), both roots, seeing they must be real, will be affected by the same sign; and therefore, since they cannot be both positive at once, they must necessarily be both negative. But this conclusion may also be obtained, independently of what we have now shown, by considering that the coefficient of the second term, which admits of being thus expressed,

$$(aabb - aaBB - bbAA)\left(\frac{1}{aa} + \frac{1}{bb}\right) + \frac{bbAA}{aa} + \frac{aaBB}{bb}$$

is evidently in this case positive.

Secondly, if the last term is negative, (or if the attracted point is situated in the plane of the ellipse *without* the curve), one root will necessarily be positive and the other negative.

Thirdly, if the last term vanish, that is if the attracted point lie in the circumference of the ellipse, the second root of the cubic will also become 0, and the third root will be

$$-\frac{bbAA}{aa} - \frac{aaBB}{bb},$$

which is negative. But this case, in itself physically impossible, and in which the attraction would become infinitely great, we shall exclude, at least in this place, from our investigation.

7. For determining the coefficients γ, γ', γ'', we find from equations 1, 3, 4, 6, 7, 9, 10,

$$\left.\begin{aligned}
\gamma &= \frac{1}{\sqrt{\left(1-\left(\frac{a\,\mathrm{A}}{a\,a+\mathrm{G}}\right)^2-\left(\frac{b\,\mathrm{B}}{b\,b+\mathrm{G}}\right)^2\right)}}\\
\gamma' &= \frac{1}{\sqrt{\left(\left(\frac{a\,\mathrm{A}}{aa-\mathrm{G}'}\right)^2+\left(\frac{b\,\mathrm{B}}{b\,b-\mathrm{G}'}\right)^2-1\right)}}\\
\gamma'' &= \frac{1}{\sqrt{\left(\left(\frac{a\,\mathrm{A}}{aa-\mathrm{G}''}\right)^2+\left(\frac{b\,\mathrm{B}}{b\,b-\mathrm{G}''}\right)^2-1\right)}}
\end{aligned}\right\} \qquad [15]$$

From these equations, when properly combined with equations 5, 8, 11, we have also

$$\left.\begin{aligned}
\gamma &= \sqrt{\frac{\mathrm{G}}{\left(\frac{\mathrm{AG}}{aa+\mathrm{G}}\right)^2+\left(\frac{\mathrm{BG}}{bb+\mathrm{G}}\right)^2+\mathrm{cc}}}\\
\gamma' &= \sqrt{\frac{\mathrm{G}'}{\left(\frac{\mathrm{AG}'}{aa-\mathrm{G}'}\right)^2+\left(\frac{\mathrm{BG}'}{bb-\mathrm{G}'}\right)^2+\mathrm{cc}}}\\
\gamma'' &= \sqrt{\frac{\mathrm{G}''}{\left(\frac{\mathrm{AG}''}{aa-\mathrm{G}''}\right)^2+\left(\frac{\mathrm{BG}''}{bb-\mathrm{G}''}\right)^2+\mathrm{cc}}}
\end{aligned}\right\} \qquad [16]$$

These last expressions show that none of the quantities G, G$'$, G$''$ can be negative, if the quantities γ, γ', γ'' are real.

In the case therefore in which C is not $= 0$, G must necessarily be put equal to the positive root of equation [13], and it appears also that — G$'$ is to be equal to one of the negative roots, and — G$''$ equal to the other *; at the same time it is altogether arbitrary which we take for — G$'$ and which for — G$''$.

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* Properly speaking, indeed, it only follows from the preceding analysis that — G$'$ and — G$''$ ought to satisfy equation 13; whence it appears doubtful whether it may not be admissible to put — G$'$ and — G$''$ both equal to the *same* negative root, the third root being altogether neglected. But it is easily perceived that, provided the second and third roots of the equation are unequal, it follows from — G$'$ = — G$''$, that $\gamma' = \gamma''$, $\alpha' = \alpha''$, $\beta' = \beta''$, and therefore that $-\alpha'\alpha'' - \beta'\beta'' + \gamma'\gamma'' = -\alpha'\alpha' - \beta'\beta' + \gamma'\gamma' = 1$, which is inconsistant with equation fourth of [1]. See what is said below concerning the case of equation 13 having two equal roots.

As often as $C = 0$, and the attracted point is situated within the curve we must necessarily assume for $-G'$ and $-G''$ the two negative roots, and put $G = 0$. But since in this case the first formula in 16 becomes indeterminate, we shall take instead of it the first formula in 15, which gives

$$\gamma = \frac{1}{\sqrt{\left(1 - \frac{AA}{aa} - \frac{BB}{bb}\right)}}.$$

But as often as $C = 0$, and the attracted point lies without the ellipse the positive root of equation [13] must be put $= G$, and either the negative root $= -G'$, and $G'' = 0$, or the negative root $= -G''$, and $G' = 0$; the coefficient γ'' or γ' will be found by means of the formula

$$\sqrt{\left(\frac{AA}{aa} + \frac{BB}{bb} - 1\right)}$$

In the case already excluded, in which the attracted point is supposed to be situated in the circumference of the ellipse, the coefficients γ and γ', or γ and γ'' become infinite which shows that our transformation is not at all applicable to it.

8. Although formulas [15] and [16] are sufficient for the determination of the coefficients $\gamma, \gamma', \gamma''$, we may, however, assign others still more elegant. For this purpose we shall multiply equation [5] by $aabb - GG$, whence arises, after a slight reduction,

$$\frac{aaAA\,(bb + G)}{aa + G} - AAG + \frac{bbBB\,(aa + G)}{bb + G} - BBG$$
$$+ \frac{aabbCC}{G} - CCG = aabb - GG.$$

But from the nature of equations we have the sum of the roots

$$G - G' - G'' = AA - BB - CC - aa - bb,$$

the product of the roots $GG'G'' = aabbCC$.

Hence the preceding equation is transformed into the following

$$\frac{aaAA\,(bb + G)}{aa + G} + \frac{bbBB\,(aa + G)}{bb + G} + G'G''$$
$$- G\,(G - G' - G'' + aa + bb) = aabb - GG.$$

which may also be put under the form

$$\frac{aaAA\,(bb + G)}{aa + G} + \frac{bbBB\,(aa + G)}{bb + G} - (aa + G)\,(bb + G)$$
$$+ (G + G')\,(G + G'') = 0.$$

Hence the value of the coefficient γ from the first formula in [15] is transformed into the following

$$\gamma = \sqrt{\frac{(aa + G)(bb + G)}{(G + G')(G + G'')}} \quad \ldots\ldots\ldots \quad [17]$$

By an analysis altogether similar, we find,

$$\gamma' = \sqrt{\frac{(aa - G')(bb - G')}{(G + G')(G'' - G')}} \quad \ldots\ldots\ldots \quad [18]$$

$$\gamma'' = \sqrt{\frac{(aa - G'')(bb - G'')}{(G + G'')(G' - G'')}} \quad \ldots\ldots\ldots \quad [19]$$

After the coefficients γ, γ', γ'' have been found, the others α, β, α', β', α'', β'' are derived from them by means of formulas, 3, 4, 6, 7, 9, 10.

9. It is easily perceived that the signs of the radical expressions by which γ, γ', γ'', are determimed may be taken at pleasure. But it is worth inquiring, in what manner the sign of the quantity ε is connected with these signs. For this end let us consider the third equation in (III.) art. 3.

$$\varepsilon\gamma = \alpha'\beta'' - \beta'\alpha'',$$

which by formulas 6, 7, 9, 10 is thus transformed

$$\varepsilon\gamma = \frac{ab\,AB\,\gamma'\gamma''}{(aa - G')(bb - G'')} - \frac{ab\,AB\,\gamma'\gamma''}{(aa - G'')(bb - G')}$$

$$= \frac{ab\,(aa - bb)\,AB\,(G'' - G')\,\gamma'\gamma''}{(aa - G')(aa - G'')(bb - G')(bb - G'')}.$$

But by considering equation 13 we easily obtain

$$(aa + G)(aa - G')(aa - G'') = aa\,(aa - bb)\,AA$$

$$(bb + G)(bb - G')(bb - G'') = -\,bb\,(aa - bb)\,BB.$$

Hence the preceding equation becomes

$$\varepsilon\gamma = \frac{(aa + G)(bb + G)(G' - G'')\,\gamma'\gamma''}{ab\,(aa - bb)\,AB}$$

which being combined with equation 17, gives

$$\gamma\gamma'\gamma'' = \frac{\varepsilon ab\,(aa - bb)\,AB}{(G + G')(G + G'')(G' - G'')},$$

Hence it appears that if that negative root of the cubic equation which is *absolutely* the greater be assumed for $-\text{G}'$, and at the same time the coefficients γ, γ', γ'' be all taken positive, ε takes the same sign which AB has; and remains unaffected though all these conditions, or any two of them, be reversed; but assumes the opposite sign, if one or three of the conditions be changed. But it is proper to note still farther the following relations, which are easily derived from the foregoing.

$$\alpha\alpha'\alpha'' = \frac{\varepsilon aab\text{AAB}}{(\text{G}+\text{G}')(\text{G}+\text{G}'')(\text{G}'-\text{G}'')}$$

$$\beta\beta'\beta'' = -\frac{\varepsilon abb\text{ABB}}{(\text{G}+\text{G}')(\text{G}+\text{G}'')(\text{G}'-\text{G}'')}$$

$$\alpha\beta = \frac{ab\text{AB}}{(\text{G}+\text{G}')(\text{G}+\text{G}'')}$$

$$\alpha'\beta' = -\frac{ab\text{AB}}{(\text{G}+\text{G}')(\text{G}'-\text{G}'')}$$

$$\alpha''\beta'' = \frac{ab\text{AB}}{(\text{G}+\text{G}'')(\text{G}'-\text{G}'')}$$

10. Our formulas may in some cases become indeterminate: these cases it is proper to consider by themselves. And first let us examine that in which the negative roots $-\text{G}'$, $-\text{G}''$ of the cubic equation becoming equal, the coefficients γ', γ'' appear from formulas 18, 19 to acquire infinite values, which however are only indeterminate.

By putting in formula 14, $g = \text{G}$, it appears that in order to render the two values of x, that is, $-\text{G}'$ and $-\text{G}''$, equal, it is necessary that

$$\text{AB} = 0, \quad aa - bb - \frac{aa\text{AA}}{aa+\text{G}} + \frac{bb\text{BB}}{bb+\text{G}} = 0.$$

Hence it is easy to perceive that, since $aa - bb$ is naturally either a positive quantity or $= 0$, we must have

$$\text{B} = 0,$$

$$aa - bb = \frac{aa\text{AA}}{aa+\text{G}}, \quad \text{or} \quad aa + \text{G} = \frac{aa\text{AA}}{aa-bb}.$$

By substituting these values in equation 14 it becomes

$$\text{G}' = \text{G}'' = bb.$$

By substituting besides, the value $x = bb$ in the cubic equation 13, it gives

$$(aa - bb)\,(\mathrm{CC} + bb) = bb\mathrm{AA}.$$

As often as this conditional equation has place at the same time with the equation $\mathrm{B} = 0$, the case occurs which we are here treating of.

And since G becomes

$$= \frac{aa\mathrm{AA}}{aa - bb} - aa = \frac{aa\mathrm{CC}}{bb},$$

formula 17 gives

$$\gamma = \sqrt{\frac{aabb\mathrm{AA}}{(aa - bb)\,(aa\mathrm{CC} + b^4)}} = \sqrt{\frac{aa\mathrm{CC} + aabb}{aa\mathrm{CC} + b^4}}.$$

and again from formulas 3 and 4, we find

$$\alpha = \frac{\gamma\,(aa - bb)}{a\mathrm{A}} = \frac{\gamma bb\mathrm{A}}{a(\mathrm{CC} + bb)} = \sqrt{\frac{bb(aa - bb)}{aa\mathrm{CC} + b^4}}$$

$$= \sqrt{\frac{b^4\mathrm{AA}}{(\mathrm{CC} + bb)\,(aa\mathrm{CC} + b^4)}}$$

$$\beta = 0.$$

The values of the coefficients γ', γ'' in the formulas 18, 19, remain in this case indeterminate, and so also do the values of the remaining coefficients α', β', α'', β''. Nevertheless by means of one of these coefficients all the other five may be expressed; for example, we have from formula 6,

$$\alpha' = \frac{\gamma'\alpha\mathrm{A}}{aa - bb},$$

and again

$$\beta' = \sqrt{(1 - \alpha'\alpha' + \gamma'\gamma')},\ \gamma'' = \sqrt{(\gamma\gamma - 1 - \gamma'\gamma')}.$$

$$\alpha'' = \frac{\gamma''a\mathrm{A}}{aa - bb},\quad \beta'' = \sqrt{(1 - \alpha''\alpha'' + \gamma''\gamma'')}.$$

But this is more neatly accomplished as follows. From

$$\gamma\gamma = 1 + \alpha\alpha, \quad \alpha\alpha' = \gamma\gamma', \quad 1 = \alpha'\alpha' + \beta'\beta' - \gamma'\gamma',$$

we obtain

$$\beta'\beta' + \frac{\gamma'\gamma'}{\alpha\alpha} = 1 - \alpha'\alpha' + \frac{\gamma\gamma\gamma'\gamma'}{\alpha\alpha} = 1$$

wherefore we may put

$$\beta' = \cos f, \quad \gamma' = \alpha \sin f, \quad \alpha' = \gamma \sin f,$$

But farther, from the formulas

$$\varepsilon\alpha'' = \beta\gamma' - \gamma\beta', \quad \varepsilon\beta'' = \gamma\alpha - \alpha\gamma', \quad \varepsilon\gamma'' = \beta\alpha' - \alpha\beta', \quad \varepsilon\varepsilon = 1,$$

we find

$$\alpha'' = -\varepsilon\gamma \cos f, \quad \beta'' = \varepsilon \sin f, \quad \gamma'' = \varepsilon\alpha \cos f.$$

The value of the angle f is here arbitrary; also, we may take at pleasure either $\varepsilon = +1$, or $\varepsilon = -1$.

11. If G' and G'' are unequal, the values of the coefficients $\gamma, \gamma', \gamma''$ in formulas 17, 18, 19 can never become indeterminate; but as often as any one of the quantities $aa - G'$, $bb - G'$, $aa - G''$, $bb - G''$ vanishes, the value of one of the coefficients $\alpha', \beta', \alpha'', \gamma''$ in formulas 6, 7, 9, 10 respectively, appears at first sight to remain indeterminate, which a little attention, however, will show not to be the case.

Let us suppose, for example, that $aa - G' = 0$, then by equation 18 we have $\gamma' = 0$, and therefore, by equation 7, $\beta = 0$, (provided that aa is not at the same time equal to bb); whence we necessarily obtain $\alpha' = \pm 1$. But if at the same time $aa = bb$, the formula which precedes equation sixth in art. 5, gives $\alpha' A + \beta' B = 0$, from which equation joined with $\alpha'\alpha' + \beta'\beta' = 1$, we derive

$$\alpha' = \frac{B}{\sqrt{(AA + BB)}}, \quad \beta' = \frac{-B}{\sqrt{(AA + BB)}},$$

These expressions evidently cannot become indeterminate, unless we have at the same time $A = 0$, and $B = 0$; but when these equations have place we are carried back to the case which we have already considered in the preceding article.

12. Having now shown fully the manner of determining the twelve quantities G, G', G'', α, α', α'', β, β', β'', γ, γ', γ'', we proceed to the consideration of the differential $d\mathrm{E}$. Let us put

$$t = \gamma + \gamma' \cos \mathrm{T} + \gamma'' \sin \mathrm{T} \;\ldots\ldots\ldots\ldots\ [20]$$

so that we obtain

$$t \cos \mathrm{E} = \alpha + \alpha' \cos \mathrm{T} + \alpha'' \sin \mathrm{T} \;\ldots\ldots\ldots (21)$$

$$t \sin \mathrm{E} = \beta + \beta' \cos \mathrm{T} + \beta'' \sin \mathrm{T} \;\ldots\ldots\ldots (22)$$

Hence we deduce

$$td\mathrm{E} = (\cos \mathrm{E}d\,.\,t \sin \mathrm{E} + \sin \mathrm{E}d\,.\,t \cos \mathrm{E} + =)$$

$$\cos \mathrm{E}\,(\beta'' \cos \mathrm{T} - \beta' \sin \mathrm{T})\, d\mathrm{T} + \sin \mathrm{E}\,(\alpha'' \cos \mathrm{T} - \alpha' \sin \mathrm{T})\, d\mathrm{T}.$$

and therefore

$$ttd\mathrm{E} = (\alpha''\beta - \alpha\beta'') \cos \mathrm{T}d\mathrm{T} + (\alpha'\beta - \beta'\alpha) \sin \mathrm{T}d\mathrm{T}$$

$$+ (\alpha'\beta'' - \beta''\alpha')\, d\mathrm{T} = \varepsilon\gamma' \cos \mathrm{T}d\mathrm{T} + \varepsilon\gamma'' \sin \mathrm{T}d\mathrm{T} + \varepsilon\gamma d\mathrm{T}$$

$$= \varepsilon t d\mathrm{T}.$$

or

$$td\mathrm{E} = \varepsilon d\mathrm{T} \;\ldots\ldots\ldots\ldots\ldots\ldots\ [23]$$

It is proper to observe that the quantity t is from its nature always positive, if the coefficient γ be positive, or always negative if γ be negative. For since $(\gamma' \cos \mathrm{T} + \gamma'' \sin \mathrm{T})^2 + (\gamma'' \cos \mathrm{T} - \gamma' \sin \mathrm{T})^2 = \gamma'\gamma' + \gamma''\gamma'' = \gamma\gamma - 1$, it follows that $\gamma' \cos \mathrm{T} + \gamma'' \sin \mathrm{T}$, if we neglect the sign, is less than γ. Hence we conclude that as often as $\varepsilon\gamma$ is a positive quantity, the variable quantities E and T always increase together; but as often as $\varepsilon\gamma$ is negative, the one of these variable quantities necessarily diminishes while the other increases.

13. The connexion between the variable quantities E and T is still better illustrated by reasoning as follows: By putting $\sqrt{(\gamma\gamma - 1)} = \delta$, so that $\delta\delta = \alpha\alpha + \beta\beta = \gamma'\gamma' + \gamma''\gamma''$, we derive from equations 20, 21, 22,

$$t(\delta + \alpha \cos \mathrm{E} + \beta \sin \mathrm{E}) = \gamma\delta + \alpha\alpha + \beta\beta + (\gamma'\delta + \alpha\alpha' + \beta\beta') \cos \mathrm{T}$$

$$+ (\gamma''\delta + \alpha\alpha'' + \beta\beta'') \sin \mathrm{T}$$

$$= (\gamma + \delta)(\delta + \gamma' \cos \mathrm{T} + \gamma'' \sin \mathrm{T})$$

In like manner from equations 21, 22 we find

$$t(\alpha \sin E - \beta \cos E) = \varepsilon(\gamma' \sin T - \gamma'' \cos T).$$

These equations, by putting

$$\frac{\alpha}{\delta} = \cos L, \quad \frac{\beta}{\delta} = \sin L, \quad \frac{\gamma'}{\delta} = \cos M, \quad \frac{\gamma''}{\delta} = \sin M,$$

assume the following form:

$$t(1 + \cos(E - L)) = (\gamma + \delta)(1 + \cos(T - M))$$

$$t \sin(E - L) = \varepsilon \sin(T - M),$$

whence, observing that $(\gamma + \delta)(\gamma - \delta) = 1$, we obtain by division

$$\tan \tfrac{1}{2}(E - L) = \varepsilon(\gamma - \delta) \tan \tfrac{1}{2}(T - M)$$

$$\tan \tfrac{1}{2}(T + M) = \varepsilon(\gamma + \delta) \tan \tfrac{1}{2}(E - L).$$

From this result not only the same conclusion is derived to which we were led in the end of the preceding article, but it farther appears that if the value of E increases by 360°, the value of T is increased or diminished by the same number of degrees, according as $\varepsilon\gamma$ is a positive or negative quantity. By putting $\delta = \tan N$, and $\gamma = \sec N$ we evidently have

$$\gamma - \delta = \tan(45^\circ - \tfrac{1}{2}N), \quad \gamma + \delta = \tan(45^\circ + \tfrac{1}{2}N).$$

14. From the combination of equations 20, 21, 22 with equations of article 5 we obtain

$$at(A - a \cos E) = \alpha G - \alpha' G' \cos T - \alpha'' G'' \sin T$$

$$bt(B - b \sin E) = \beta G - \beta' G' \cos T - \beta'' G'' \sin T.$$

By putting therefore, for the sake of brevity,

$$(\alpha G - \alpha' G' \cos T - \alpha'' G'' \sin T)(\gamma - e\alpha + (\gamma' - e\alpha') \cos T +$$
$$(\gamma'' - e\alpha'') \sin T) = aX,$$

$$\beta G - \beta' G' \cos T - \beta'' G'' \sin T)(\gamma - e\alpha + (\gamma' - e\alpha') \cos T +$$
$$(\gamma'' - e\alpha'') \sin T) = bY,$$

$$C\,\{\gamma + \gamma' \cos T + \gamma'' \sin T\}\,\{\gamma - \varepsilon\alpha + (\gamma' - \varepsilon\alpha') \cos T +$$

$$\gamma'' - \varepsilon\alpha'' \sin T) = Z,$$

we have

$$d\xi = \frac{\varepsilon X dT}{2\pi t^3 \varrho^3}, \quad d\eta = \frac{\varepsilon Y dT}{2\pi t^3 \varrho^3}, \quad d\zeta = \frac{\varepsilon Z dT}{2\pi t^3 \varrho^3}.$$

But

$$t\varrho = \pm \sqrt{(G + G' \cos T^2 + G'' \sin T^2)},$$

the upper or lower sign having place according as t is a positive or negative quantity (for ϱ is from its nature always considered as positive), that is, according as the coefficient γ is positive or negative. Hence

$$\frac{\varepsilon dT}{2\pi t^3 \varrho^3} = \pm \frac{dT}{2\pi (G + G' \cos T^2 + G'' \sin T^2)^{\frac{3}{2}}},$$

where the ambiguous sign depends upon the sign of $\gamma\varepsilon$.

To obtain now the values of ξ, η, ζ themselves, it is necessary to take the integrals of the differentials from the value of T, to which corresponds E = 0, to that value to which corresponds E = 360°; or, (what evidently comes to the same thing), from a value of T, to which corresponds an arbitrary value of E, to that value to which corresponds the said value of E increased by 360°; it is therefore admissible to integrate from T = 0 to T = 360°, as often as $\varepsilon\gamma$ is a positive quantity, or from T = 360° to T = 0 as often as $\varepsilon\gamma$ is negative. It is evident then that, independently of the sign of $\varepsilon\gamma$, we have

$$\xi = \int \frac{X dT}{2\pi (G + G' \cos T^2 + G'' \sin T^2)^{\frac{3}{2}}}$$

$$\eta = \int \frac{Y dT}{2\pi (G + G' \cos T^2 + G'' \sin T^2)^{\frac{3}{2}}}$$

$$\zeta = \int \frac{Z dT}{2\pi (G + G' \cos T^2 + G'' \sin T^2)^{\frac{3}{2}}},$$

the integrations being extended from T = 0 to T = 360°.

15. It is not difficult to perceive that the integrals

$$\int \frac{\cos \mathrm{T} d\mathrm{T}}{(\mathrm{G} + \mathrm{G}' \cos \mathrm{T}^2 + \mathrm{G}'' \sin \mathrm{T}^2)^{\frac{3}{2}}}$$

$$\int \frac{\sin \mathrm{T} d\mathrm{T}}{(\mathrm{G} \times \mathrm{G}' \cos \mathrm{T}^2 + \mathrm{G}'' \sin \mathrm{T}^2)^{\frac{3}{2}}}$$

$$\int \frac{\cos \mathrm{T} \sin \mathrm{T} d\mathrm{T}}{(\mathrm{G} + \mathrm{G}' \cos \mathrm{T}^2 + \mathrm{G}'' \sin \mathrm{T}^2)^{\frac{3}{2}}},$$

extended from T = 180° to T = 360° have values equal to those which they would acquire if extended from T = 0 to T = 180°, but affected with opposite signs; wherefore these integrals when extended from T = 0 to T = 360° evidently become = 0. Hence we conclude that

$$\xi = \int \frac{((\gamma - e\alpha)\alpha \mathrm{G} - (\gamma' - e\alpha')\alpha' \mathrm{G}' \cos \mathrm{T}^2 - (\gamma'' - e\alpha'')\alpha'' \mathrm{G}'' \sin \mathrm{T}^2) d\mathrm{T}}{2\pi a (\mathrm{G} + \mathrm{G}' \cos \mathrm{T}^2 + \mathrm{G}'' \sin \mathrm{T}^2)^{\frac{3}{2}}}$$

$$\eta = \int \frac{((\gamma - e\alpha)\beta \mathrm{G} - (\gamma' - e\alpha')\beta' \mathrm{G}' \cos \mathrm{T}^2 - (\gamma'' - e\alpha'')\beta'' \mathrm{G}'' \sin \mathrm{T}^2) d\mathrm{T}}{2\pi b (\mathrm{G} + \mathrm{G}' \cos \mathrm{T}^2 + \mathrm{G}'' \sin \mathrm{T}^2)^{\frac{3}{2}}}$$

$$\zeta = \int \frac{((\gamma - e\alpha)\gamma - (\gamma' - e\alpha')\gamma' \cos \mathrm{T}^2 - (\gamma'' - e\alpha'')\gamma'' \sin \mathrm{T}^2) \mathrm{C} d\mathrm{T}}{2\pi (\mathrm{G} + \mathrm{G}' \cos \mathrm{T}^2 + \mathrm{G}'' \sin \mathrm{T}^2)^{\frac{3}{2}}},$$

the integrals being extended from T = 0 to T = 360°. If therefore we denote by P and Q the two following integrals, taken between the same limits

$$\int \frac{\cos \mathrm{T}^2 d\mathrm{T}}{2\pi ((\mathrm{G} + \mathrm{G}') \cos \mathrm{T}^2 + (\mathrm{G} + \mathrm{G}'') \sin \mathrm{T}^2)^{\frac{3}{2}}}$$

$$\int \frac{\sin \mathrm{T}^2 d\mathrm{T}}{2\pi ((\mathrm{G} + \mathrm{G}') \cos \mathrm{T}^2 + (\mathrm{G} + \mathrm{G}'') \sin \mathrm{T}^2)^{\frac{3}{2}}}$$

we shall obtain

$$a\xi = ((\gamma - e\alpha)\, \alpha \mathrm{G} - (\gamma' - e\alpha')\, \alpha' \mathrm{G}')\, \mathrm{P} +$$
$$((\gamma - e\alpha)\, \alpha \mathrm{G} - (\gamma'' - e\alpha'')\, \alpha'' \mathrm{G}'')\, \mathrm{Q},$$
$$b\eta = ((\gamma - e\alpha)\, \beta \mathrm{G} - (\gamma' - e\alpha')\, \beta' \mathrm{G}')\, \mathrm{P} +$$
$$((\gamma - e\alpha)\, \beta \mathrm{G} - (\gamma'' - e\alpha'')\, \beta'' \mathrm{G}'')\, \mathrm{Q},$$
$$\zeta = ((\gamma - e\alpha)\, \gamma + (\gamma' - e\alpha')\, \gamma')\, \mathrm{CP} +$$
$$((\gamma - e\alpha)\, \gamma + (\gamma'' - e\alpha'')\, \gamma'')\, \mathrm{CQ},$$

in which manner our problem is completely resolved.

16. With regard to the quantities P and Q, they both evidently become

$$= \frac{1}{2\,(G + G')^{\frac{3}{2}}}$$

as often as $G' = G''$, but in all other cases they must be referred to transcendentals. In what manner the transcendental quantities may be expressed by series, is abundantly obvious. But we hope it will be agreeable to our readers, if upon this occasion we explain the determination of these and of other transcendentals by means of a peculiar and most convenient method, which we have now for many years frequently employed; and of which we propose to treat more fully in another place.

Let m, n be two positive quantities, and let us put

$$m' = \tfrac{1}{2}(m + n), \quad n' = \sqrt{mn}$$

so that m', n' may be respectively the arithmetical and geometrical mean between m and n. Moreover let

$$m'' = \tfrac{1}{2}(m' + n'), \quad n'' = \sqrt{m'n'}$$

$$m''' = \tfrac{1}{2}(m'' + n''), \quad n''' = \sqrt{m''n''},$$

and so on; so that the series m, m', m'', m''', &c. and the series n, n', n'', n''', &c. will converge very rapidly toward a *common limit*, which we shall denote by μ, and shall denominate simply *the arithmetico-geometrical mean* between m and n. We shall now demonstrate that $\frac{1}{\mu}$ is the value of the integral

$$\int \frac{dT}{2\pi\,\sqrt{(mm\cos T^2 + nn\sin T^2)}},$$

extended from $T = 0$, to $T = 360°$.

DEMONSTRATION. Let us suppose that the variable quantity T is so expressed by means of another variable T', that

$$\sin T = \frac{2m\sin T'}{(m + n)\cos T'^2 + 2m\sin T'^2}$$

and it is easy to perceive that while T' increases from the value 0 to 90°, 180°, 270°, 360°, T will also increase (though by unequal intervals) from 0 to 90°, 180°, 270°, 360°. Now, after proper reductions we find

$$\frac{d\mathrm{T}}{\sqrt{(mm\cos\mathrm{T}^2 + nn\sin\mathrm{T}^2)}} = \frac{d\mathrm{T}'}{\sqrt{(m'm'\cos\mathrm{T}'^2 + n'n'\sin\mathrm{T}'^2)}},$$

and therefore the values of the integrals

$$\int\frac{d\mathrm{T}}{2\pi\sqrt{(mm\cos\mathrm{T}^2 + nn\sin\mathrm{T}^2)}},\ \int\frac{d\mathrm{T}'}{2\pi\sqrt{(m'm'\cos\mathrm{T}'^2 + n'n'\sin\mathrm{T}'^2)}}$$

if each is extended from the value 0 of its variable quantity to the value 360°, are equal to one another. Since we may proceed in the same manner (employing m', m'', n', n'', T', T'' instead of m, m', n, n', T, T', — and so on successively) as far as we please, it appears that to these values the value of the integral

$$\int\frac{d\Theta}{2\pi\sqrt{(\mu\mu\cos\Theta^2 + \mu\mu\sin\Theta^2)}},$$

from $\Theta = 0$ to $\Theta = 360°$, is also equal; which value is evidently $\frac{1}{\mu}$. *Q. E. D.*

17. From the equation expressing the relation between T and T', namely

$$(m - n)\sin\mathrm{T}\,.\,\sin\mathrm{T}'^2 = 2m\sin\mathrm{T}' - (m + n)\sin\mathrm{T},$$

is easily deduced

$$\sqrt{(mm\cos\mathrm{T}^2 + nn\sin\mathrm{T}^2)} = m - (m - n\sin\mathrm{T}\,.\,\sin\mathrm{T}',$$

$$\sqrt{(m'm'\cos\mathrm{T}'^2 + n'n'\sin\mathrm{T}'^2)} = m\,\mathrm{cotang}\,\mathrm{T}\,.\,\tan\mathrm{T}',$$

and hence, by help of the same equation,

$$\sin\mathrm{T}\,.\,\sin\mathrm{T}'\,.\,\sqrt{(mm\cos\mathrm{T}^2 + nn\sin\mathrm{T}^2)} + m'(\cos\mathrm{T}^2 + \sin\mathrm{T}^2) =$$

$$\cos\mathrm{T}\,.\,\cos\mathrm{T}'\,.\,\sqrt{(m'm'\cos\mathrm{T}'^2 + n'n'\sin\mathrm{T}'^2)} - \tfrac{1}{2}(m - n)\sin\mathrm{T}'^2,$$

this equation being multiplied by

$$\frac{d\mathrm{T}}{\sqrt{(mm\cos\mathrm{T}^2 + nn\sin\mathrm{T}^2)}} = \frac{d\mathrm{T}'}{\sqrt{(m'm'\cos\mathrm{T}'^2 + n'n'\sin\mathrm{T}'^2)}}$$

gives

$$\frac{m'(\cos\mathrm{T}^2 - \sin\mathrm{T}^2)\,d\mathrm{T}}{\sqrt{(mm\cos\mathrm{T}^2 + nn\sin\mathrm{T}^2)}} = \frac{-\tfrac{1}{2}(m - n)\sin\mathrm{T}'^2\,d\mathrm{T}'}{\sqrt{(m'm'\cos\mathrm{T}'^2 + n'n'\sin\mathrm{T}'^2)}}$$

$$+ d\,.\,\sin\mathrm{T}'\cos\mathrm{T},$$

By multiplying this last equation by $\frac{m-n}{\pi}$, by substituting

$$m'(m-n) = \tfrac{1}{2}(mm-nn), \quad (m-n)^2 = 4(m'm'-n'n'),$$

$$\sin T'^2 = \tfrac{1}{2} - \tfrac{1}{2}(\cos T'^2 - \sin T'^2),$$

and by integrating from the value T and T′ = 0 to T and T′ = 360°, we have

$$(mm-nn)\int \frac{(\cos T^2 - \sin T^2)\,.\,dT}{2\pi\sqrt{(mm\cos T^2 + nn\sin T^2)}} = -\frac{2(m'm'-n'n')}{\mu}$$

$$+ 2(m'm'-n'n')\int \frac{(\cos T'^2 - \sin T'^2)\,.\,dT'}{2\pi\sqrt{(m'm'\cos T'^2 + n'n\sin T'^2)}},$$

And since the definite integral on the right-hand side of this equation may be in like manner transformed, it is manifest that the integral

$$\int \frac{(\cos T^2 - \sin T^2)\,dT}{2\pi\sqrt{(mm\cos T^2 + nn\sin T^2)}},$$

will be expressed by an infinite series which converges very rapidly, namely,

$$-\frac{2(m'm'-n'n') + 4(m''m''-n''n'') + 8(m'''m'''-n'''n''')\ \&c.}{(mm-nn)\,\mu}$$

which we shall put equal to $-\frac{\nu}{\mu}$.

The numerical calculation is most conveniently performed by logarithms, if we put

$$\tfrac{1}{4}\sqrt{(mm-nn)} = \lambda, \ \tfrac{1}{4}\sqrt{(m'm'-n'n')} = \lambda', \ \tfrac{1}{4}\sqrt{(m''m''-n''n'')} = \lambda'', \&c.$$

whence we shall obtain

$$\lambda' = \frac{\lambda\lambda}{m'}, \quad \lambda'' = \frac{\lambda'\lambda'}{m''}, \quad \lambda''' = \frac{\lambda''\lambda''}{m'''}, \ \&c. \text{ and}$$

$$\nu = \frac{2\lambda'\lambda' + 4\lambda''\lambda'' + 8\lambda'''\lambda''' + \&c.}{\lambda\lambda}.$$

§ 18. By the method now explained even *indefinite* integrals (which begin from the value 0 of the variable quantity) may be assigned with the greatest neatness. For example, if T″ is supposed to be determined by means of m', n', T′, as T′ is by m, n, T; and again T‴ by m'', n'', T″; and so on, then also, for any determinate value of T, the values of the terms of the series T, T′, T″, T‴, &c. will converge very rapidly to the limit Θ, and we shall have

$$\int \frac{d\text{T}}{\sqrt{(mm \cos \text{T}^2 + nn \sin \text{T}^2)}} = \frac{\Theta}{\mu}$$

$$\int \frac{(\cos \text{T}^2 - \sin \text{T}^2\, d\text{T}}{\sqrt{(mm \cos \text{T}^2 + nn \sin \text{T}^2)}} = -\frac{\nu\Theta}{\mu}$$

$$+ \frac{\lambda' \cos \text{T} \sin \text{T}' + 2\lambda'' \cos \text{T}' \sin \text{T}''' + 4\lambda''' \cos \text{T}'' \sin \text{T}''' + \&c.}{\lambda\lambda}$$

But it is sufficient to have noticed these things in passing, as they are not necessary to our present purpose.

19. If we now put $m = \sqrt{(\text{G} + \text{G}')}$, $n = \sqrt{(\text{G} + \text{G}'')}$, the values of the quantities P and Q will be easily reduced to the transcendental quantities μ, ν. For since P, Q are the values of the integrals.

$$\int \frac{\cos \text{T}^2 d\text{T}}{2\pi(mm \cos \text{T}^2 + nn \sin \text{T}^2)^{\frac{3}{2}}}, \int \frac{\sin \text{T}^2 d\text{T}}{2\pi(mm \cos \text{T}^2 + nn \sin \text{T}^2)^{\frac{3}{2}}},$$

from T = 0 to T = 360°, it is at once evident that

$$mm\text{P} + nn\text{Q} = \frac{1}{\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad [24]$$

Farther, we have

$$\frac{(\cos \text{T}^2 - \sin \text{T}^2)\, d\text{T}}{2\pi\sqrt{(mm \cos \text{T}^2 + nn \sin \text{T}^2)}} + \frac{(mm \cos \text{T}^2 - nn \sin \text{T}^2)\, d\text{T}}{2\pi(mm \cos \text{T}^2 + nn \sin \text{T}^2)^{\frac{3}{2}}},$$

$$= \frac{(mm \cos \text{T}^4 - nn \sin \text{T}^4)\, d\text{T}}{\pi\, (mm \cos \text{T}^2 + nn \sin \text{T}^2)^{\frac{3}{2}}}$$

$$= d\, \frac{\cos \text{T} \sin \text{T}}{\pi\sqrt{(mm \cos \text{T}^2 + nn \sin \text{T}^2)}}.$$

By integrating this equation from T = 0 to T = 360°, it gives

$$-\frac{\nu}{} + mm\text{P} - nn\text{Q} = 0 \quad \ldots\ldots \quad [25]$$

From the combination of equations 24, 25 we finally derive

$$\text{P} = \frac{1 + \nu}{2mm\mu}, \quad \text{Q} = \frac{1 - \nu}{2nn\mu}.$$

*** On pages 6 and 7 of this article for $e = +1$ and $e = -1$, read $\mathrm{E} = +1$ and $\mathrm{E} = -1$.

ARTICLE II.

A New Solution of the Problem, in which it is required to inscribe three circles in a given triangle so that each of them may touch the other two, and also two sides of the triangle.

By Mr. LECHMÜTZ.

Taken from the Annales de Mathematiques.

Let A, B, C be the three angular points of a triangle, O the centre of its inscribed circle the radius of which we shall suppose equal to unity.

From this centre let fall upon the sides BC, CA, AB, the perpendiculars $OA' = OB' = OC' = 1$, and also from the same point to the angular points, draw the lines OA, OB, OC.

Making

$$\begin{aligned} \text{angle } AOB' &= \text{angle } AOC' = \alpha, \\ \text{angle } BOC' &= \text{angle } BOA' = \beta, \\ \text{angle } COA' &= \text{angle } COB' = \gamma; \end{aligned}$$

we shall have

$$\begin{aligned} AB' = AC' &= \text{tang.}\,\alpha, & OA &= \text{sec.}\,\alpha, \\ BC' = BA' &= \text{tang.}\,\beta, & OB &= \text{sec.}\,\beta, \\ CA' = CB' &= \text{tang.}\,\gamma, & OC &= \text{sec.}\,\gamma. \end{aligned}$$

Because $2\alpha + 2\beta + 2\gamma$ equal four right-angles we shall also have

$$\text{tang.}\,\gamma = -\,\text{tang.}\,(\alpha + \beta) = -\frac{\text{tang.}\,\alpha + \text{tang.}\,\beta}{1 - \text{tang.}\,\alpha\,\text{tan.}\,\beta};$$

whence, by taking away the denominator and transposing,

$$\text{tang.}\,\alpha + \text{tang.}\,\beta + \text{tang.}\,\gamma = \text{tang.}\,\alpha\,\text{tang.}\,\beta\,\text{tang.}\,\gamma.$$

Now to inscribe in this triangle three circles such that one of them may touch the other two and two sides of the triangle, it is clear that the centres of these circles must be situated on the lines OA, OB, OC which bisect the angles of the triangle. Let X, Y, Z be these centres and x, y, z their corresponding radii.

If the centres X, Y be projected orthogonally on the side $AB = AC' + BC' = \text{tang.}\,\alpha + \text{tang.}\,\beta$, their projections divide this side into three segments the extremes of which are $x\,\text{tang.}\,\alpha$, $y\,\text{tang.}\,\beta$. The intermediate segment is the projection of the distance of the centres $XY = x + y$, and is consequently

$$\sqrt{\left\{(x + y)^2 - (x - y)^2\right\}} = \sqrt{(4xy)} = 2\sqrt{(xy)};$$

equating therefore the sum of these three parts with the first expression for the side AB, we have

$$x \text{ tang. } \alpha + 2\sqrt{(xy)} + y \text{ tang. } \beta = \text{tang. } \alpha + \text{tang. } \beta.$$

From the other two sides we get analogous equations, and if, for abridging, we make

$$\begin{aligned} \text{tang. } \alpha &= a, \\ \text{tang. } \beta &= b, \\ \text{tang. } \gamma &= c; \end{aligned}$$

the problem will be reduced to resolve, relatively to x, y, and z the three equations

$$ax + 2\sqrt{(xy)} + by = a + b, \ldots\ldots\ldots\ldots(1)$$
$$by + 2\sqrt{(yz)} + cz = b + c, \ldots\ldots\ldots\ldots(2)$$
$$cz + 2\sqrt{(zx)} + ax = c + a \ldots\ldots\ldots\ldots(3)$$

with the condition

$$a + b + c = abc. \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

By multiplying cross-ways equations (2, 3) and reducing, we have

$$(b + c)(ax + 2\sqrt{(xz)}) - (a + c)(by + 2\sqrt{(yz)}) = c(a - b)z;$$

but from equation (4)

$$c = -\frac{a + b}{ab - 1},$$

whence

$$b + c = \frac{a(1 + b^2)}{ab - 1}, \quad a + c = \frac{b(1 + a^2)}{ab - 1};$$

Substituting therefore, and leaving out the common denominator, we have

$$a(1 + b^2)(ax + 2\sqrt{(xz)}) - b(1 + a^2)(by + 2\sqrt{(yz)}) = (a^2 - b^2)z,$$

or

$$a(1 + b^2)(ax + 2\sqrt{(xz)}) - b(1 + a^2)(by + 2\sqrt{(yz)})$$
$$= \{(1 + a^2) - (1 + b^2)\}z:$$

or, by transposing,

$$(1 + b^2)(a^2x + 2a\sqrt{(xz)} + z) = (1 + a^2)(b^2y + 2b\sqrt{(yz)} + z),$$

or, again

$$(1 + b^2)(a\sqrt{x} + \sqrt{z})^2 = (1 + a^2)(b\sqrt{y} + \sqrt{z})^2;$$

and, by extracting the square root and dividing

$$\frac{a\sqrt{x} + \sqrt{z}}{\sqrt{(1 + a^2)}} = \frac{b\sqrt{y} + \sqrt{z}}{\sqrt{(1 + b^2)}}.$$

By a simple permutation of the letters, we conclude that

$$\frac{b\sqrt{y}+\sqrt{x}}{\sqrt{(1+b^2)}}=\frac{c\sqrt{z}+\sqrt{x}}{\sqrt{(1+c^2)}},\quad \frac{c\sqrt{z}+\sqrt{y}}{\sqrt{(1+c^2)}}=\frac{a\sqrt{x}+\sqrt{y}}{\sqrt{(1+a^2)}};$$

adding these two last member to member, z disappears and we have

$$\left\{\frac{1}{\sqrt{(1+b^2)}}-\frac{a}{\sqrt{(1+a^2)}}-\frac{1}{\sqrt{(1+c^2)}}\right\}\sqrt{x}$$

$$=\left\{\frac{1}{\sqrt{(1+a^2)}}-\frac{b}{\sqrt{(1+b^2)}}-\frac{1}{\sqrt{(1+c^2)}}\right\}\sqrt{y};$$

but, because

$$c=\frac{a+b}{ab-1},$$

we have

$$\frac{1}{\sqrt{(1+c^2)}}=\frac{ab-1}{\sqrt{\{(1+a^2)(1+b^2)\}}};$$

substituting therefore, and taking away the denominator, the equation becomes

$$\{1-ab+\sqrt{(1+a^2)}-a\sqrt{(1+b^2)}\}\sqrt{x}$$

$$=\{1-ab+\sqrt{(1+b^2)}-b\sqrt{(1+a^2)}\}\sqrt{y}.$$

The coefficients of the two members, may at first be written thus

$$\{1-a+\sqrt{(1+a^2)}\}+a\{1-b-\sqrt{(1+b^2)}\},$$

$$\{1-b+\sqrt{(1+b^2)}\}+b\{1-a-\sqrt{(1+a^2)}\};$$

or, by considering that

$$a=\tfrac{1}{2}.2a=-\tfrac{1}{2}\{(1-a)^2-(1+a^2)\}$$

$$=-\tfrac{1}{2}\{1-a+\sqrt{(1+a^2)}\}\{1-a-\sqrt{(1+a^2)}\},$$

$$b=\tfrac{1}{2}.2b=-\tfrac{1}{2}\{(1-b)^2-(1+b^2)\}$$

$$=-\tfrac{1}{2}\{1-b+\sqrt{(1+b^2)}\}\{1-b-\sqrt{(1+b^2)}\},$$

they take this new form

$$\{1-a+\sqrt{(1+a^2)}\}\{1-\tfrac{1}{2}[1-a-\sqrt{(1+a^2)}][1-b-\sqrt{(1+b^2)}]\};$$

$$\{1-b+\sqrt{(1+b^2)}\}\{1-\tfrac{1}{2}[1-a-\sqrt{(1+a^2)}][1-b-\sqrt{(1+b^2)}]\};$$

where they have one factor common, by suppressing which, the equation becomes simply

$$\{1-a+\sqrt{(1+a^2)}\}\sqrt{x}=\{1-b+\sqrt{(1+b^2)}\}\sqrt{y}.$$

Supposing therefore, for abridging,

$$1-a+\sqrt{(1+a^2)}=A,$$
$$1-b+\sqrt{(1+b^2)}=B,$$
$$1-c+\sqrt{(1+c^2)}=C,$$

we draw from these, by a simple permutation of letters

$$B\sqrt{y}=C\sqrt{z},\qquad C\sqrt{z}=A\sqrt{x};$$

whence

$$\sqrt{x}=\frac{C}{A}\sqrt{z},\qquad \sqrt{y}=\frac{C}{B}\sqrt{z}\,\ldots\ldots\ldots(5)$$

Going back now to the primitive equations, if from the sum of the equations (2, 3) we subtract equation (1) and suppress the factor 2, which is common to all the terms of the resulting equation it becomes

$$cz+\sqrt{(xz)}+\sqrt{(yz)}-\sqrt{(xy)}=c;$$

putting in this for $\sqrt{x}$ and $\sqrt{y}$ their values (5) it becomes

$$\left(c+\frac{C}{A}+\frac{C}{B}-\frac{C^2}{AB}\right)^2 z=c,$$

or $$\left\{cAB+C(A+B-C)\right\}z=cAB.\ldots\ldots\ldots\ldots(6)$$

Or, we have, from the values of A, B, C

$$cAB=c(1-a)(1-b)+c(1-b)\sqrt{(1+a^2)}+c(1-a)\sqrt{(1+b^2)}+$$
$$c\sqrt{\left\{(1+a^2)(1+b^2)\right\}},$$

or, putting for $\sqrt{\left\{(1+a^2)(1+b^2)\right\}}$ its equal $\left\{ab-1\sqrt{(1+c^2)}\right\}$,

$$cAB=c(1-a)(1-b)+c(1-b)\sqrt{(1=a^2)}+c(1-a)\sqrt{(1+b^2)}+$$
$$c(ab-1)\sqrt{(1+c^2)}.$$

We have again

$$A+B-C=1-a-b+c+\sqrt{(1+a^2)}+\sqrt{(1+b^2)}-\sqrt{(1+c^2)};$$

whence by multiplying by $C=1-c+\sqrt{(1+c^2)}$, and putting for $\sqrt{\left\{(1+a^2)(1+c^2)\right\}}$ and $\sqrt{\left\{(1+b^2)(1+c^2)\right\}}$ their equals $(ac-1)\sqrt{(1+b^2)}$ and $(bc-1)\sqrt{(1+a^2)}$,

$$C(A+B-C)=-(1-c)(a+b)-2c^2-c(1-b)\sqrt{(1+a^2)}-$$
$$(a+b-2c)\sqrt{(1+c^2)}-c(1-a)\sqrt{(1+b^2)}$$

therefore, by adding and reducing,

$$cAB+C(A+B-C)=c-a-b+abc-2c^2$$
$$+(c-a-b+abc)\sqrt{(1+c^2)}.$$

Replacing abc by its equivalent $a+b+c$, the above becomes

$$cAB + C(A+B-C) = 2c\left\{1 - c + \sqrt{(1+c^2)}\right\} = 2cC;$$

Substituting therefore this value in equation (6) and suppressing the factor c common to the two members, it becomes simply

$$2Cz = AB,$$

By a simple permutation of the letters, we obtain equations for x and y; so that we have finally

$$x = \frac{BC}{2A}, \quad y = \frac{CA}{2B}, \quad z = \frac{AB}{2C}.$$

This done, produce AO, BO, CO, to meet the circumference of the inscribed circle again in A″, B″, C″; from the points A, B, C taken respectively for centres, and with the radii AB′ = AC′, BC′ = BA′, CA′ = CB′ let arcs be described cutting AO, BO, CO respectively in A‴, B‴, C‴ we shall then have

$$\begin{aligned}\mathrm{A''A'''} &= \mathrm{AO} + \mathrm{OA''} - \mathrm{AB'} = \mathrm{OA''} - \mathrm{AB'} + \mathrm{AO}\\ &= 1 - \text{tang.}\,\alpha + \text{sec.}\,\alpha = A\\ \mathrm{B''B'''} &= \mathrm{BO} + \mathrm{OB''} - \mathrm{BC'} = \mathrm{OB''} - \mathrm{BC'} + \mathrm{BO}\\ &= 1 - \text{tang.}\,\beta + \text{sec.}\,\beta = B\\ \mathrm{C'C''} &= \mathrm{CO} + \mathrm{OC''} - \mathrm{CA'} = \mathrm{OC''} - \mathrm{CA'} + \mathrm{CO}\\ &= 1 - \text{tang.}\,\gamma + \text{sec.}\,\gamma = C.\end{aligned}$$

The three lengths A, B, C being thus determined, we can obtain by a simple construction, the three radii x, y, z. For this purpose construct a triangle DEF, the three sides EF, FD, DE of which are respectively equal to these three lengths; through the summits draw straight lines terminating in the opposite sides in D′, E′, F′, and in such directions as to make

angle FDD′ = angle E,
angle DEE′ = angle F,
angle EFF′ = angle D;

then, by virtue of the proportionality of the homologous sides of the similar triangles, the lengths, DD′, EE′, FF′ are the diameters of the circles sought, having their centres respectively in X, Y, Z.

These expressions for the radii of the circles once found, it is easy, by substituting their values, to find any of the other unknown quantities that we please.

Taking, for example, the distances of the points A, B, C to the points where the required circles touch the sides of the triangle, for the unknown quantities, and which are $= x$ tang. $\alpha = ax$; y tang. $\beta = by$, z tang. $\gamma = cz$, and we have

$$ax = \frac{aBC}{2A}, \quad by = \frac{bCA}{2B}, \quad cz = \frac{cAB}{2C}.$$

Or from the values found above for cAB and $C(A + B - C)$, we have

$$cAB = C\left\{2c - (A + B - C)\right\};$$

therefore

$$cz = \frac{cAB}{2C} = \tfrac{1}{2}\left\{2c - (A + B - C)\right\};$$

that is to say,

$$cz = \tfrac{1}{2}\left\{a + b + c - 1 + \sqrt{(1+c^2)} - \sqrt{(1+a^2)} - \sqrt{(1+b^2)}\right\},$$

or again

$$cz = \tfrac{1}{2}(\mathrm{AB}' + \mathrm{BC}'' - \mathrm{CA}' - \mathrm{OC}' + \mathrm{OC} - \mathrm{OA} - \mathrm{OB}):$$

from which we get exactly the construction of Malfatti.

If we take for the unknowns, the distances AX, BY, CZ, the summits of the corresponding centres, these unknowns are respectively $x\sqrt{(1+a^2)}$, $y\sqrt{(1+b^2)}$, $z\sqrt{(1+c^2)}$ and we find, for example, from what has preceded

$$z\sqrt{(1+c^2)} + cz \,.\, \frac{\sqrt{(1+c^2)}}{c}$$

$$= \frac{\sqrt{(1+c^2)}}{2c}\Big(a+b+c-1+\sqrt{(1+c^2)}-\sqrt{(1+a^2)}\surd-(1+b^2)\Big).$$

Lastly if we take for the unknowns, the distances OX, OY, OZ, these are respectively $(1-x)\sqrt{(1+a^2)}$, $(1-y)\sqrt{(1+b^2)}$, $(1-z)\sqrt{(1+c^2)}$, and we find, for example,

$$(1-z)\sqrt{(1+c^2)} = (c-cz)\,\frac{\sqrt{(1+c^2)}}{c}$$

$$= \frac{\sqrt{(1+c^2)}}{2c}\Big((1-a-b+c+\sqrt{(1+a^2)}+\sqrt{(1+b^2)}-\sqrt{(1+c^2)}\Big).$$

By varying the signs of the radicals in a proper manner, and substituting for the inscribed circle, properly so called, each of the other three circles which can be described to touch at once the three sides of the triangle we may obtain all the solutions of which the problem is susceptible.

Berlin 23rd Jan. 1820.

CAMBRIDGE PROBLEMS.

THE SENATE-HOUSE PROBLEMS,

Given to the Candidates for Honors during the Examination for the degrees of B. A.

BY THE TWO MODERATORS.

MONDAY, JANUARY 17, 1820.

MONDAY MORNING.—MR. WHEWELL.

First and Second Classes.

1. GIVEN two sides and the included angle, find an expression for the area, (1) in a plane, and (2) in a spherical triangle.

2. A straight line cuts a parabola, whose vertex is *A*, in two points *P* and *Q*, and its axis in *O*; ordinates *PM*, *QN*, being drawn, shew that *AO* is a mean proportional between *AM* and *AN*.

3. The force varies inversely as (distance)$^{\frac{10}{3}}$. A body is projected from any point in any direction, with a velocity equal to that from infinity. Find the position of the apse, and the whole angle described.

4. On a horizontal dial the angle corresponding to a second of time at 4 o'clock, is double the angle for a second at noon. Find the latitude of the place.

5. The equation to a curve is $y = x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$. Trace it; find its maximum ordinate, and its area.

6. The earth revolving round a fixed axis, shew that a body let fall from the top of a high tower will not strike the ground exactly at the foot of the tower. Between what cardinal points of the compass will the point struck be situated with respect to the foot of the tower?

7. Express the distance of a point from the earth's centre in terms of the latitude.

8. A point T moves uniformly along a straight line; another point P, with *three* times the velocity, always moves towards T, so as to describe the *curve of pursuit*. Trace the curve, and shew that the path described by T from the time when the paths are at right angles till it is overtaken by P is $\frac{3}{8}$ of their distance at that time.

9. The equation to the elliptical paraboloid being $ax^2 + by^2 + abz = abc$, draw a normal to it; and determine the points where this line cuts the three co-ordinate planes. Also find the solid content of a portion contained by planes parallel to the planes of xz and yz.

10. Find right-angled triangles such that all the sides shall be rational numbers.

11. If a, b, c be the sides of a plane triangle and C the angle opposite to c; prove that

$$\text{hyp. log. } c = \text{hyp. log. } a - \frac{b}{a}\cos C - \frac{b^2}{2a^2}\cos 2C - \frac{b^3}{3a^3}\cos 3C - \&c.$$

12. Integrate the differential

$$dx\sqrt{\frac{1 - e^2x^2}{1 - x^2}},$$

in a series which converges rapidly when e is near $= 1$; and the equation

$$(x + y)\frac{dx}{dy} = x + y - x\frac{dy}{dx}.$$

13. Define the Moon's *variation*. Give Newton's construction for it, and hence shew how it varies.

MONDAY AFTERNOON.—MR. WHEWELL.

Fifth and Sixth Classes.

1. Find the value of ·151636363, &c. of £1.

2. Find in what time at compound interest, at 5 per cent. a sum will become 10 times its original value. (N.B. the log. of 105 is 2·0211893.)

3. Solve the equations

$$x + \sqrt{(x^2 + \sqrt{(x^2 + 96)})} = 11$$

$$\left.\begin{aligned} x(y+z) &= a \\ y(x+z) &= b \\ z(x+y) &= c \end{aligned}\right\}$$

$x^3 - 6x - 40 = 0$, by Cardan's method.

$$3^{2x} \cdot 2^{3x} = 10.$$

4. A beam rests with one end on a horizontal plane, and the other against a vertical wall; find the horizontal force necessary to prevent its lower end from sliding outwards.

5. A projectile is to be thrown across a plain 120 feet wide, to strike a mark 30 feet high, the velocity of projection being that acquired down 80 feet; find at what angle it must be projected.

6. A piece of wood weighs 12 lbs. and when annexed to 22 lbs. of lead, and immersed in water, the whole weighs 8 lbs. The specific gravity of lead being 11 times that of water, find the specific gravity of the wood

7. A cylinder whose axis is horizontal empties itself by a hole in the lowest part; find the time?

8. A trapezium has two opposite sides equal, and the other two parallel; compare the resistance upon it, when it moves in

the direction of the parallel sides, and when it moves in a direction perpendicular to them.

9. Explain why all parts of the field of view of a telescope are not equally bright; and find the proportion of the bright part to the whole in the astronomical telescope.

10. Having observed the elongation of a planet when stationary, shew how its distance from the sun may be found?

11. Find the integrals of

$$\frac{dx}{\sqrt{a}-\sqrt{x}}, \quad \frac{x\,dx}{\sqrt{(2ax-x^2)}}, \quad dx \,.\, \log. x.$$

12. Find an expression to determine the force by which a body may be retained in a given curve: apply it to the curve whose equation is

$$p^2=\frac{b^2x^4}{a^4+x^4}.$$

13. The force varying inversely as the distance, find the angle between the apsides in an orbit nearly circular, and prove the method.

14. Sum the following series to n terms,

$$\frac{1}{\sqrt{x}}-\frac{\sqrt{2}}{x}+\frac{2}{x\sqrt{x}}-\&c.$$

and continue the harmonic progression .... 3, 4, 6, ... upwards and downwards. How far can it be continued either way?

15. Form the equation whose roots are

$$2+\sqrt{-3},\quad 2-\sqrt{-3},\quad 1,\ \text{and}-5.$$

16. In any plane triangle of which the sides are a, b, c, and the opposite angles A, B, C, prove that

$$\sin A=\frac{a\,.\sin C}{\sqrt{(a^2-2ab\,.\cos C+b^2)}}.$$

17. How high will a given balloon ascend? When it floats in the air, supposing that a given weight of ballast is thrown out, to what additional height will it rise, and how much will a barometer in it sink?

MONDAY AFTERNOON.—MR. WILKINSON.

Third and Fourth Classes.

1. Find the discount of £.100 for one year at 5 *per cent.* and then calculate the interest on this discount for the same time.

2. Solve the equations

$$\frac{x^4+1}{(x+1)^4} = \frac{1}{2}, \text{ and } \sqrt{x^4-1}+\sqrt{x^2-1} = x^3.$$

3. If a rigid sphere, revolving round an axis, become fluid and therefore change its figure, the whole moment of inertia will remain the same as before.

4. Find the centres of oscillation of a square suspended by one corner and oscillating *flatways* and *edgeways*.

5. What is the reason that waves always break against the shore whatever be the direction of the wind.

6. The horizon of any place being taken as the plane of projection, find the figure and dimensions of the path of the diurnal motion of a given star orthographically projected.

7. What is the meaning of the astronomical term equation? The equation of time (arising from what causes?) is a maximum about the beginning of November, is it then additive or subtractive?

8. The number of impossible roots of any equation $x^n - px^{n-1} + \&c. = 0$ is not increased by multiplying its terms by the successive terms of the series 0, 1, 2, 3, 4, &c.

9. Integrate these expressions

$$\frac{dz}{1-\tan^2 z}, \text{ and } \frac{dx}{1+x+x^2}.$$

10. If a tangent be drawn at the extremity of the latus rectum of a conic section meeting the tangent at the vertex, the part of this latter tangent thus cut off shall be equal to the distance the vertex and the focus of the curve.

11. Find in what curve a body must revolve round a repulsive force varying as the distance from a point, so that its velocity may always equal that in a circle at the same distance, round an equal attractive centre of force.

12. Transform the equation of the lemniscata $(x^2 + y^2)^2 = a^2(x^2 - y^2)$, from rectangular to polar co-ordinates.

13. If in an equation between x and y the sum of the indices be the same in every term, the loci of the corresponding values of x and y are straight lines. Find what lines are defined by the equation $y^3 - 2xy^2 + x^3 = 0$.

Monday Evening.—Mr. Wilkinson.

1. What is the purchase of £1034. 15*s*. stock in the 3 per cents. at 62½.

2. If two straight lines in space be parallel, their projections on any plane will be parallel.

3. Find the solidity of an octahedron.

4. Required the present worth of an annuity of (a) pounds for n years payable every instant in equal portions, interest also being convertible into principal as fast as it becomes due.

5. R and r being the radii of the circumscribed and inscribed circles of the triangle whose sides are a, b, c, shew that

$$Rr = \frac{abc}{2(a + b + c)}.$$

6. If n be a prime number and a and b integers not divisible by n, then $\dfrac{a^{n-1} - b^{n-1}}{n}$ is a whole number.

7. State D'Alembert's principle, and the principle of virtual velocities, and employ them in deducing this theorem

$4m\,(Ax + By + Cz) = Aa^2 + Bb^2 + Cc^2$; A, B, C, being *weights* which put a system in motion, x, y, z, the spaces perpendicularly descended by them respectively, and a, b, c, their actual velocities.

8. A slender rod of uniform thickness revolves round an axis passing through one of its extremities; find,

(1) At what point an obstacle must be opposed to it that there may be no stress on the axis from the shock?

(2) What quantity of matter should be collected in this point that the impulse on the obstacle may be the same as that of the rod?

(3) At what distance from the axis the obstacle must be opposed that the impulse may be the same as if the whole matter in the rod were collected in that point?

9. The Earth being supposed a sphere of uniform density, shew that the pressure on each side of a plane passing through its centre : the whole weight of the Earth :: 3 : 16.

10. A piston is thrust down uniformly into a cylinder full of air, having a small orifice at the end; find the quantity of air discharged in a given time into a vacuum.

11. If parallel rays fall upon a spherical refracting surface, the distance from the axis of the geometrical focus of a small pencil which does not pass through the centre, is proportional to the cube of the distance at which it is incident from the axis.

12. The Sun's R. A. on two successive days at noon was $6^h. 40'. 25''$, and $6^h. 45'. 13''$, by the Nautical Almanack (and therefore in sidereal time); the Moon's R. A. at the same time was from the Nautical Almanack (and therefore expressed in degrees) $5^s. 9^\circ. 32'$, and $5^s. 20^\circ. 9'$. Required the time of the Moon's transit in the interval.

13. The length of a degree perpendicular to the meridian is always greater than the degree of the meridian corresponding.

14. Define the axis of a curve, and draw the axis to the curve of which $y^3 - 3axy + x^3 = 0$ is an equation.

15. T and t are the parts of the tangent at the vertex (A) of a rectangular hyperbola, (whose semiaxis $= 1$) cut off by lines (CP, Cp) drawn from the centre to the curve; shew that if the sector CAP be n times the sector CAp,

$$\frac{1 - T}{1 + T} = \left(\frac{1 - t}{1 + t}\right)^n.$$

16. A's skill is double B's, and their stakes equal; find what

C, whose skill is equal to *A*'s, must stake, that *A*'s advantage may be as great as if he played with *B*.

17. If it be an even wager that D wins (n) successive games of E, what is E's chance of winning the first game?

18. Integrate the differential equations

$$\frac{dy}{dx} = a \sin x + by; \text{ and } \frac{d^2y}{dx^2} - \frac{2dy}{dx} + 2y = 0;$$

and shew how to separate the variable quantities in

$$\frac{dy}{dx} = \frac{a^2 + x^2 - y^2}{a^2}.$$

19. Apply the method of increments to sum the series

$$\frac{3}{1 \cdot 2} \cdot \frac{1}{2} + \frac{4}{2 \cdot 3} \cdot \frac{1}{2^2} + \frac{5}{3 \cdot 4} \cdot \frac{1}{2^3} + \text{\&c. (to } n \text{ terms)},$$

and also to shew that

$$\sin x + \sin 3x + \sin 5x + \ldots\ldots\ldots \text{(to } n \text{ terms)}$$

$$= \frac{\sin^2 nx}{\sin x}.$$

20. If tangents be drawn at the extremities of the major axis of an ellipse, the rectangle under the parts of these tangents intercepted by the tangent at any other point of the curve is a constant quantity.

21. Conversely, if AT, Vt, be two perpendiculars to a given line AV and on the same side of it, and also the rectangle $AT \times Vt$ be constant; to investigate the nature of the curve which the line Tt perpetually touches.

22. The axes of two equal cylinders intersect each other at right angles, find the content of the solid cut out of each by the surface of the other.

23. The attractions of *ellipsoids* on particles in the surface in directions perpendicular to any of the principal sections are as the distances.

24. Two bodies connected by an inflexible rod without weight, and to one of which a certain velocity is communicated, are constrained to move along two grooves at right angles to each

other; required the circumstances of motion; and shew that when the bodies are equal the line which joins them revolves uniformly.

TUESDAY MORNING.—MR. WILKINSON.

First and Second Classes.

1. What will the ceiling of a room come to, whose length is 24 f. 6 in. and width 16 f. 3 in. at 3*s*. per yard.

2. $u = \text{arc}\left(\tan \dfrac{x-y}{x+y}\right)$ find the differential of u.

3. Every positive number has an infinite number of logarithms, only one of which is real, and every negative number has only imaginary logarithms.

4. When a system of bodies connected in any manner is in equilibrium the centre of gravity is as high or as low as is possible.

5. The resistance on a cube moving in a fluid in the direction of its diagonal : resistance on the same cube moving in a direction perpendicular to its side :: 1 : $\sqrt{3}$.

6. A conical vessel is filled with water; find that heavy sphere which when put into it shall force out the greatest quantity of fluid.

7. Draw a tangent to the curve formed by the intersection, of a right cone with the cylinder erected on the radius of the base as a diameter.

8. Investigate the nature of the curve trisecting all the arcs described on the same chord.

9. Integrate

$$\frac{ydx - xdy}{(x+y)^2}, \text{ and } xdy - ydx = \frac{axdy - ydx}{\sqrt{(x^2+y^2)}}.$$

10. *A*, and *B* whose skill is *m* times *A*'s, agree to play with this law, that *A* shall continue to stake against *B* so long as *B* wins without interruption; shew that *B*'s expectation is worth $(m-1)$ times the stake.

11. A circle of curvature is described at the vertex of a parabola, and another circle which touches that and both the

arcs of the curve, and so on continually; compare the sum of all the areas of these circles with that of the parabola.

12. If D and D' be the lengths of a degree of a meridian at the equator and in latitude λ respectively, a and b the equatorial and polar diameters,

$$\frac{a}{b} = \frac{\sin\lambda}{\sqrt{\left(\frac{D}{D'}\right)^{\frac{2}{3}} - \cos^2\lambda}}.$$

13. Explain how the comparative density of the Sun and Moon have been deduced from the phænomena of precession and nutation.

TUESDAY AFTERNOON.—MR. WILKINSON.

Fifth and Sixth Classes.

1. Required the value of 1 cwt. 2 qrs. 3 lbs. at £4. 5*s*. 6*d*. per cwt.

2. Find the value of

$$\frac{x - x^5}{1 - x^2} \text{ when } x = 1.$$

3. Write down the 4 first terms of the expansion of

$$\left(x - \frac{1}{x}\right)^{-\frac{1}{2}}.$$

4. The two bases of any oblique prism are reciprocally proportional to the sines of the angles which they make with the axis.

5. Given the forces of many agents, to find the time in which they will all produce a given effect.

6. Find the time of one oscillation in a cycloid.

7. A row of non-elastic balls whose magnitudes increase in geometrical progression are placed at equal distances in a straight

line, and a given velocity is communicated to the first; required the time before the n^{th} is put in motion.

8. Explain the method of finding the specific gravity of a body lighter than water.

9. Construct Newton's telescope, and shew that objects appear inverted through it.

10. The aberration of a given star in R. A. is not necessarily nothing when that in declination is a maximum.

11. Integrate

$$\frac{dx}{x^2+1}, \quad \frac{dx}{\sqrt{x^2+1}}, \text{ and } \frac{dx}{x\sqrt{x^2+1}}.$$

12. Find the area and point of contrary flexure of the curve, whose abscissa is always equal to the arc of a circle, the versed sine of which is the ordinate.

13. If $S = 1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} +$ &c. *in inf.*

and $s = 1 - \frac{1}{2^m} + \frac{1}{3^m} - \frac{1}{4^m} +$ &c. *in inf.*

Shew that $S : s :: 2^{m-1} : 2^{m-1} - 1.$

14. The total number of odd combinations that can be formed out of any number of things is greater by unity than the total number of even.

15. If at any point P in an ellipse the ordinate NP be produced to meet the tangent at the extremity of the latus rectum, the whole line thus produced is equal to SP the focal distance.

16. The force to a plane must vary inversely as the cube of the distance in order that the trajectory may be a semicircle.

17. If two bodies describe about each other and about their common centre of gravity similar and concentric ellipses, the forces with which they attract each other are proportional to their distances.

TUESDAY AFTERNOON.—MR. WHEWELL.

Third and Fourth Classes.

1. In the common system of notation explain why the number of digits cannot be more or less than the local value 10.

2. To find a point within a given triangle from which the three sides shall subtend equal angles.

3. A body is projected with the velocity due to a height h, at an angle A with the horizon. Find an expression for the latus rectum of the parabola described.

4. Shew how the points of contrary flexure in spirals may be found; and apply the method to find the point of flexure of the *lituus* whose equation is $\theta = \frac{a^2}{r^2}$, θ being the angle, and r the radius vector.

5. Shew that a body cannot move so that the velocity shall vary as the space from the beginning of motion. And if the velocity vary as the cube root of the space, find how the time and the force vary.

6. Two bodies P and Q, whose specific gravities are m and n, balance each other on a given straight lever. When the whole is immersed in water whose specific gravity is 1, what alterations must be made in the place of the fulcrum, that they may continue to balance?

7. If a right cone, the diameter of whose base is BC, and vertex A, be cut by a plane so that the section may be an ellipse whose major axis is PQ, the solid content of the part towards the vertex is to that of the whole, as $(APQ)^{\frac{3}{2}}$ to $(ABC)^{\frac{3}{2}}$; APQ and ABC being the areas of those triangles. Also find the equation to the surface of a cone referred to three rectangular co-ordinates.

8. A beam hangs, by means of a given cord fastened to its upper end, from a fixed point in a vertical wall. Against what point of the wall must its lower end be placed, that it may have no tendency either to ascend or descend? Within what limits for the length of the beam is this equilibrium possible?

9. The equation $x^4 - 45x^2 - 40x + 84 = 0$, has two roots whose difference is 3; find them.

10 The force varies inversely as the square of the distance.

A body is projected in a direction which makes an angle of 60° with the distance, with a velocity which is to the velocity from infinity as $1 : \sqrt{3}$. Find the major axis, the position of the apse, and the excentricity, of the ellipse which will be described; and the periodic time.

11. Integrate

$$\frac{dx}{(1-x^3)^{\frac{1}{3}}}; \quad \varepsilon^x x dx; \quad \sin mx \,.\, \sin nx \,.\, \cos px \,.\, dx;$$

$$\frac{dy}{dx} = \frac{a+2x-y}{a-x+2y}.$$

12. Sum the series

$$\frac{1}{1 \cdot 3} - \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} - \text{\&c. } in\ inf.$$

$$\frac{2}{1 \cdot 3 \cdot 7} + \frac{4}{3 \cdot 7 \cdot 15} + \frac{8}{7 \cdot 15 \cdot 31} + \text{\&c. to } n \text{ terms}$$

by increments:

and prove $1^2 + n^2 + \left(\frac{n(n-1)}{1 \cdot 2}\right)^2 +$

$$\left(\frac{n(n-1) \cdot (n-2)}{1 \cdot 2 \cdot 3}\right)^2 + \text{\&c.} = \frac{1 \cdot 2 \cdot 3 \cdots 2n}{(1 \cdot 2 \cdot 3 \cdots n)^2}.$$

13. If we divide a, a^2 a^3, ... by a prime number p, (a being any number) we shall obtain a remainder 1 before we have taken p terms. Also after this remainder the remainders will recur.

TUESDAY EVENING.—Mr. WHEWELL.

1. The area of any right-angled triangle is equal to the rectangle of the semi-perimeter and the excess of the semi-perimeter above the hypothenuse. Required proof.

2. A sets off from London to York, and B at the same time from York to London: they travel uniformly; A reaches York 16 hours, and B London 36 hours, after they have met on the road; find in what time each has performed the journey.

3. The surface of a right cone being given, to find its form that the solid content may be the greatest possible.

4. The equation to a straight line being $y = ax + b$, find the equation to another straight line drawn perpendicular to the first and passing through a given point. Also, solve the same problem when the lines are referred to *three* rectangular coordinates.

5. Two given spheres are moving in given straight lines with given uniform velocities; find where they will meet, (1) when their directions are in the same plane, (2) when they are not.

6. A slender cylinder, whose specific gravity is $\frac{1}{2}$ that of water, is placed in the fluid in an oblique position: find the magnitude, direction, and point of application of the force which must act on the cylinder to keep it immersed $\frac{5}{6}$ of its length.

7. A gate of given weight and form is hung by hinges to a post inclined to a given angle from the vertical. When it swings freely, find the time of its small oscillations.

8. The perihelion distance of a comet is $\frac{1}{3}$ the distance of the earth from the sun; and its orbit, which is parabolical, and the earth's, which is circular, are in the same plane: how many days is the comet within the earth's orbit?

9. Describe the *Repeating Circle*, and the method of observing with it; and explain its advantages.

10. Find the time of a body falling through any space towards a centre of force, (1) when the force varies inversely as the square root, (2) when it varies inversely as the cube root, of the distance. For what laws of force is the integration, which gives the time, practicable?

11. The force varies inversely as the fifth power of the distance. A body is projected with a velocity which is to the velocity from infinity as 5 to 3, and in a direction which make with the line drawn to the centre an angle whose sine is $\frac{2\sqrt{6}}{5}$. Find the orbit, and the time of describing a given angle; and determine whether it has an apse.

12. AM, MP, MT, are the abscissa, ordinate, and subtangent of a curve, of which the property is that $AM : MP :: MT : TA$; find its equation, and trace it.

13. Integrate the following differentials and differential equations,

$$\frac{dx \sqrt{(1 - x^2)}}{1 + x^2}; \quad \varepsilon^{ax} \sin^2 x \,.\, dx;$$

$$x^m (ydx + xdy) = y^n (ydx - xdy);$$

$$d^3y - 3d^2ydx + 3dydx^2 - ydx^3 = 0;$$

$$x\frac{dz}{dx} + y\frac{dz}{dy} = z.$$

14. If P and S, attracting each other with forces proportional to their masses, revolve round their centre of gravity; their periodic time is to the periodic time of an indefinitely small particle, which describes a similar orbit, round S at the same distance, as $\sqrt{S}$ to $\sqrt{(S + P)}$. Also, if to P be annexed an equal mass which is *not attracted* by S, what alteration will be made in this proposition?

15. In any polyhedron the number of solid angles together with the number of plane faces, exceed by 2 the number of *edges*.

16. On two straight lines at right angles to each other, two points move respectively from given positions, with equal uniform velocities; find the curve to which the line which joins them is always a tangent.

17. In a bag are 8 bank notes, viz, 1 of twenty pounds, 2 of five, and 5 of one: a person is allowed to take out three indiscriminately: what is the value of his expectation.

18. The radiating point and the caustic being given, shew that there are an infinite number of reflecting curves which will produce the caustic.

19. A body falls towards a centre of force which varies directly as the distance, in a medium of which the resistance varies as the square of the velocity: find the velocity at any point. Shew from your result that when the resistance vanishes, the velocity coincides with that in a non-resisting medium.

20. Let $yz = \varphi\left(\frac{y}{x}\right)$ be the equation to a curve surface, where φ is any function whatever: shew that the part which the tangent plane to any point cuts off from the axis of z, is twice the value of z for that point.

21. Sum the series,

$$\frac{1}{1\cdot 4}-\frac{1}{3\cdot 6}+\frac{1}{5\cdot 8}-\text{\&c. } in\ inf.$$

$$x\cdot\cos A+x^2\cdot\cos 2A+x^3\cdot\cos 3A+\text{\&c. } in.\ inf.$$

Also prove that

$$\frac{1}{1^2+1}+\frac{1}{2^2+1}+\frac{1}{3^2+1}+\text{\&c. } in\ inf.$$

$$=\frac{\pi}{2}\cdot\frac{\varepsilon^{2\pi}+1}{\varepsilon^{2\pi}-1}-\frac{1}{2};$$

and find an expression for the n^{th} term of a series where each term is the product of the two preceding.

22. *AB* is the axis of a cycloid, (of which *A* is the vertex,) and *C* its middle point. An ordinate is drawn meeting the axis in *M*, the cycloid in *P*, *P′*, and the generating circle in *Q*, *Q′*. *CN* is taken towards *A* equal to *AM*. Then the cycloidal sector *PNP′* = triangle *QBQ′*. Required proof.

23. If a chain of given length be suspended from two points, shew that its centre of gravity is lowest when its form is a catenary.

24. Expand the radius vector of an ellipse about the focus

$\left(= r = \dfrac{a(1-e^2)}{1+e\,.\cos\theta}\right)$ in a series of the form

$$A+B\,.\cos\theta+C\,.\cos 2\theta+D\,.\cos 3\theta+\text{\&c.}$$

CAMBRIDGE PROBLEMS.

THE SENATE-HOUSE PROBLEMS,

Given to the Candidates for Honors during the Examination for the degrees of B. A.

BY THE TWO MODERATORS.

Monday, January 15, 1821.

Monday Morning.—Mr. Chevallier.

First and Second Classes.

1. The number 803 expressed in a different scale of notation becomes 30203; required the radix of the scale?

2. Of all sections made by planes passing through both sides of an oblique cone, two are circles, and all the rest ellipses.

3. Why does the apparent distance of two fixed stars increase as they approach the horizon?

4. A clepsydra is constructed to mark equal portions of time, in the form of a paraboloid having its vertex downwards, the equation to the generating curve being $y^4 = ax$. How must the scale on the axis be graduated?

5. a and b are the arms of a straight lever which turns on an axis, the radius of which is r. P would maintain the equilibrium acting perpendicularly at the distance (a), if there were no friction: but a weight p must be added to it in order to overcome the friction. Find the proportion of the friction to the pressure?

6. S being the focus of an hyperbola, and (pm) the perpendicular upon its directrix from a point (p) in the *opposite hyperbola*, $Sp : pm :: SC : AC$, a given ratio.

7. Find the law of force by which a body may describe a rectangular hyperbola, the force acting in parallel lines perpendicular to one of its asymptotes.

8. Two pendulums, the length of which are L and l, begin to oscillate together, and are again coincident after n oscillations of the first pendulum. Given L to find l.

9. The moon revolves in a circle about the earth, and the quantity of matter in the earth is suddenly doubled. Compare the eccentricity of the orbit now described with its axis major, and with the original radius of the moon's orbit.

10. Shew that the hour of sun-rise and sun-set together = 12 hours nearly: and find the correction necessary if the sun's declination shall have changed by a given quantity.

11. Explain the construction of the steam engine; and having given the weight upon the piston, the quantity of steam admitted, and the content of the cylinder, find the velocity of the piston at any point, and the time of describing the cylinder.

12. Prove that

$$n^n - n.(n-1)^n + n.\frac{n-1}{2}.(n-2)^n - \&c. = 1.2.3\ldots n.$$

13. Find the force of the sun to disturb the motions of the moon.

MONDAY AFTERNOON.—MR. CHEVALLIER.

Fifth and Sixth Classes.

1. Find x from the equation

$$\sqrt{x} + \sqrt{a+x} = \frac{2a}{\sqrt{a+x}}.$$

2. The common difference of 4 numbers in arithmetical progression is 1, and their product 120; find the numbers?

3. A pendulum virbrating seconds at the earth's surface is carried to a distance from the centre of the earth equal to that of the moon. What is the time of its vibration?

4. Shew that the image of a straight line placed between the centre and principal focus of a concave mirror is an hyperbola.

5. In latitude 45°, find the time of sun-rise on the longest day.

6. Insert three harmonic means between a and b.

7. Compare the pressure on the surface of a sphere filled with water, with the weight of a sphere of mercury of the same magnitude.

8. A beam 30 feet long, balances itself upon a point at $\frac{1}{3}^{rd}$ of its length from the thicker end. But when a weight of 10lbs. is suspended at the other end, the prop must be moved two feet

towards it to maintain the equilibrium. What is the weight of the beam?

9. Prove Demoivre's formula

$$(\cos A \pm \sqrt{-1} \,.\, \sin A)^m = \cos mA \pm \sqrt{-1} \,.\, \sin mA.$$

10. Given the three angles of a spherical triangle, to find its surface.

11. Sum the series

$$1 + 1 + \frac{3}{4} + \frac{1}{2} + \frac{5}{16} + \frac{3}{16} + \text{\&c. to } n \text{ terms.}$$

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \text{\&c. to } n \text{ terms,}$$

by the method of increments.

12. Solve $x^3 - \frac{5}{2} x^2 + x - \frac{1}{6} = 0$, the roots being in harmonical progression.

13. Find the area of a curve in which the abscissa $= \theta$, and the ordinate $= \frac{\cos^2 \theta}{r}$, between the values of $\theta = 0$, and $\theta = 90°$.

14. Find $d \left\{ \frac{1}{\sqrt{-1}} \cdot 1 \left(x \sqrt{-1} + \sqrt{1 - x^2} \right) \right\}$.

$$\int \frac{dx \sqrt{2ax + x^2}}{x},$$

$$\int \frac{dx}{(1 + x^2)^3}.$$

15. Explain how wheels assist the motion of a carriage.

16. In what direction must a body be projected with a given velocity from a point in a given inclined plane, that the range may be the greatest possible?

17. Explain Archimedes' screw?

18. Find the time in which the moon would fall to the earth, if suddenly deprived of its angular motion?

MONDAY AFTERNOON.—MR. PEACOCK.

Third and Fourth Classes.

1. In the solution of right-angled spherical triangles by Napier's rules, what cases are ambiguous?

14. Integrate the equations

$$(1)\quad \frac{dx}{1+x+x^2} + \frac{dy}{1+y+y^2} = 0.$$

$$(2)\quad y = \frac{x\,dy^2}{dx^2} + \frac{dy}{dx}.$$

$$(3)\quad \frac{d^2z}{dx^2} + \frac{3d^2z}{dxdy} + \frac{2d^2z}{dy^2} = x + y.$$

15. The curve which is expressed by the particular solution of a differential equation of the first order, is the locus of the intersections of the curves which arise from giving every possible value to the constant in the general solution.

16. A tetrahedron, whose base is an equilateral triangle, the side of which is equal to one half of each of the remaining edges, is thrown upon a horizontal table: what is the probability of its resting upon its base, excluding all consideration of the mechanical action, which arises from the rotation of the solid?

17. Supposing γ Draconis to be affected by a sensible annual parallax, in what manner will its apparent place be affected both by aberration and parallax on March 20, June 21, September 23, and December 23, its right ascension being 18 hours nearly?

18. The level of a transit instrument is suspended by hooks from its axis: in what manner are the errors which arise from the axis not being horizontal, or from the level not being parallel to the axis, distinguished from each other, and by what adjustments are they corrected?

19. A ray of light, after being refracted through media possessing *equal dispersive* powers, will always appear coloured at its emergence, unless the incident and emergent rays are parallel.

20. Two equal bodies *A* and *B* are connected by a string of given length: *A* is placed in a horizontal groove, and *B* hangs freely down, the string passing through an aperture which is continued along the bottom of the groove; a given velocity is given to *A*; find the position of *B* at the end of t'.

21. A cylinder of given length, is pressed down in a vertical position into a fluid, so that its upper end is on a level with the surface, the specific gravity of the cylinder being one half that of the fluid: the pressure being removed, to find the greatest height to which the upper end of the cylinder will rise above the surface of the fluid.

22. Find an expression for that part of the force of jupiter to disturb saturn, which acts in the direction of a tangent to saturn's orbit.

23. If the accelerating gravity of a satellite of jupiter to the sun was greater than that of jupiter at the same distance, in the

ratio of $e : 1$, where e differs little from unity, then the distance of the centres of the sun and of the orbit of the satellite, would be greater than the distance of the centres of the sun and jupiter in the ratio of $\sqrt{e} : 1$.

24. If the sun and moon be both supposed to be in the equator: to compare the lengths of a lunar and a tide day, for a given position of the luminaries.

25. Shew that all the sections of an ellipsiod made by parallel planes are similar ellipses.

TUESDAY MORNING.—MR. PEACOCK.

First and Second Classes.

1. If a and b be prime numbers, the number of numbers prime to ab and less than ab, is equal to

$$(a-1)(b-1),$$

unity being considered as one of them.

2. In what case are solid figures said to be similar and equal, according to Euclid? to what objections are these definitions liable?

3. Mention some of the experiments and observations, from which we may infer the truth of the second law of motion.

4. A given globe *rolls* down a given inclined plane in a medium resisting as the square of the velocity; to find the time of describing a given space.

5. Find the equation of the curve traced out by the extremity of the perpendiculars upon the tangents of a circle, drawn from a point in its circumference; and find its greatest ordinate.

6. An object appears brighter, *cæteris paribus*, when seen through a convex lens, than when seen through a concave lens.

7. Explain the method of determining accurately, when the first point of aries is upon the meridian.

8. Explain the method of determining the sun's meridian altitude by means of a sextant: 1st on the open sea: 2d when the sun's altitude is not less than 60°, but a neighbouring coast is on the same side of the ship with the sun: 3dly on land?

9. In elliptical orbits of small eccentricity, the diminution of angular velocity in moving from the lower apse to the higher, is nearly proportional to the increase of distance.

10. A degree of latitude in latitude 45°, is nearly an arithmetic mean between a degree at the equator and the pole.

11. Given the logarithm of n, to find the logarithm of $n+1$.

12. The integration of the partial differential equation

$$Pp + Qq = R,$$

where $p = \frac{dz}{dx}$ and $q = \frac{dz}{dy}$, and P, Q and R are functions of the variables x, y and z, is reduced to the integration of equations of two variables, when any one of the equations

$$Pdy - Qdx = 0,$$

$$Pdz - Rdx = 0,$$

$$Qdz - Rdy = 0,$$

involves two variables only.

13. Two bodies A and B are placed upon a horizontal plane, and connected by a rigid rod without weight: a body C impinges upon a given point of the rod, in a given direction and with a given velocity: define the motions of A and B, the body C not being connected with the system after the impact.

14. The attraction of a spherical shell upon a particle placed without it, is the same as if the whole matter in the shell were placed in its centre.

TUESDAY AFTERNOON.—MR. PEACOCK.

Fifth and Sixth Classes.

1. A person paid a tax of 10 per cent. upon his income: what must his income have been, when after he had paid the tax, he had £1250 remaining?

2. Prove the rule for extracting the square root in numbers?

3. The difference between any number, and that number inverted, is divisible by 9.

4. If two straight lines be at right angles to the same plane, they are parallel to one another.

5. Given two sides and the included angle of a plane triangle: find the remaining parts, and reduce the results to logarithmic computation?

6. Every equation has at least as many changes of sign from + to — and from — to +, as it has positive and possible roots; and as many continuations of sign from + to +, and from — to —, as it has negative and possible roots.

7. Two tangents to a parabola drawn from the same point of the directrix, are at right angles to each other.

8. Prove that the sides of the polar or supplemental triangle are supplements of the angles of the given triangle?

9. When a body is uniformly accelerated from rest, to find the space described in a given time.

10. Shew, when P sustains W upon a screw, if a slight motion be given to the machine, that P's velocity : W's velocity :: $W : P$.

11. Distinguish between the centre of gravity and centre of pressure, and shew that the former is always nearer to the surface of the fluid than the latter.

12. Graduate a thermometer according to Fahrenheit's scale?

13. Prove that objects appear erect in Galileo's telescope?

14. By what experiments is it proved, that light consists of rays differing in colour and refrangibility?

15. What are the principal phænomena presented by the sun and earth in the course of a month, to a lunar observer?

16. Explain the mode of constructing a catalogue of the fixed stars?

17. When is the planet venus stationary, and when retrograde?

18. A body describes an ellipse, the force being in the centre: given the force at a given distance, to find the actual periodic time.

19. A body is projected from a given point, in a given direction with a given velocity, when the force varies inversely as $(\text{dist.})^2$: find the latus rectum of the orbit described?

20. Investigate the differential expression for the length of a curve.

21. Trace the curve whose equation is

$$x^4 - a^2x^2 + a^3y = 0,$$

and find its points of contrary flexure.

22. Find the integrals of

$$\frac{dx}{\sqrt{(a - x b^2)}}, \quad \frac{x\,dx}{(x + a)(x^2 + b^2)},$$

$$d\theta\,(\sin\theta)^3, \quad a^2x^3dx.$$

23. Prove that

$$\log(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \&c.$$

24. Two equal weights are fixed, one at the middle point, and the other at the extremity of an inflexible and imponderable rod, which is suspended at the other extremity: if this compound pendulum be made to vibrate through small arcs, to find the time of its vibrations.

TUESDAY AFTERNOON.—MR. CHEVALLIER.

Third and Fourth Classes.

1. Shew how the logarithms of the natural numbers from 1 to 12 may be computed.

2. The sides of a plane triangle are 3, 5, 6: compare the radii of the inscribed and circumscribed circles.

3. If two straight lines intersect each other in a circle, the sum of the arcs cut off between their extremities is the same as that cut off by any two lines respectively parallel to them, and intersecting each other within the circle. Prove this property, and shew its use in correcting observations made with circular instruments inaccurately centered

4. If a ladder slides down a perpendicular wall, shew that each stave describes a quadrant of an ellipse, except the middle one, which describes a quadrant of a circle.

5. Apply D'Alembert's principle to find the velocity and time when p draws q over a fixed pulley.

6. Explain the phases of the earth as they would be seen from the moon.

7. A small aperture (a) is made in the vertical side of a cylindrical vessel filled with a fluid; the area of its horizontal section being A. Compare the latus-rectum of the parabola first described by the spouting fluid with the length of a pendulum vibrating once while the surface of the fluid descends to the orifice.

8. The earth being a sphere, and its radius 4000 miles, what must be its diurnal rotation that a body at the equator may lose half its weight?

9. A sphere of given radius is suspended by a point at a distance from its centre equal to its diameter. Find the time of its oscillation, and the point within the sphere, at which it must be suspended so as to oscillate in the same time.

10. In a revolving spheroid of small eccentricity, if polar gravity : equatoreal sensible gravity :: radius of equator : semi-axis, gravity is every where perpendicular to the spheroidal surface.

11. Apply the duodenary scale of notation to find the solidity of a cube the side of which is $13^{f}.\ 7^{in}.\ 7^{pts}$.

12. On board a ship in north latitude, jupiter is observed on the meridian at $3^{h}.\ 4^{m}.\ 56^{s}$. and his corrected altitude is 29° 6′ 42″. One of his satellites is at the same instant eclipsed. His tabulated declination is 5° 4′ 35″ north, and the tabulated time of the eclipse $7^{h}.\ 0^{m}.\ 32^{s}$. Required the latitude and longitude of the ship.

TUESDAY EVENING.—Mr. CHEVALLIER.

1. Prove the rule for the multiplication of duodecimals.

2. Represent $\sqrt{2n\sqrt{-1}}$ as a binomial surd?

3. Find two numbers such that their sum, product, and the difference of their squares may be all equal.

4. How many different ways may £100 be paid in crowns and guineas?

5. A body describes a logarithmic spiral, and approaches the centre by a space which is small compared with the whole distance. Compare the time of one revolution with the time to the centre.

6. In the expansion of $(a + b + c + \&c.)^m$ where $m = p + q + r + \&c.$: find the coefficient of the term involving $a^p \,.\, b^q \,.\, c^r$, &c.

7. Shew that the image of an indefinite straight line perpendicular to the axis of a convex lens, and nearer to its centre than the principal focus of parallel rays incident in the opposite direction, forms the arcs of two opposite hyperbolas; and find the semi-axes.

8. Explain *fully* the method of finding the longitude at sea by the observed distance between the moon and a star and their altitudes.

9. A vessel is kept filled with a fluid; and an aperture is made in its perpendicular side in the form of a parabola, the vertex of which coincides with the surface of the fluid. Find the depth of a horizontal section such that if the whole fluid issued with *its* velocity, the quantity discharged in a given time would be the same as when each horizontal section flows with its own velocity.

10. A recurring series may generally be resolved into two or more geometric series.

11. Find the solid content of a sphere by referring it to three rectangular co-ordinates.

12. Shew, by measuring the area *ANB* in the 12th section, that if a sphere be composed of particles the attraction of which $\propto \dfrac{1}{(\text{dist.})^2}$, the attraction of the whole sphere on an external particle varies in the same law.

13. At what distance from its centre must the earth, considered as a sphere, receive a single impulse, so as to produce its diurnal and annual rotation?

14. A body oscillates in a cycloid: compare the whole tension of the string at any point with the weight of the body.

15. Give Clairaut's approximation to the solution of a cubic equation in the irreducible case.

16. Trace the curve the equation to which is

$$xy + ay + bx = c.$$

17. Sum the series

$$1^2 - 2^2 + 3^2 - 4^2 + \&c. \pm x^2.$$

18. A comet describes 90° from the perihelion in 100 days. Compare its perihelion distance with the radius of a planet's circular orbit which revolves about the sun in 942 days.

19. Explain Borda's circle of repetition: and the method of finding the latitude by the zenith distances of stars near the meridian.

20. Find $\Delta^3 . u_x$, $\Sigma . x^3$, and $\Sigma . \sin x\theta$.

21. Integrate

$$\frac{dx}{\sqrt{1+x^2}} + adx + 2bydy = 0, \quad \frac{d^4y}{dx^4} = \frac{d^2y}{dx^2}.$$

22. If a body is projected in any direction, and acted upon continually by two forces tending to fixed centres, not both in the same plane with the direction of projection, it will describe by lines drawn from the two fixed points equal solids in equal times about the line joining the two points.

23. Given, in the equation $\frac{d^2u}{dv^2} + u = 0$, $u = a \sin v + b . \cos v$, to solve the equation $\frac{d^2u}{dv^2} + u + \Pi = 0$, by the *Variation of the Parameters.*

24. A body oscillates in a cycloid, in a medium the resistance of which $\propto (\text{vel.})^2$; construct for the resistance at any point.

THE SENATE HOUSE PROBLEMS,

Proposed by the Two Moderators, to the Candidates for Honors, during the Examination for the Degree of B. A.

MONDAY, JANUARY 14, 1822.

MONDAY MORNING.—MR. CHEVALLIER,

First and Second Classes.

1. Find a Series of fractions converging to $\sqrt{17}$.

2. If $ABCD$ is a parallelogram, and AC a diameter; and from B there be drawn a straight line cutting the diameter in E and the two sides, or the sides produced, in F, f respectively; shew that $EF . Ef = BE^2$.

3. If $1, \rho, \rho^2, \ldots \rho^{n-1}$ are the roots of the equation $x^n - 1 = 0$; find the value of $1^m \cdot \rho^r + \rho^m 1^r + \&c. + \rho^m \cdot \rho^{2r} + \rho^{2m} \cdot \rho^r + \&c.$

4. Integrate the quantities

$$\frac{dx}{x^n\sqrt{1+x^2}}, \text{ n being an even number; } \frac{dx}{x^2\sqrt{a+bx+cx^2}}.$$

5. What must be the solid of revolution, so that when suspended by its vertex the centre of oscillation may be in its base?

6. If a triangular prismatic beam is supported at both ends, shew that it is twice as strong when the edge is uppermost, as when the base is.

7. A person wishes to see distinctly when under water. What kind of glasses must he use, and of what focal length?

8. The autumnal equinox takes place at 6 in the evening, the Moon being full at the same instant, and in her ascending node. The next night the Moon rises at the *same hour*: required the north latitude of the place.

9. Given the base of a plane triangle, and the difference of the angles at the base to find the curve traced by the vertex.

10. Two equal heavy balls are suspended by wires of the same given length from the vertical axis of a machine, and are just in contact. How far will they separate from one another when a given angular velocity is communicated to the system?

11. Shew that if M and S represent the height of the tide produced by the Moon and Sun respectively; retardation of tide at new and full Moon : retardation in quadratures :: $M - S$: $M + S$.

12. Find the mean horary motion of the Moon's nodes, when the line of the nodes is in octants.

MONDAY AFTERNOON.—MR. CHEVALLIER.

Fifth and Sixth Classes.

1. The first term of a geometric series continued in infinitum is 1; and any term is equal to the sum of all the succeeding terms. Required the series.

2. Transform $x^3 - 2x^2 + 2x - 4 = 0$ into an equation, the roots of which are the squares of the roots of the original equation.

3. Sum the series

$$\frac{1}{2.4.6} + \frac{1}{4.6.8} + \&c. \text{ to n terms, and in infinitum.}$$

4. Trace the curve, the equation to which is $y^2 - ax - ab = 0$. Draw a tangent to it at any point; and determine the angle at which the curve cuts the axis.

5. In a spherical triangle, having given two angles and the included side, it is required to find the other angle.

6. Given

$$\left.\begin{array}{l}\log.\ 6753 = 3\cdot8294967\\ \log.\ 6754 = 3\cdot8295611\end{array}\right\}\text{ to find log. }67532.$$

7. Shew how the true value of a fraction may be found, the numerator and denominator of which both vanish upon assigning a particular value to the variable quantity: and find the value of

$$\frac{\tan.x + \sec.x - 1}{1 + \tan.x - \sec.x}\text{ when }x = 0.$$

8. Find the radius of curvature at any point of the catenary.

9. A body falls down a given inclined plane; and, at the instant it begins to fall, another is projected upwards, from the bottom of the plane, with a velocity equal to that in falling through n times its length. Where will they meet?

10. Find the position of the centre of gravity of the quadrant of a circular area.

11. An isosceles triangle is immersed perpendicularly in a fluid, with its vertex coincident with the surface, and its base parallel to it. How must it be divided by a line parallel to the base, so that the pressure upon the upper and lower parts respectively may be in the ratio of 1 : 7?

12. When the same string passes over any number of pulleys, and the parts of the string supporting any pulley at the lower block are not parallel to one another, find the proportion between P and W in equilibrio.

13. A hollow cylinder is viewed by an eye placed in its axis produced. Compare its apparent capacity when empty and when filled with water.

14. By what methods may the variation of the compass be determined: and to what point does the true north correspond when the variation is 22° 30′ west?

15. Find the latitude of the place in which the longest day contains 16 hours.

16. A body is projected in a given direction, with a given velocity from a given point; and is acted upon by a *repulsive* force which varies as the distance from another given point. Required the curve which it will describe.

17. How may the phenomena of the Trade Winds be explained?

18. A bucket descends into a well, unwinding a string from a cylinder of given weight and radius. What is the velocity acquired in falling through a given space, and the time of descent; the weight of the string being neglected?

19. Integrate the quantities

$$\frac{dx}{(1-x^2)^{\frac{3}{2}}}, \quad \frac{dx}{a-bx^2}, \quad \frac{dx}{x^3-3x^2+2x}.$$

20. A body describes a circle about a fixed point, the force varying inversely as the square of the distance: another body, the attractive force of which varies in the same law, is introduced into the system. How will this affect the velocity of the body, the form of the orbit, and the periodic time?

MONDAY AFTERNOON.—MR. HIND.

Third and Fourth Classes.

1. Two weights sustain each other on two inclined planes having a common altitude by means of a string parallel to the planes: compare the pressures.

2. Prove that

$$2\,(\sin.^2 A \,.\, \sin.^2 B + \cos.^2 A \cos.^2 B) = 1 + \cos.\, 2A \,.\, \cos.\, 2B.$$

3. PSp is any parameter of a parabola whose focus is S, and latus rectum L, prove that

$$4SP \,.\, Sp = L\,(SP + Sp).$$

4. A parallelogram and a triangle upon the same base and between the same parallels revolve round the base as an axis; prove that the solid generated by the triangle equals one-third of that generated by the parallelogram.

5. How much of the earth's surface may be seen by a person raised n radii above it?

6. The ordinate to the axis of an ellipse is produced till it equals the corresponding subtangent: find the equation to the curve thus traced out and the area.

7. Given the sum of $2n$ quantities in arithmetical progression, and the sum of their squares, to find the quantities themselves.

8. Three equal weights are placed at the angles of an equilateral triangle without weight, which is suspended by an axis perpendicular to its plane bisecting one of the sides; find the centre of oscillation.

9. Prove Newton's 4th Lemma, (Sect. 1.) and by means of it find the contents of an oblate spheroid.

10. Differentiate

$$(1) \quad \log. \frac{\sqrt{4ax+x^2}+x}{\sqrt{4ax+x^2}-x}. \qquad (2) \quad \frac{e^x\sqrt{c+x}}{e^{-x}\sqrt{c-x}}.$$

and integrate

(1) $\frac{dx}{\sqrt{a+b\sqrt[3]{x}}}$. (2) $\frac{dx\sqrt{a^2+x^2}}{x^3}$. (3) $e^{ax}\cos. x \, . \, dx$.

11. A vessel of given altitude empties itself through an orifice of given dimensions in its lowest point, and the upper surface descends with a given uniform velocity: find the content of the vessel.

12. A ray of light issuing from a point in the extreme ordinate of a parabola is incident in a direction parallel to the axis, and after two reflections at the curve meets the ordinate again: prove that the length of the path described will be the same from whatever point in the ordinate the ray proceeds.

13. Find the area included between any two radii of a spiral where the angle contained between them is the measure of their ratio.

14. Given the velocity of projection equal to the velocity in a circle at the same distance, (force $\propto \frac{1}{D^2}$), required the direction in which a body must be projected at a given distance, that the focus of the conic section described may bisect the semi-axis major; and determine the magnitude and position of the axes.

15. The plane of a vertical dial is inclined at an angle of 45° to the plane of the meridian in a latitude whose sine is $= \frac{1}{\sqrt{3}}$: find the position of the substile, the altitude of the stile, and the hour lines.

MONDAY EVENING.—MR. HIND.

1. Shew that $y^{mn} - 1$ is divisible by either of the quantities $y^m - 1$ and $y^n - 1$ without a remainder.

2. In an isosceles plane triangle, prove, *geometrically*, that the versed sine of the vertical angle : radius :: the square of the base : twice the square of either side.

3. Given the m^{th} and n^{th} terms of an harmonical progression, to find the $(m+n)^{th}$ term.

4. The earth being considered a perfect sphere, prove that at any place the length of a degree of latitude : the length of a degree of longitude :: radius : cosine of latitude.

5. Prove that $\cos. (A+B) . \sin. (A-B) + \cos. (B+C) . \sin. (B-C) + \cos. (C+D) . \sin. (C-D) + \cos. (D+A) . \sin. (D-A) = 0$.

Investigate the following formulæ,

$$\cos.\,\theta = \frac{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}}{2},\ \sin.\,\theta = \frac{e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}}{2\sqrt{-1}}.$$

and thence prove that $\dfrac{\sin.\,A}{1 - \cos.\,A} = \cot.\,\dfrac{A}{2}$.

6. Find the time of emptying a sphere filled with fluid through an orifice in its lowest point.

7. Prove that $\log.\,(n+1) = \log.\,n + 2\log.\left(\dfrac{2n+2}{2n+1}\right) + \log.\,\dfrac{(2n+1)^2}{(2n+1)^2 - 1}$.

8. Find the surface of an equilateral and equi-angular spherical polygon of n sides, and determine the value of each of the angles when the surface equals half the surface of the sphere.

9. Required the radius of curvature of the curve whose equation is $\dfrac{x}{y} = \dfrac{a+y}{a-x}$, and determine the co-ordinates of the centre.

10. A ball, whose elasticity : perfect elasticity :: n : 1, is projected with a given velocity in a direction making an angle of 60° with the horizon, and when at its greatest height, is reflected by a vertical plane: determine where the ball will again strike the horizon, and the whole time of flight.

11. Investigate a differential expression for the surface of a solid of revolution, and apply it to find the surface of the solid generated by the figure of tangents revolving round its axis.

12. Shew how every case of oblique spherical triangles may be solved by Napier's rules *only*.

13. Investigate the apsidal equation, and shew what number of possible positive roots it can have, when the velocity is acquired from a finite distance and the force varies as D^{n-1}.

14. Given the logarithms of $(1+x)$ and $(1+2x)$, shew how the logarithm of $(1+3x)$ may be computed.

15. Define the centre of spontaneous rotation; shew generally how it may be found; and determine it when a straight rod of uniform density and given length is struck perpendicularly at a given point.

16. If the area of a curve between any two values of one abscissa can be expressed in finite terms; shew that the area between two values of any other abscissa of the same curve can be found.

17. The density of different parts of a circle varies as the square of the distance from the centre: find the velocity ac-

quired by a corpuscle attracted towards this circle in a line passing through its centre and perpendicular to the plane, the attractive force of each particle varying as $\frac{1}{D^2}$.

18. State generally the principal of *virtual velocities*, and from it deduce the position of equilibrium of a straight rod of uniform density placed on two inclined planes.

19. Find the n^{th} differential of $\sqrt{1-x^2}$.

Find also $\int \frac{dx}{\sqrt{1-x^2}} \cdot \int \frac{dx}{\sqrt{1-x^2}}$,

and $\int dx\,(a^2 + b^2 - 2ab \cos. x)^n$ from $x = 0$ to $x = 180°$, and determine the relation between x and y when

$$\frac{d^2y}{dx^2} + A.\frac{dy^2}{dx^2} + B.\frac{dy^2}{dx^3} = 0.$$

20. Find the whole times of ascent and descent of a body urged by the force of gravity in a medium whereof the resistance varies as the square of the velocity, and give *Newton's* constructions.

21. Sum the following series

(1) $\frac{2x}{1 \cdot 3} + \frac{3x^2}{3 \cdot 5} + \frac{4x^3}{5 \cdot 7} +$ &c. ad inf.

(2) $\frac{\cos. \theta}{1^2} - \frac{\cos. 2\theta}{2^2} + \frac{\cos. 3\theta}{3^2} -$ &c. ad inf.

(3) $\tan. \theta . \left(\tan. \frac{\theta}{2}\right)^2 + 2 \tan. \frac{\theta}{2} . \left(\tan. \frac{\theta}{4}\right)^2$

$+ 4 \tan. \frac{\theta}{4} . \left(\tan. \frac{\theta}{8}\right)^2 +$ &c. to (n) terms.

22. Find the horary motion of the moon's nodes in a circular orbit.

Newton, Prop. 30. Book iii.

23. A ray of homogeneal light is refracted through a double convex lens: compare the densities in different parts of the circle of chromatic dispersion.

24. The sum of the areas described by any number of bodies round a given point, multiplied by the respective masses of the bodies, is proportional to the time, supposing them to be acted upon *only* by their mutual attractions, and by the force tending to the given point.

TUESDAY MORNING.—MR. HIND.

First and Second Classes.

1. *A* and *B* can do a piece of work in *m* days; *B* and *C* in *n* days: in what time can *A* and *C* do the same, supposing that *A* can do *p* times as much as *B* in a given time?

2. A normal drawn to a cissoid at the point where it cuts the generating circle, meets the axis produced in a certain point: prove that the line intercepted between this point and the vertex of the cissoid is divided into three equal parts by the centre and the further extremity of the diameter, of the generating circle.

3. The distance of the centre of gravity of a cycloid from the vertex $= \frac{7}{12}$ of the axis: compare, from this, the contents of the solids generated by its revolution round the base and a tangent at the vertex.

4. Prove that the cube of any number, and the number itself being divided by 6 leave the same remainder.

5. Two straight rods equal in length are suspended by their extremities, one being of uniform density, and the density of the other varying as the n^{th} power of the distance from the point of suspension, and they make small oscillations in times which are as $\sqrt{5} : \sqrt{6}$. Required the value of n.

6. An upright cylindrical vessel empties itself through an orifice in the base: compare the pressures upon the concave surface at first, and when half the time of emptying has elapsed.

7. Given the magnitude of a spherical surface, find the radius of the sphere so that the corresponding spherical segment may be the greatest possible.

8. Compare the momentum of a paraboloid with that of its inscribed cone, having the same base and vertex, when they both revolve round their common axis.

9. Investigate the equation to the curve, in which the area has the same ratio to the square of the ordinate that the ordinate has to the abscissa.

10. Given the latitudes and longitudes of two places on the Earth's surface, to find their distance.

11. Find the integral of

$$x^m . dx . (\log. x)^m \text{ from } x = 0 \text{ to } x = 1.$$

12. A string wrapped round a cylindrical annulus of uniform density whose radii are *R* and *r*, passes over a fixed pulley, and

has a weight attached to it: find the space descended by the annulus in a given time.

13. Given the time, construct for the inclination of the lunar orbit to the plane of the ecliptic.

Newton, Prop. 35. Book iii.

TUESDAY AFTERNOON.—MR. HIND.

Fifth and Sixth Classes.

1. Find the present worth of *P£* due *n* years hence, at *r* per cent. discount.

2. Investigate the rule for the extraction of the square root in whole numbers, and determine *generally*, the limit which the remainder after any operation cannot exceed.

3. Shew that the content of a sphere : the content of the greatest cone that can be inscribed in it :: $3^3 : 2^3$.

4. Given the ratio of the sines of the angles of a plane triangle, and the radius of the inscribed circle; to construct the triangle.

5. The orifices in the equal bases of two upright prismatic vessels are in the ratio of 2 : 1, and the vessels are emptied in equal times; compare their altitudes.

6. Prove that *Waring's* solution of a biquadratic equation fails when all the roots are impossible, and of the form $a \pm b\sqrt{-1}$.

7. Determine the point in *P*'s orbit, (Sect. 11,) where the tangential ablatitious force is a mean proportional between the addititious and central ablatitious forces.

8. The roots of the equation $6x^4 - 43x^3 + 107x^2 - 108x + 36 = 0$ are of the form a, b, $\frac{a}{b}$ and $\frac{b}{a}$, find them: and shew what relation exists between the coefficients of a cubic whose roots are of the form $+a$, $-a$, and $+b$.

9. Find the point at which a ray of light parallel to the axis must be incident upon a concave spherical reflector, so that after two reflections it may cut the axis in a given angle.

10. The equation to a curve is $y = x^3 - 9x^2 + 24x + 16$; determine the values of the abscissa when the ordinate is a maximum, and when a minimum, and find the area included between those ordinates.

11. Differentiate

(1) $\log.(2x + 1 + 2\sqrt{1 + x + x^2})$. (2) $(\cos.\theta + \sec.\theta)$.

Integrate

(1) $\dfrac{x^2\,dx}{(1 + x^2)^2}$. (2) $\dfrac{x^{\frac{1}{2}}\,dx}{\sqrt{a + b\sqrt{x}}}$. (3) $a\,dx \int b\,dx \int c\,dx$,

and find the values of

(1) $\dfrac{a^{\log.x} - 1}{\log.x}$ when $x = 1$. (2) $\dfrac{\log.\tan.x}{\log.\tan.\frac{x}{2}}$ when $x = 0$.

12. The magnitudes of three perfectly elastic bodies are in harmonical progression, prove that the momentum communicated to either of the extremes by the impact of the other equals the momentum of the mean moving with the velocity of the impinging body before impact.

13. Given the latitude of the place, and the length of the day, to find the time of the year.

14. If a body describe the arc of a cycloid by a force acting parallel to its base, prove that the force varies inversely as $2\sin.\theta - \sin.2\theta$, θ being the corresponding arc of the generating circle reckoning from the vertex.

TUESDAY AFTERNOON.—MR. CHEVALLIER.

Third and Fourth Classes.

1. $S_1, S_2, S_3, \ldots S_n$ being the sums of n geometric series continued in infinitum, the first term of which is 1, and the common ratio $\frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3} \ldots \frac{1}{r_n}$, respectively. Required the sum of their reciprocals, $\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} \ldots + \frac{1}{S_n}$.

2. Shew that cones and cylinders upon equal bases are to one another as their altitudes.

3. P supports W upon an axle, by means of a perpetual screw acting upon the circumference of the wheel. Required their proportion.

4. A sphere of glass and another of water being placed in air, what must be the proportion of their radii, that their magnifying powers may be the same?

5. A life-boat contains 100 cubic feet of wood, specific gra-

vity 8; and 50 feet of air, specific gravity .0012. When filled with fresh water, what weight of iron ballast, specific gravity 7.645, must be thrown in before it will begin to sink?

6. Explain the different uses of a fly-wheel in machinery.

7. The velocity in an ellipse at the greatest distance is half that with which a body would move in a parabola at the same distance. What is the eccentricity of the ellipse?

8. Shew that the stereographic projection of a great circle of a sphere is a circle; and find the radius.

9. P descends vertically, drawing Q over a fixed pulley. Find the pressure upon the axis of the pulley; and its value when P is indefinitely increased.

10. A pendulum is composed of two thin wires of equal length, at right angles to each other at the point of suspension, and vibrating in their own plane. Find the time of a small oscillation; and the angle at which they must be inclined to each other, so that the time of oscillation may be doubled.

11. What effects are produced by aberation in the apparent places of the moon and the planets.

12. Shew that the inclination of the moon's orbit is the greatest when the lines of the nodes is in syzygy; and the least, when the nodes are in quadrature and the moon in syzygy.

TUESDAY EVENING.—MR. CHEVALLIER.

1. Prove the rule for single position: state to what limitations it is subject: and apply it to find such a number that when divided by 3, 4, and 5, respectively, the sum of the quotients may be 94.

2. Shew that if an equation have two equal roots, and the terms are multiplied by the terms of an equal progression, the result will $= 0$.

3. Find a quantity, which when multiplied into $a^{\frac{1}{3}} - b^{\frac{1}{3}}$ renders the product rational.

4. Given the place of a planet at noon, on March 20th, in Libra 3° 4′ 30″; on the 21st in 8° 7′ 7″; on the 22d in 13° 19′ 30″; and on the 23d in 18° 41′ 44″; to find its place on the 22d at 6h.

5. Sum the series

$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \&c.$; to n terms and in infinitum.

$1^2 + 3^2 + 5^2 + \&c.$; to n terms.

$1 + 2x + 11x^2 + 43x^3 + \&c.$; in infinitum.

6. If Jupiter and Saturn are in conjunction with one another, and in opposition to the Sun, on a given day; and their periodic times are 12 years and 29.5 years respectively; when will they again be in the same position.

7. A weight W is raised upon a moveable pulley. The two extremities of the cord are wound in different directions about two cylinders, which have a common axis but different radii; and the power P descends unwinding a string from a wheel of a given radius upon the same axis. What is the force which accellerates P's descent, when the strings are parallel to each other?

8. In the scale of Reaumur's thermometer, the freezing point of water is 0, and the boiling point 80°. In the centigrade thermometer those points are 0 and 100° respectively. What will be the degree of heat marked by each, when Fahrenheit's thermometer stands at 59°?

9. Given the altitude of the Sun, and the breadth of the penumbra which the top of a mountain throws upon a horizontal plane, to find the height of the mountain.

10. Explain Atwood's machine; and mention some of the facts which it establishes in the theory of motion uniformly accellerated or retarded.

11. A plane of given form and area is supported in the air as a kite, the wind acting in a direction parallel to the horizon: the weight of the string and materials being (w), and the horizontal pressure of the wind equivalent to a weight (p) upon each square foot, required the angle made by the plane with the horizon; and the greatest weight which it can support.

12. When the force by which a watch-balance is actuated varies as the n^{th} power of the distance from the point of the spiral spring's quiescence, find the alteration in the daily rate, in consequence of a given change in the arc of vibration.

13. Explain the method of deducing the Sun's parallax from the transit of an inferior planet.

14. A seconds pendulum of given length, in the form of a thin rectangular bar, suspended at the middle of its extremity by an axis perpendicular to its plane, is carried to the top of a mountain. The length of the bar is diminished by a given quantity in consequence of a change of temperature, the breadth remaining the same, and it loses t'' in a day. What is the height of the mountain?

15. A cannon ball weighing 24lbs. strikes a wall with a velocity of 1700 feet. Find the weight of a beam, terminated by a hemisphere of the same diameter as that of the ball, which, when moved with a velocity of 10 feet, may penetrate to the same depth; and the weight of a similar beam, which may have the same effect in *shaking* the wall.

16. A body not affected by gravity falls down the axis of a thin cylindrical tube infinite in length, the particles of which attract with a force which varies inversely as the square of the distance. Find the velocity acquired in falling through a given space.

17. Four persons A, B, C, D, in order, cut a pack of cards, replacing them after each cut, on condition that the first who cuts a heart shall win. What are their respective probabilities of success?

18. A ship (P) begins to sail towards another (B) from a fixed point (C). At the same instant B begins to move in a direction perpendicular to P's first motion: P is always found in the line joining C and B; but can only accelerate her rate of sailing so as to retain the same distance from B as at first. What is the curve traced by P?

19. Integrate

$$\frac{dx}{\sqrt{x^2+y^2}} - \frac{xdy}{y\sqrt{x^2+y^2}} = 0;$$

$$dy + ydx = ax^3 dx;$$

$$dx\,dy - (x+a)\,d^2y - \frac{xdy^2}{b} = 0.$$

20. Two material points S and P, the mass of the first being twice that of the second, attract each other with a force which varies inversely as the square of the distance. When they have approached each other by half their original distance, P receives a new perpendicular impulse, which communicates to it a velocity equal to that which S has acquired. What curve is now described by each about the other?

21. Find the whole variation in the inclination of the Moon's orbit, as the Moon moves from quadrature to syzygy; the line of the nodes lying in quadrature.

Newt. Lib. iii. Prop. 34. Cor. 4.

22. Shew that if a body oscillates in a cycloid, in a medium the resistance of which is constant, the successive altitudes to which it will rise are in arithmetical progression.

23. The axis of Archimedes' screw is inclined at a given angle to the horizontal section of the water. Find the highest and lowest points of the spiral tube, its point of inflection; and the quantity of water which can be raised in a given time by a given power.

24. If a body is acted upon by any forces which would, if separately communicated, cause it to revolve about given axes with given angular velocities; find its axis of rotation, and its angular velocity: and apply the conclusion where there are three axes at right angles to each other.

THE SENATE HOUSE PROBLEMS,

Proposed by the Two Moderators, to the Candidates for Honors, during the Examination for the Degree of B. A.

MONDAY, JANUARY 13, 1823.

MONDAY MORNING.—MR. HIND.

First and Second Classes.

1. A given annuity which is to continue (3 n) years, is left equally between A and B: A receives the whole for (n) years, and B the whole for the remainder of the time; it is required to find the present worth of the annuity, and the rate of compound interest.

2. Find the values of θ which satisfy the equation

$$2 \sin^2 3\theta + \sin^2 6\theta = 2, \qquad \text{radius being} = 1.$$

3. Two bodies A and B descend from the same extremity of the vertical diameter of a circle, one down the diameter, and the other down the chord of 30°. Find the ratio of A to B, when their centre of gravity moves along the chord of 120°.

4. Given the sum of three quantities in geometrical progression, and the sum of their reciprocals, to find the quantities themselves.

5. The angles of a plane triangle are A, B, C; it is required to prove that the perimeter of the triangle : the diameter of its inscribed circle $:: \text{rad}^3 : \tan. \frac{A}{2} . \tan. \frac{B}{2} . \tan. \frac{C}{2}$.

6. In an equation of (n) dimensions, the second and third terms may be taken away by the same transformation when the square of the sum of the roots : the sum of their squares $:: n : 1$. Required a proof.

7. Three points being given in position; it is required to draw a straight line through one of them, so that the rectangle of the perpendiculars let fall upon it from the other two may be the greatest possible.

8. A ray of light issues from the extremity of the diameter of a semi-circle, and is reflected by the circumference: determine the point of incidence, so that after reflection, the ray may pass through a given point in the diameter produced.

9. A semicircular area is placed with its vertex upon a horizontal plane; find the time of one of its small oscillations.

10. A body projected in a direction parallel to the horizon, and acted upon by the force of gravity, describes a common cycloid; shew that the resistance of the medium, and the velocity at any point, vary respectively as $\sin.\frac{\theta}{2}$, and $\cos.\frac{\theta}{2}$, θ being the corresponding arc of the generating circle.

11. A chain of uniform density is suspended at its extremities by means of two tacks in the same horizontal line at a given distance from each other; find the length of the chain so that the stress upon either tack may be equal to the chain's weight.

12. Integrate $\frac{dx}{\sqrt[m]{1 - x^m}}$.

13. Find the horary variation of the inclination of the lunar orbit to the plane of the ecliptic. *Newton*, Prop. 34. Book 3.

14. Solve the functional equation

$$(\psi a^{x^2} . \psi \left(\frac{1 - a^x}{1 + a^x}\right) = b^2 a^x.$$

MONDAY AFTERNOON.—MR. HIND.

Fifth and Sixth Classes.

1. In what time will the amount of P£ at (r) per cent. simple interest be equal to (p) times the interest of the same sum, and what is the rate per cent. when the required time is (q) years?

2. Prove *geometrically* that

$$\text{ver. sin. } A : \sin. A :: \tan. \frac{A}{2} : \text{rad.}$$

3. Of the two quantities $a^6 + a^4 b^2 + a^2 b^4 + b^6$ and $(a^3 + b^3)^2$, shew which is the greater.

4. Divide the length of a given inclined plane into three parts so that the times of descent down them may be equal.

5. In any plane triangle, prove that the sines of the angles are inversely as the perpendiculars let fall from them upon the opposite sides.

6. Two bodies, A and B, whose elasticity is m, moving in opposite directions with velocities a and b impinge directly upon each other: find the distance between them when t' from the moment of impact have elapsed.

7. The roots of the equation $x^4 - 10 x^3 + 35 x^2 - 50 x + 24 = 0$ are of the form $a + 1$, $a - 1$, $b + 1$, $b - 1$: find them.

8. Where must a ray of light parallel to the axis of a concave spherical reflector be incident, that after reflection it may divide the radius in the ratio of $\sqrt{3} - 1 : 1$?

9. The sum of a series of quantities in geometrical progression wanting the first term, is equal to the sum of all the terms except the last, multiplied by the common ratio. Required a proof.

10. Find the angular distance of a body from the vertex of a common parabola where the velocity is equal to half the greatest velocity.

11. Determine *geometrically* that point in the hypothenuse of a given right-angled triangle whose base is parallel to the horizon, from which the time of a body's descent to the right angle may be the least possible.

12. Shew that Cardan's rule for the solution of a cubic equation is applicable when all the roots are possible and two of them equal, and by means of it, find the roots of the equation $x^3 + 6x^2 - 32 = 0$.

13. Find the differential of an arc the tan. of whose half is x, and integrate

$$\frac{xdx}{\sqrt{a^4x^{-1} + 1}}, \quad \frac{x^{\frac{1}{2}}\,dx}{a^3 - 5x^3} \text{ and } \frac{d\theta}{(\tan.\theta)^2}.$$

14. Sum the series:

$$\frac{1}{m} + \frac{2}{m(m+a)} + \frac{2(a+2)}{m(m+a)(m+2a)} + \text{\&c. } in\ inf. \text{ when}$$

m is greater than 2.

15. Divide the arc of a cycloid into two parts so that the times of a body's oscillating through them may be in the ratio of $1 : 5$.

16. Two places in the same latitude whose difference of longitude is l, are distant a miles from each other: find their latitude.

17. The difference of the forces on P and p (*Newt.* Sect. 9.) $\propto \frac{1}{CP^3}$. Required a proof.

18. A given paraboloid filled with fluid is placed with its vertex downwards and its axis vertical; determine the time of emptying one half of its content through a given orifice in the vertex.

MONDAY AFTERNOON.—MR. HIGMAN.

Third and Fourth Classes.

1. Transform the equation $x^3 - px^2 + qx - r = 0$ whose roots are a, b, c, into one whose roots are

$$\left(\frac{a}{b}+\frac{b}{a}\right),\quad \left(\frac{a}{c}+\frac{c}{a}\right),\quad \left(\frac{b}{c}+\frac{c}{b}\right).$$

2. Find the number of different triangles into which a polygon of n sides may be divided by lines joining the angular points.

3. If the velocities of two balls A and B, whose elasticity is e, be a and b before impact, and u and v after; also if α and β be the velocities lost and gained, then will

$$Aa^2 + Bb^2 = Au^2 + Bv^2 + \frac{1-e}{1+e}\left\{A\alpha^2 + B\beta^2\right\}.$$

4. Find the attraction of a rectangle on a corpuscle situated in one of its sides, produced, in a direction perpendicular to the other side; the force tending to each particle of the rectangle varying inversely as the square of the distance.

5. If there be taken the evolute of a logarithmic spiral, the evolute of that evolute, and so on ad infinitum, find the sum of the arcs of all the successive evolutes.

6. Sum the series:

(1) $\dfrac{1}{3.5.7} + \dfrac{2}{4.6.8} + \dfrac{3}{5.7.9} +$ &c. to n terms.

(2) $\dfrac{1}{1.4.7} + \dfrac{1}{2.6.9} + \dfrac{1}{3.8.11} +$ &c. *ad inf.*

(3) $1 + n\cos\theta + \dfrac{n.(n-1)}{1.2}\cos 2\theta + \dfrac{n.(n-1)(n-2)}{1.2.3}\cos 3\theta +$ &c.

7. If a ray of light refracted into a sphere, emerge from it after any given number of reflections; determine the distance of the incident ray from the axis, when the arc of the circle intercepted between the axis and the point of emergence is a minimum.

8. Find the centre of resistance of a semicircle revolving round its diameter in a medium, whose resistance $\propto$ (velocity)^2.

9. If z = true zenith distance of a planet, p = its parallax at that distance, and P = horizontal parallax, then

$$\tan\left(\frac{z}{2}+p\right) = \tan\frac{z}{2}\tan^2\left(45^\circ + \frac{P}{2}\right).$$

10. If the force $\propto \frac{1}{\text{dist.}^4}$, and a body be projected at an apse with the velocity acquired in descending from an infinite distance to that point, construct the curve described, and find the time of descent to the centre.

11. A body descends down the convex side of a logarithmic curve placed with its asymptote parallel to the horizon, find where it leaves the curve.

12. If S be the momentum of inertia of a system, in respect of an axis which passes through its centre of gravity, S' the momentum of inertia of the same system, in respect of an axis parallel to the first and distant from it by the space k, and M the mass of the system, then

$$S' = S + Mk^2.$$

MONDAY EVENING.—MR. HIGMAN.

1. Prove that $(Aa + Bb + Cc + \dots\dots\dots\dots)^2 = (A + B + C + \dots\dots\dots)(Aa^2 + Bb^2 + Cc^2 + \dots\dots\dots) - AB(a-b)^2 - AC(a-c)^2 - BC(b-c)^2 - \dots\dots$

2. If through any point O within a triangle, three straight lines be drawn from the angles A, B, C meeting the opposite sides in a, b, c, then will $\frac{Oa}{Aa} + \frac{Ob}{Bb} + \frac{Oc}{Cc} = 1$.

3. If from any point in a rectangular hyperbola whose axis is vertical, two lines be drawn to the extremities of the axis major, the times of descent down them will be equal.

4. Prove that the periodic time of a body revolving in an ellipse round the focus $= \frac{2\pi a^{\frac{3}{2}}}{\sqrt{m}}$, where a = semi-axis, and m = force at a distance 1; and apply this result to deduce the actual time of falling down AC in Prop. 32. Sect. 7.

5. The sine of half the angle that measures the duration of the shortest twilight $= \frac{\sin.\ 9^\circ}{\cos.\ \text{lat.}}$.

6. Find the equation of the caustic of the parabola, when the rays are incident perpendicular to the axis. Trace the caustic, and find the angles at which it cuts the axis, its maximum ordinate, its area, and its length.

7. If a be a root of Des Cartes's reducing cubic, then will the four roots of the equation $x^4 + qx^2 + rx + s = 0$ be

$$-\frac{\sqrt{a}}{2} \pm \sqrt{\left(-\frac{q}{2}-\frac{a}{4}+\frac{r}{2\sqrt{a}}\right)};$$

$$\text{and } +\frac{\sqrt{a}}{2} \pm \sqrt{\left(-\frac{q}{2}-\frac{a}{4}-\frac{r}{2\sqrt{a}}\right)}.$$

8. Two cylindrical vessels of given dimensions, containing given quantities of water, are made to communicate with each other by means of a small orifice at their bases; find the time elapsed before the water stands at the same altitude in both vessels.

9. If a comet move in a hyperbola whose semi-axis $= a$, and eccentricity $= a\,e$, its place at the end of t'' after leaving the perihelion may be determined from the equations

$$(1)\quad t = \frac{Pa^{\frac{3}{2}}}{2\pi}\left\{e \tan\theta - \log.\tan.\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}$$

$$(2)\quad \tan\frac{v}{2} = \sqrt{\left(\frac{e+1}{e-1}\right)}.\tan\frac{\theta}{2}$$

where $v =$ true anomaly reckoned from the perihelion, and $P =$ Earth's periodic time, her mean distance being 1.

10 Shew that in consequence of the mean disturbing force of the Sun in the direction of the radius vector, the distance of the Moon from the Earth is increased by a 358th part, and her angular velocity diminished by a 179th part.

11. Integrate

$$(1)\quad \frac{x^4dx}{\sqrt{(2ax-x^2)}}.$$

$$(2)\quad \frac{dx}{\sqrt{(1-x^2)}}.\log x \text{ between } x = 0, \text{ and } x = 1.$$

$$(3)\quad aydy - by^2dx + cxdx = 0.$$

12. If AB be an elliptic quadrant, CP, CD semi-conjugate diameters, PF perpendicular to CD, K the point, in which CD produced meets the circumscribing circle, and KMQ a line perpendicular to the major axis meeting the ellipse in Q, then will arc BP — arc $AQ = CF$.

13. The square of the area of any one of the faces of a triangular pyramid is equal to the sum of the squares of the other three, minus twice the rectangle contained by the product of every two and the cosine of their inclination.

14. Sum the following series:

$$(1)\quad \frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots ad\ inf. \text{ when } n \text{ is infinite.}$$

$$(2)\quad \frac{x^2}{1.2}+\frac{x^5}{1.2.3.4.5}+\frac{x^8}{1.2.3.4.5.6.7.8}+\ldots\ldots ad\ inf.$$

(3) $\frac{1}{2.1^2} + \frac{1}{2^2.2^2} + \frac{1}{2^3.3^2} + \ldots\ldots ad\ inf.$

16. Amongst all the axes passing through the centre of gravity of a triangle in its own plane, find that for which the momentum of inertia is a maximum or a minimum.

16. If a ray of light *QACS* be refracted through a prism *IKL* in a plane perpendicular to its axis, and if the vertical angle $KIL = \alpha$, $QAK = \theta$, $ACL = \phi$, and the whole deviation of the ray $= \delta$, then will

$$\tan\left(\phi - \frac{\alpha}{2}\right) = \frac{\tan\left(\theta + \frac{\delta + \alpha}{2}\right)\tan\frac{\delta + \alpha}{2}}{\tan\frac{\alpha}{2}}.$$

17. Find the different positions of equilibrium of a parabola floating in a fluid with its vertex immersed.

18. If the longitudes of a planet in three different points of its orbit be denoted by a, b, c, and its latitudes at those points by α, β, γ; then will

$$\tan\beta . \sin(c - a) = \tan\alpha . \sin(c - b) + \tan\gamma . \sin(c - a).$$

19. If x be a very large number, and e the base of Napier's system, then will

$$1.2.3.4 \ldots\ldots\ldots x = \left(\frac{x}{e}\right)^x \sqrt{(2\pi x)} \text{ nearly.}$$

20. *A* and *B* are at play together, and the latter having lost p stakes, is determined to play till he has won them again; find the probability that this never takes place, supposing the play to continue without limitation; his number of chances b for winning any assigned game being less than a that for the contrary.

21. If the force $\propto \frac{1}{\text{dist.}}$, then the time of descent to the centre $= a\sqrt{\frac{\pi}{2m}}$, where $a =$ whole distance, and $m =$ force at a distance 1 from the centre.

22. Required the curve, which within its own arc, its evolute and radius of curvature shall contain the least area.

23. When pulses are propagated through an elastic medium, the several parts going and returning by a very short reciprocal motion; they are accelerated and retarded according to the law of a pendulum oscillating in a cycloid. *Newton*, Lib. 2. Prop. 47.

24. If two bodies be projected at equal angles of elevation, and with equal velocities, one in a non-resisting medium, and the other in a medium whose resistance corresponding to the velocity $v = nv^2$; also, if s' be the arc of the parabola, and s the arc of the other curve described by the bodies when they are moving in directions making equal angles with the horizon, then will $e^{2ns} = 1 + 2ns'$.

TUESDAY MORNING.—MR. HIGMAN.

First and Second Classes.

1. Find the greatest term of the expansion of $(a+b)^n$.

2. The product of all the lines, that can be drawn from one of the angles of a regular polygon of n sides, inscribed in a circle whose radius is a, to all the other angular points $= na^{n-1}$.

3. If the sun's longitude $= c$, and the obliquity of the ecliptic $= \varphi$, then will the equation of time arising from the obliquity of the ecliptic

$$= \tan^2\frac{\varphi}{2}\sin 2c - \frac{1}{2}\tan^4\frac{\varphi}{2}\sin 4c + \frac{1}{3}\tan^6\frac{\varphi}{2}\sin 6c \text{—\&c.}$$

ad inf. converted into time.

4. A beam of given length and weight is placed with one end on a vertical, and the other on a horizontal plane; find the force necessary to keep it at rest, and the pressures on the two planes.

5. Find the perihelion distance of the Comet, moving in the plane of the ecliptic, that stays the longest time within the Earth's Orbit.

6. If P be any rational function of x, in which the highest power of x is less than n; and if A, B, C, D,......K be the

values of P, $\frac{dP}{dx}$, $\frac{d^2P}{dx^2}$, $\frac{d^3P}{dx^3}$ ...... $\frac{d^{n-1}P}{dx^{n-1}}$ when $x = a$,

$$\text{then } \frac{P}{(x-a)^n} = \frac{A}{(x-a)^n} + \frac{B}{(x-a)^{n-1}} + \frac{C}{1.2.(x-a)^{n-2}}$$

$$+ \frac{D}{1.2.3.(x-a)^{n-3}} + \ldots\ldots + \frac{K}{1.2\ldots(n-1)(x-a)}.$$

7. Explain fully the motion of the apsides, and the variation of the eccentricity of P's Orbit. Cor^s. 8 and 9, Prop. 66.

8. A sphere of given weight and dimensions, descends in a

fluid; find the pressure on the bottom of the vessel that contains it, arising from the action of the sphere on the fluid.

9. If $f, f', f'' \ldots\ldots$ be the focal lengths of any number of contiguous lenses, the thickness of each being very small, and F be the focal length of the compound lens, then

$$\frac{1}{F} = \frac{1}{f} + \frac{1}{f'} + \frac{1}{f''} + \ldots\ldots$$

10. If in the spherical triangle ABC, c and C be constant, and the other angles and sides variable, then will AC and BC be the corresponding values of θ and φ in the differential equation

$$\frac{d\,\varphi}{\sqrt{(1 - e^2 \sin^2 \phi)}} + \frac{d\,\theta}{\sqrt{(1 - e^2 \sin^2 \theta)}} = 0 \text{ where } e = \frac{\sin C}{\sin c}.$$

11. Putting A and B for the sectors CAP, CAp of a rectangular hyperbola, whose semi-axis $CA = 1$, and calling the abscissas CN, Cn, the *cosines*, and the ordinates PN, pn the *sines* of A and B, then will

$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B.$$

$$\text{and } \cos (A \pm B) = \cos A \cos B \pm \sin A \sin B.$$

12. If a body describing a spiral in a medium whose density $\propto \frac{1}{\text{dist.}}$, cut the radius vector at B in the same angle as at A, and with a velocity, which is to that at $A :: \sqrt{SA} : \sqrt{SB}$; then will the distances at which the body cuts the radius vector, and the times of successive revolutions be in geometric progression. *Newton*, Cor. 7. Prop. 15. Lib. 2.

TUESDAY AFTERNOON.—MR. HIGMAN.

Fifth and Sixth Classes.

1. In every geometrical progression consisting of an odd number of terms, the sum of the squares of the terms is equal to the sum of all the terms multiplied by the excess of the odd terms above the even.

2. Take away the third term of the equation

$$x^3 - 6x^2 + 9x - 20 = 0.$$

3. On a lever of uniform density, every inch weighing w oz. a weight of W oz. is suspended at a given distance from the fulcrum which is placed at one extremity. What must be the length of the lever, so that the whole may be supported by the

least possible power acting in an opposite direction at the other extremity?

4. The initial force accelerating a body down a circular arc is to the force accelerating it down the chord as a to 1 ultimately.

5. If S = surface of a portion of the earth $ABCD$, AB being an arc of the equator, and AC, BD two arcs of circles of latitude; also if $AB = c$, $AC = a$, $BD = b$, then will

$$\tan \frac{S}{2} = \frac{\sin \frac{a+b}{2}}{\cos \frac{a-b}{2}} \cdot \tan \frac{c}{2}.$$

6. Determine that point of the cubical parabola where the curvature is the greatest.

7. If in an ellipse there be taken three abscissas in arithmetic progression, the radius vectors drawn from the focus to the extremities of the ordinates at those points will also be in arithmetic progression.

8. Find the sum of $1 + \frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{4} + \&c.$

α, β, γ, &c. being the coefficients of the expansion of $(a+b)^n$.

9. Compare the resistances on a globe and the circumscribing cylinder moving in a fluid with equal velocities in the direction of the axis of the cylinder.

10. A given weight is suspended by a string and immersed in a river whose inclination to the horizon is known. Having observed the angle which the string makes with a vertical line, it is required to determine the velocity of the stream.

11. Sum the series:

(1) $1 + 3x + 5x^2 + 7x^3 + \&c.$ *ad inf.*

(2) $\frac{1}{1.2.4} + \frac{1}{2.3.5} + \frac{1}{3.4.6} + \&c.$ to n terms by increments.

12. Find the integrals of $\frac{x\,dx}{\sqrt{(a^4+x^4)}}$; and of $v^2 dx$ where $v = \log. x$.

13. If the focus of incident rays be placed any where in the axis of an elliptic reflector, find the geometrical focus of reflected rays.

14. If A, A' denote the intensities of two lights, and n, n' the number of laminæ of any substance, through which when the lights are viewed they appear equally obscure; also the light lost in passing through each lamina being supposed to be $\frac{1}{m^{th}}$ of that which entered it; then will

$$A : A' :: \left(\frac{m-1}{m}\right)^{n'} : \left(\frac{m-1}{m}\right)^{n}.$$

15. In consequence of the aberration of light, every star appears to describe an ellipse in the heavens, of which the true place of the star is the centre. Prove this, and find the axes of the ellipse.

16. If the force vary inversely as the square of the distance; and m = force at a distance 1, and l = latus rectum of the conic section, then the area described in 1″ $= \sqrt{\frac{lm}{8}}$.

17. Trace the curve whose equation is $y = \sec x$, draw a tangent to it, and find its area.

18. Find the value of Q in *Newton*, Prop. 41.

TUESDAY AFTERNOON.—MR. HIND.

Third and Fourth Classes.

1. Extract the square root of $\cos. 4A \pm \sqrt{-1}. \sin. 4A$.

2. If tangents drawn to any two points of an ellipse meet each other; shew that their lengths are inversely as the sines of the angles which they make with the lines drawn to either focus.

3. There are four numbers, the first three of which are in arithmetical, and the last three in harmonical progression; it is required to prove that the first has to the second the same ratio which the third has to the fourth.

4. If 2 S be equal to the sum of the sides of a plane triangle, A, B the angles opposite to the sides a, b, respectively, then is

$$\sin.^2 \frac{A}{2} : \sin.^2 \frac{B}{2} :: \frac{a}{b} : \frac{S-a}{S-b}.$$ Required a proof.

5. The beam of a false balance being of uniform density and thickness, it is required to shew that the lengths of the arms are respectively proportional to the differences between the true and apparent weights.

6. In the equation $x^3 - px^2 + qx - r = 0$, prove that the sum of the products of the roots and their reciprocals taken three and three together : product of the roots :: $1 + p^2 + q^2 + r^2 : r^2$.

7. In any right-angled plane triangle, prove that twice the side of the inscribed square is an harmonical mean between the sides containing the right angle.

8. Shew that $\log. \sec. \theta = \frac{\tan^2\theta}{2} - \frac{\tan^4\theta}{4} + \frac{\tan^6\theta}{6} - \&c.$

and thence deduce the sum of the series

$$1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\text{\&c. } \textit{in inf.}$$

9. Determine the angular distance of a body from the vertex of an ellipse whose eccentricity $= \frac{1}{2}$, at which the velocity : greatest velocity :: 1 : $\sqrt{3}$.

10. A cylinder of given length is just immersed vertically in two fluids whose specific gravities are as 1 : 2, and the pressures of the fluids upon the convex surface are as 2 : 3; find the length of the cylinder immersed in each fluid.

11. There are three plane reflectors, two of which are at right angles to each other, and a ray of light is incident upon the third, and reflected successively by each of them; it is required to shew that the angle between the first incident and last reflected rays is equal to twice the angle of incidence upon the first surface.

12. Determine the length of a straight line drawn through the centre of gravity of a given isosceles triangle, making a given angle with the base, and terminated by the sides.

13. AN and NP are the abscissa and ordinate of a cissoid, the diameter of whose generating circle is AB, AP is joined and NQ always taken equal to it. Prove that the whole area of the curve traced out by $Q : AB^2 :: 4 : 3$.

14. Given the latitude of a place, find the time of the year when a given star rises at a given hour.

TUESDAY EVENING.—MR. HIND.

1. Extract the fourth root of $m^2.(m^2-3n^2)+n^2.(n^2-3m^2)$ $+4.(m-n).(m+n).mn\sqrt{-1}$.

2. Prove the following Formulæ:

$$\tan^2\frac{A}{2}=\frac{2\sin.A-\sin.2A}{2\sin.A+\sin.2A} \text{ and}$$

$$\tan nA=\frac{\sin.A+\sin.3A+\text{\&c.}\ \ldots\ (n)}{\cos.A+\cos.3A+\text{\&c.}\ \ldots} \text{ and from the}$$

former, deduce the tan of 15°.

3. Find three fractions having different denominators, whose sum shall be $1\frac{23}{30}$.

4. If three roots of an equation be nearly equal to one another, and much less than all the others, shew that an approximation may be made to them by the solution of a cubic.

5. Every prism having a triangular base, may be divided into three pyramids, that have triangular bases, and are equal to one another. *Euclid*, Prop. 7, Book 12.

6. Explain the construction, adjustment, and use of the zenith sector.

7. Find the greatest triangle that can be inscribed in a given circle.

8. Two given spheres are situated at the extremities of the diameter of a given circle: determine the position of an eye in the circumference, where the surface seen is the greatest possible.

9. A triangle is described about an ellipse, prove that the products of the alternate segments of the sides, made by the points of contact, are equal.

10. Prove that $\sin. A + \sin. 2A + \sin. 3A + \&c.$ is a recurring series; find the scale of relation, and by means of it the sum of (n) terms.

11. P descending vertically, draws Q up an inclined plane by means of a string passing over a pulley above it: find the velocity acquired by P in describing a given space.

12. Shew that the velocity acquired by a body in falling from infinity to the earth's centre, is to the velocity of a secondary at the earth's surface :: $\sqrt{3} : 1$.

13. Find where an inferior planet will appear stationary, supposing the force of gravity to vary inversely as the cube of the distance, and the orbits of the earth and planet to be circular.

14. In Cotes's first spiral, it is required to shew that the successive distances at which the curve cuts the apsidal line may be represented by a series of the form

$$\frac{c}{c^2+1},\ \frac{c^2}{c^4+1},\ \frac{c^3}{c^6+1},\ \ldots\ \frac{c^n}{c^{2n}+1}.$$

15. Determine the pressure upon the axis of any vibrating body in any given position.

16. Investigate the nature of the surface, such that if two lights of given intensities be placed in two given points, every point in it may be proportionally illuminated by each.

17. Integrate

$$\frac{dx}{(a+b\,.\sin.x)^2},\ \text{and}\ \frac{x^m dx}{(x+\sqrt{1+x^2})^2},$$

and find the relation of x and y in the equation

$$\frac{d^2y}{dx^2} + 2x\,.\frac{dy}{dx} + (y-m)\,.x^2 = 0.$$

18. Find the equation to a curve surface in which the normals from every point meet a given plane in a given straight line.

19. A given cylinder filled with fluid revolves round

its vertical axis with a given uniform velocity; find what quantity of fluid will escape if a small orifice be made in the centre of its base.

20. Determine the conditions of equilibrium of a material point situated in a canal of indefinitely small dimensions and acted upon by any number of forces.

21. If the force of gravity be uniform and act perpendicularly to the plane of the horizon; it is required to determine the motion of a projectile in a medium whose resistance is proportional to the velocity. *Newton*, Prop. 4. Book 2.

22. If a prism be turned round its axis which is perpendicular to the incident rays of the sun, till the spectrum neither ascends nor descends, shew that the refractions at the points of incidence and emergence are equal.

23. To determine the mean motion of the moon's nodes. *Newton*, Prop. 32. Book 3.

24. If three planes be at right angles to each other, find the equation to the surface to which if a tangent plane be drawn, the content of the solid formed by this and the other three planes may be constant.

THE SENATE HOUSE PROBLEMS,

Proposed by the Two Moderators, to the Candidates for Honors, during Examination for the Degree of B. A.

MONDAY, JANUARY 14, 1804.

MONDAY MORNING.—MR. HUGHES.

First and Second Classes.

1. THE area of a floor is 403 ft. 7 in. and length 27 ft. 10 in. find the width by duodecimals.

2. Find the value of A in the equation

$$\tan A + 2\cot 2A = \sin A\left(1 + \tan A . \tan\frac{A}{2}\right).$$

3 Given the direction and velocity of projection, find the direction and velocity of a projectile at the end of t'', and its height above the horizontal plane passing through the point of projection.

4. A ray of light proceeding from a moveable point is

reflected by a fixed plane mirror to a given point; find the locus of the first point when the path of the ray is of given length.

5. Two equal cylinders balance at the extremities of equal arms of a straight lever, when immersed in fluids, whose densities vary as the depth. The surface of one coincides with that of the fluid, and the depth of the upper surface of the other is equal to n times its altitude. Compare the densities of the fluids at equal depths.

6. Given the precession in right ascension of a star, find the corresponding change in the angle of position.

7. If two given spheres touch each other internally, and the interior be taken away, find a point within the remainder such that a particle being placed there shall remain at rest. $\left(\text{Attraction of each particle} \propto \frac{1}{\text{dist.}^2}\right)$.

8. Divide a given paraboloid into two parts in the ratio of $m : n$, by a plane inclined at a given angle to the axis.

9. Integrate $ydydx = (x + a)\,dy^2 + adx^2$.

10. Find the sum of the m^{th} powers of the reciprocals of the roots of an equation in terms of the inferior powers.

11. A and B are two material points connected by an inflexible rod without weight; B moves on the horizontal plane CB, A descends along the inclined plane AC, the motion of the rod being in a vertical plane. Compare the velocity which A has at the point C with the velocity it would have were it to descend freely down AC.

12. The equation of the curve made by the intersection of a plane with a surface of the second degree is a quadratic equation.

MONDAY AFTERNOON.—MR. HUGHES.

Fifth and Sixth Classes.

1. If a be greater than b, $a^n - b^n$ is greater than $nb^{n-1}(a-b)$, and less than $na^{n-1}(a-b)$.

2. If two equal parabolas have a common axis, a straight line touching the interior and bounded by the exterior will be bisected in the point of contact.

3. Force varies as $\frac{bA^m + cA^n}{A^3}$, find the angle between the apsides.

4. In a triangle whose sides are a, b, c, and A, B, C, the angles opposite, having given B, a, and the area of the triangle, find the remaining sides and angles.

5. A body is projected upwards from the lower extremity of a given vertical line with a given velocity; after what time must another be projected downwards from the upper extremity with the same velocity, so as to meet the former in the middle point of the line?

6. A cylinder placed with its axis vertical in a fluid rests with an m^{th} part immersed; when placed in another fluid it rests with an n^{th} part immersed; to what depth would it sink in a mixture composed of equal quantities of these fluids?

7. If the particles of two spheres attract with forces varying as the distance, the force with which the spheres attract each other is as the distance between their centres.

8. Define the centre of a lens, and prove it to be a fixed point.

9. Trace the curve whose equation is $(y-c)^2=(x-a)^4.(x-b)$, and determine the position of its tangent at the point where $x=a$.

10. Integrate $\dfrac{dx}{(a^2+x^2)^{\frac{3}{2}}}$, $\dfrac{dx}{a+bx+cx^2}$, $a^x x^3 dx$.

11. Inscribe a semicircle in a quadrant.

12. If a pole rests with one end on the ground against a wall, and the other attached to a string fixed in the wall, find the tension of the string.

13. The altitude of the mercury in a barometer placed in a given cylindrical diving bell is observed at the beginning and end of a descent; find the depth descended.

14. A body which is half elastic descends along the arc of an inverted cycloid, and is reflected by the axis which is vertical; find the space described in the time of n oscillations of a pendulum vibrating in the same cycloid.

15. Find the height of a lunar mountain.

16. To make a body oscillate in a given hypocycloid.

MONDAY AFTERNOON.—MR. HIGMAN.

Third and Fourth Classes.

1. If a be the approximate square root of any number n, and $n-a^2=\pm b$, then will $\sqrt{n}=a\pm\dfrac{2ab}{4a^2\pm b}$ very nearly.

2. Prove the properties of the *complemental* triangle; and from these properties, and the expressions for the cosine of an angle in terms of the sides, and for the cosine of a side in terms of the angles, deduce Napier's rules for the solution of right-angled spherical triangles.

3. A and B are two balls whose elasticity $= e$, and A strikes B at rest; prove that if B be infinitely greater than A, A's momentum before impact : momentum communicated to B :: 1 : $1 + e$.

4. If two lines revolving in the same plane round the points S and H, intersect one another in the point P in such a manner that (1) $SP^2 + HP^2 =$ constant quantity. (2) that SP be to HP in the given ratio of n to 1; prove that in each case the locus of the point P is a circle.

5. In the second case of the last problem, suppose rays diverging from the point S, to be incident on a refracting surface, generated by the revolution of the concave circumference of the circle round its axis; then if sin. incidence : sin. refraction :: n : 1, and n be > 1, all rays diverging from S will after refraction diverge *accurately* from H.

6. Find the time of the Sun's passing the vertical wire of a telescope.

7. The lower end of a barometer is immersed in a bason of mercury, and the upper is suspended from the extremity of the beam of a common balance; what weight suspended at the other extremity of the beam, will keep it in equilibrium?

8. If α, β, γ be the angles, which the resultant of three forces, acting at right angles to one another, makes with each of them respectively, then will

$$\cos. 2\alpha + \cos. 2\beta + \cos. 2\gamma = -1.$$

9. In Newton's second Lemma, if the ordinate vary as the m^{th} power of the abscissa, find the limit of the sum of the areas of the circumscribing parallelograms.

10. Explain the nature of centrifugal force, and shew that in all curves it $= \frac{h^2}{\text{dist.}^3}$ where $h =$ twice the area described in one second of time.

11. Let Q be a point in a semicircle whose diameter is AB, join AQ, and in AQ produced, take AP a mean proportional between AB, and AQ; find the equation of the curve, which is the locus of the point P; its area; the content of the solid generated by its revolution; and the radius of curvature at its vertex.

12. Prove that the curve in the last problem possesses these properties.

(1) That a particle placed any where in its perimeter as at P, will attract a particle at A, in the direction AB, with a constant force; supposing the force of attraction to vary as $\frac{1}{\text{dist.}^2}$.

2. That if it be made the revolving orbit in the ninth Section of Newton, and $G = \frac{F}{2}$, the orbit traced out in fixed space will be the lemniscate of Bernoulli.

MONDAY EVENING.—MR. HIGMAN.

1. If a right cone, whose vertical angle is 90°, be cut by a plane which is parallel to one touching the slant side, prove that the latus rectum of the section is equal to twice its distance from the vertex.

2. Let a be the distance from the centre of force, from which a body must fall externally to acquire the velocity in a circle whose radius $= r$ when the force $\propto \frac{1}{\text{dist.}}$; and let b be the distance to which it must fall internally to acquire the same velocity; then will a, r, b be in geometric progession.

3. Describe the conic section whose equation is $y^2 - 2xy + x^2 - 8x + 16 = 0$.

4. In what latitude is the angle between the 3 and 4 o'Clock hour lines, on a horizontal dial, a maximum?

5. If when the mercury in a true barometer stands at an altitude a, the mercury in an imperfect one of given length stand at the altitude b; what will be the height of the mercury in the true barometer, when it stands at an altitude c in the imperfect one?

6. Deduce the equation to the catenary $2y = a(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$, where $a =$ tension at the lowest point; and prove that its radius of curvature is equal to its normal.

7. If at the distance a from the centre of force, a body be projected at an angle of 45° with a velocity, which is to that in a circle at the same distance as $\sqrt{2}$ to $\sqrt{3}$, and the force $= \frac{2a^2n}{\text{dist.}^5} + \frac{n}{\text{dist.}^3}$, required the curve described.

8. Prove that the locus of the intersection of the lines SP and CQ, which cut off the true and eccentric anomalies is an ellipse.

9. If the sides of a triangle ABC be bisected in the points D, E, F; then the centre of the circle inscribed in the triangle DEF, is the centre of gravity of the perimeter of the triangle ABC.

10. The sum of the squares of the coefficients of the expansion of $a^x = \frac{1}{\pi}\int e^{2k\cos x}dx$ taken between $x = 0$, and $x = \pi$; k being $= \log. a$.

11. If the radius vector and perpendicular on the tangent in any curve described by a revolving body, be denoted by r and p; and in the curve of a star's apparent aberration, as seen from this body, by r' and p', then will $r : p :: r' : p'$.

12. If a prism be laid on its base in the open air, and the eye be placed in a proper position; the base will appear to be divided into two parts, the one much brighter than the other, and separated from one another by a coloured bow, concave towards the eye. Give Newton's explanation of the cause of this phaenomenon.

13. Let $\frac{A}{A}, \frac{B'}{B}, \frac{C'}{C} \ldots\ldots$ be the 1st, 2d, 3d, ... approximations to the value of a fraction, when the continued fraction terminates; then will the fraction $= \frac{A'}{A} + \frac{1}{AB} - \frac{1}{BC} + \frac{1}{CD} - \frac{1}{DE} + \ldots\ldots$

14. Sum the series:

$$\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \text{\&c. to } n \text{ terms;}$$

$$\frac{1}{1.2} - \frac{3}{4.5} + \frac{5}{7.8} - \text{\&c. to infinity;}$$

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \text{\&c. to infinity.}$$

15. Find the integrals of

$$\frac{(x^3 + x^2 + 2)dx}{x^5 - 2x^3 + x};\ a^x \sin^3 x dx;$$

and the relation between x and y in the equation

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0.$$

16. Prove that the area of a regular polygon *inscribed* in a circle, is a *geometric* mean between the areas of an inscribed, and of a circumscribed regular polygon of half the number of sides; and that the area of a regular polygon *circumscribed* about a circle, is an *harmonic* mean between the areas of an inscribed regular one of the same number of sides, and of a circumscribed regular one of half that number.

17. In what latitude will a ring surrounding the earth, and parallel to the equator, attract a particle placed in the earth's centre with the greatest force possible?

18. A body acted on by gravity moves on the surface of a cone, whose axis is vertical; find the law of force tending to the vertex, by which the projection of its path on a horizontal plane passing through the vertex may be described, and hence deduce the angle between the apsides when the orbit is nearly circular.

19. If the radius of the anterior surface of a concave glass speculum of inconsiderable thickness $(c) = a$; then if the radius of the second surface $= a + \frac{13c}{9}$, the image of a distant object formed by reflection at the first surface will coincide with the image formed by reflection at the second surface, and by refraction at the first.

20. A rod of given length and weight, and of uniform density rests with one end in water, and the other on the edge of the vessel which contains it, find the magnitude of the part immersed, and the pressure on the side of the vessel.

21. If $\alpha, \beta, \gamma, \dots$ be the roots of the equation $X = 0$, and $A, B, C \dots$ the results when these roots are substituted for x in the limiting one; then will $y = \frac{X}{A(x-\alpha)}a + \frac{X}{B(x-\beta)}b + \frac{X}{C(x-\gamma)}c + \dots$ be the equation of the parabolic curve which passes through points of which $\alpha, \beta, \gamma \dots$ are the abscissas, and $a, b, c \dots$ the corresponding ordinates.

22. If a circle whose diameter is equal to the whole tide in any given latitude be placed vertically, and so as to have the lower extremity of its diameter coincident with the level of low water, prove that the tide will rise or fall over equal arcs in equal times.

23. From a bag containing two balls, a white ball is drawn twice following, the ball having been replaced in the bag after the first drawing; required the probability that both balls are white, and that the ball being a second time replaced, a white ball will be drawn at the third trial.

24. If a body, whose elasticity $= m$, descends from rest through an altitude h, by the uniform force of gravity, in a medium whose resistance to a velocity $v = gkv^2$, and impinging on a horizontal plane, rise and fall alternately; prove that the whole space described by the body

$$= \frac{1}{gk^2} \log. \left\{ \frac{m^2 e^{-2gk^2 h} - 1}{m^2 - 1} \right\}.$$

TUESDAY MORNING.—MR. HIGMAN.

First and Second Classes.

1. PROVE that $\log. x = n(x^{\frac{1}{n}} - 1)$ nearly, when n is very great.

2. If two lines SP, HP revolve about the points S, H, so that $SP \times HP = CS^2$, (C being the middle point of SH) then the locus of the point P is the lemniscate of Bernoulli.

3. A and B are two given points in a horizontal line, to which are fastened two strings of given length: the string BC passes through a ring at C, and is fastened to a given weight W; find the position in which the weight will rest.

4. A wine glass in the form of a paraboloid, is partly filled with water, and then inverted on a table; given the weight of the glass, required the greatest quantity of water that can be contained without running out.

5. If $\delta =$ distance of a plane from the origin of the three rectangular co-ordinates, AX, AY, AZ; and if α, β, γ be the angles which δ makes with these three respectively, then will the equation to the plane be

$$x \cos \alpha + y \cos \beta + z \cos \gamma = \delta.$$

6. Supposing a Comet of the same magnitude and density as the Moon, on its nearest approach to the Earth, to be distant thirty radii from the Earth's centre; required the magnitude of the tide raised by the Comet.

7. Describe the experiment by which Newton shewed that the more refrangible rays are the more reflexible.

8. If a body describe the spiral of Archimedes, the force being in the pole, and its motion beginning from that point; then will the times of the successive revolutions be as the differences of the cubes of the natural numbers; and the excess of the time of the $\overline{n+1}^{\text{th}}$ revolution above that of the $n^{\text{th}} = n \times$ excess of the 2^{nd} above the 1^{st}.

9. If S be the Sun, and A, B two planets that appear

stationary to one another, then tan. SBA : tan. SAB : : periodic time of A : periodic time of B.

10. If $N = n^{th}$ term of the expansion of a^x, determine n when the series reckoned from that term, begins to converge; and shew that the sum of all the terms which follow N is less than $\dfrac{Nn}{1 - x \log. a}$.

11. If a be an approximate value of x in any equation, and b, c be the results, when a is substituted for x in the original and in the limiting equation; then will $x = a - \dfrac{b}{c}$ nearly.

12. If any number of forces $p, q, r \ldots$ in different planes, act on a point making the angles $\alpha, \beta, \gamma \ldots$ with the resultant, then will $p \cos \alpha + q \cos \beta + r \cos \gamma + \ldots$ be a maximum.

TUESDAY AFTERNOON.—MR. HIGMAN.

Fifth and sixth Classes.

1. PROVE geometrically, that the side of a square is incommensurable with its diagonal.

2. Draw a line perpendicular to two straight lines not in the same plane.

3. If c be the hypothenuse of a right-angled spherical triangle, prove that $\sin^2 \dfrac{c}{2} = \sin^2 \dfrac{a}{2} \cos^2 \dfrac{b}{2} + \cos^2 \dfrac{a}{2} \sin^2 \dfrac{b}{2}$.

4. In a parabola find the area included between the curve, its evolute, and its radius of curvature.

5. A given weight P draws another given weight W up an inclined plane, of a given height and length, by means of a string parallel to the plane; when and where must P cease to act that W may just reach the top?

6. Prove that the chord of curvature $= \dfrac{PQ^2}{QR}$.

7. In a given latitude, on a given day of the year, determine the time during which a primary, and secondary rainbow may be seen.

8. Investigate the expression for the area of a plane triangle in terms of the sides; and apply the result to the case where the sides are 24, 30, 18.

9. Find x so that $\dfrac{\tan^3 x}{\tan 3x}$ may be a maximum.

10. Find the integrals of $\frac{x^6 dx}{1+x^3}$; $\frac{x^{\frac{3}{2}} dx}{\sqrt{(2a-x)}}$; $\frac{dx}{\sin x}$.

11. If the periodic time of a body revolving in a circle round the centres of force S and R be the same, compare the force tending to S with that tending to R.

12. Find the centre of oscillation of an isosceles triangle, vibrating about an axis which is perpendicular to its plane, and passes through the angle contained by the equal sides of the triangle.

13. A circle whose plane is vertical, is just immersed in a fluid; divide it by a horizontal line into two such parts, that the pressures on them may be equal.

14. On a given day, in a given latitude, the Sun being on the meridian, determine geometrically the angle at which a rod of given length must be inclined to the horizon, that its shadow may be the greatest possible.

15. Transform 8978 from a local value 11, and 3256 from a local value 7, to a system in which the local value is 12; and multiply the numbers together in that system.

16. The image of a circular arc concentric with the reflector, subtends the same angle as the object both from the surface and the centre.

17. PSp be any line drawn through the focus S of a conic section, meeting the curve in the points P and p, and SL be the semi-latus rectum, then will SP, SL, Sp be in harmonic progression.

18. If P' be a point so taken in the radius vector SP of a parabola, that $SP' = SY$ the perpendicular on the tangent, then will the locus of the point P' be the elliptic spiral; prove this, and compare the times of two bodies describing AP, and AP' the absolute forces being the same in both cases.

TUESDAY AFTERNOON.—MR. HUGHES.

Third and Fourth Classes.

1. D is a point within a triangle ABC: having given AB, AC and the angles ABD, ACD, BDC, find BC.

2. Describe a circle about a given segment of a parabola made by an ordinate perpendicular to the axis.

3. Compare the pressures on the upper and lower halves of a hemispherical vessel filled with fluid.

4. If (i) be the angle of incidence of a ray passing through a prism in a plane perpendicular to its axis, (e) the angle

of emergence, (a) the vertical angle of the prism, and $1 : n :: \sin I : \sin R$ out of the ambient medium into the prism, then

$$\sin e = \frac{1}{n} \sin a \sqrt{(1 - n^2 \sin^2 i)} - \cos a \, . \sin i.$$

5. A hemispherical vessel rests with its base on a horizontal plane; having given the weight, and inner radius of the basin, find the specific gravity of a fluid which when just filling it shall begin to run out of the bottom.

6. Find the velocity acquired by an inelastic body descending through a system of three planes, the first being vertical, the second inclined at 45°, and the third at 15° to the horizon respectively.

7. Find the isosceles triangle of a given area, which vibrating about an axis passing through its vertex perpendicular to its plane shall oscillate in the least time possible.

8. Integrate $x^2 dx \, (\log x)^2$, $\frac{x^3 dx}{\sqrt{(x^2+1)}}$, and $dy + \frac{ny dx}{\sqrt{1+x^2}} = adx$.

9. Prove that $(1 + x)^m$ may in all cases be expressed by a series of the form

$$1 + ax + bx^2 + \&c.$$

10. The altitudes of two stars as they cross the prime vertical are observed and the difference of their right ascensions is known; find the latitude of the place.

11. If a body describe an oval round a centre of force, the distance at which the angular velocity is equal to the mean angular velocity is $\sqrt{\frac{A}{\pi}}$, where A is the area of the figure, and $\pi = 3.14159$.

12. Having given the centre of gyration of a circle revolving about a diameter, find the centre of gyration of an ellipse revolving about either axis.

Tuesday Evening.—Mr. Hughes.

1. Having given the area, the base, and the sum of the angles at the base, find the difference of the angles at the base.

2. A vertical cylindrical tube is connected by a horizontal branch with a cubical vessel of water, and the water is made to ascend in the tube by a condensing syringe applied to the top of the vessel. Having given the dimensions of the vessel,

tube, and syringe, and the elevation of the water in the tube and vessel; find the number of descents of the piston.

3. A body half elastic moving along a horizontal plane is reflected by a hard plane inclined at an obtuse angle to its course; prove that time of flight on the inclined plane : time of acquiring the velocity before impact by a body descending vertically : : tangent of plane's inclination : radius.

4. If u be a function of x, and in the equation $\frac{du}{dx} = 0$ there be m roots equal to a, and n roots equal to b, then there will be one minimum value of u for each of the roots a and b if m and n be odd, and neither maxima nor minima values when they are even.

5. If an object be placed between two plane reflectors inclined at any angle; find the locus of an eye such that the length of the ray by which any given image is seen may be equal to a given line.

6. If $\frac{A}{B}$ be a fraction in its lowest terms, B greater than A, and of the form $B'.2^m.5^n$; the quotient will be a mixed circulating decimal, and the higher of the indices m, n will be the number of figures in the part which does not recur.

7. ACB is a quadrant of a circle whose centre is C, CA, CB its radii, AD, BE equal arcs, DE the chord of the arc DE; shew that the solid generated by the revolution of the circular segment DE about either radius is equal to twice the sphere whose diameter is $\sqrt{2}.\ \sin \frac{1}{2} DE$.

8. An imperfectly elastic body revolving in an ellipse whose eccentricity is $\frac{1}{2}$ is reflected at the mean distance by a plane coincident with the distance, so as to move after impact in the direction of the axis minor; find the degree of elasticity, and compare the periodic times in the two ellipses $\left(F \propto \frac{1}{D^2}\right)$.

9. A cylindrical tube is filled with fluid and closed at both ends; compare the pressures on its sides at the Earth's surface and at a given altitude above it supposing the bulk of the fluid from change of temperature to be diminished an n^{th} part, and the axis to be vertical in both positions.

10. State how the Sun, Planets, and fixed Stars are affected by aberration; and shew that the part of the aberration arising from the motion of the planet varies as $\frac{\cos SPT}{\sqrt{SP}}$, S being the Sun, T the Earth, and P the Planet.

11. If particles of a spherical shell attract with forces varying as $\frac{1}{D^3}$, and a cylindrical rod of uniform density whose length equals n times the radius of the sphere pass through the shell; find the pressure on the shell when the rod is at rest, the part of it within the shell being equal to the radius of the sphere.

12. From the vertex of a parabola a straight line is drawn inclined at 45° to the tangent at any point; find the equation to the curve which is the locus of their intersections.

13. If $x^n + Ax^{n-1} \ldots\ldots - Px^p \ldots\ldots - Sx^s \ldots\ldots + Tx + V = 0$ when P is the greatest and S the last negative coefficient, then $\frac{V^{\frac{1}{s}}}{V^{\frac{1}{s}} + P^{\frac{1}{s}}}$ is an inferior limit of the positive roots.

14. Integrate $\frac{dx}{(1-x)(1-2x)^{\frac{1}{2}}}$, $\frac{dx}{(x-a)x^{\frac{2n+1}{2}}}$, and $x^2d^2y = xdxdy + nydx^2$.

15. A body acted on by gravity descends along the concave part of a vertical circle from the extremity of the horizontal radius; find its initial velocity so that after quitting the circumference it may pass through the centre.

16. Find the centre of gyration of a right cone revolving about an axis passing through its centre of gravity parallel to one of its sides.

17. To find the horary motion of the Moon's nodes in a circular orbit.

18. Find the correction due to the length of a pendulum for the thickness of its axis.

19. If the inscribed sphere be taken away from the Earth, find the time in which a particle situated in the plane of the Earth's equator within the space occupied by the inscribed sphere will reach the inner surface of the remaining meniscus; the Earth being supposed an oblate spheroid of small ellipticity.

20. If I be the obliquity of the ecliptic, L the Sun's longitude, A his right ascension, then

$$A = L - \left\{\tan^2\frac{I}{2}\cdot\frac{\sin 2L}{\sin 1''} - \tan^4\frac{I}{2}\cdot\frac{\sin 4L}{\sin 2''} + \tan^6\frac{I}{2}\cdot\frac{\sin 6L}{\sin 3''} - \&c.\right\}.$$

21. Q is the focus of incidence of a pencil of rays which passes nearly perpendicularly through the sides of a prism

whose vertical angle A is small, q the focus of emergent rays, and QC, qc perpendicular to the first and second surfaces respectively. Having given the ratio of $\sin I : \sin R$ out of the ambient medium into the prism, and also QC, CA and angle A, find qc and Ac.

22. P and Q are two material points connected by an inflexible line PQ; P moves along a groove and PQ on a smooth horizontal plane. Having given the initial position of the rod and the quantity and direction of the motion communicated to P, find the angular velocity of the rod when it coincides with the groove.

23. Resistance varies partly as the velocity and partly as the velocity squared; construct for the time when a body is projected downwards with a velocity greater than the greatest it can acquire from rest.

24. Every surface of the second degree may be generated by a circle of variable radius moving parallel to itself, the centre moving along a diameter of the surface. Prove it in the case of the ellipsoid.

THE SENATE HOUSE PROBLEMS,

Proposed by the Two Moderators, to the Candidates for Honors during the Examination for the Degree of B. A.

MONDAY, JANUARY 17, 1825.

MONDAY MORNING.—MR. CHEVALLIER.

First and Second Classes.

1. REPRESENT a million according to the duodenary scale of notation.

2. Divide 29^2 into two other square integer numbers.

3. Required the sum of

$$\frac{1^2}{2 \cdot 3 \cdot 4 \cdot 6} + \frac{2^2}{3 \cdot 4 \cdot 5 \cdot 7} + \text{\&c. to } n \text{ terms.}$$

4. If A_1 is the area of a given logarithmic spiral from a distance r to the centre, A_2 the corresponding area of the curve traced out by the perpendicular upon its tangent, A_3 that of the

curve traced out by the perpendicular on *its* tangent, and so on continually, find the value of

$$A_1 + A_2 + A_3 + \&c.$$

5. A seconds pendulum is carried to the height of one radius above the earth, and another is sunk to the depth of half a radius. Compare the times of their oscillations.

6. If the latitude of a place be determined by observing the altitude of the Sun at 6 o'clock, and the tabulated declination be affected by a small error, find the corresponding error in latitude.

7. A weight P is *supported* by a string, which passes over a fixed pulley, is wound several times round a cylinder Q, and attached to a pin on the other side, the strings being all parallel. If the string be now cut between the cylinder and this point of support, required the motions of P and Q.

8. Given the radius of the Moon and her mass compared with that of the Earth, to find the density of the atmosphere at the Moon's surface, supposing it to be similar to our own.

9. Shew that a sphere is emptied through a small orifice at the lowest point in less time than any other spherical segment of the same capacity.

10. A flood-gate moves upon a vertical axis, the area on one side of the axis being the quadrant of a circle, and on the other side a parallelogram of the same altitude. Required the width of the parallelogram so that the gate may *just* open by the pressure of the water when it has risen to the top.

11. Given the distance of Jupiter from the Sun, his radius, and the time of his diurnal and annual revolution, to compare the aberration of a given Star when it passes the meridian of an observer in his equator at mid-day and at midnight.

12. Prove that, in the direct impact of perfectly hard bodies, the difference of the *vires vivæ* before and after impact is equal to the sum of the *vires viva* of the bodies moving with the velocities lost and gained respectively.

MONDAY AFTERNOON.—MR. CHEVALLIER.

Fifth and Sixth Classes.

1. Required the present worth of £75 due 15 months hence at 5 per cent.

2. Shew that, if the sum of the digits in the odd places be subtracted from any number, expressed in decimal notation, and the sum of the digits in the even places be added to

the same number, the result is divisible by 11.

3. The difference of the means of 4 numbers in geometrical progression is 2, and the difference of the extremes is 7. Required the numbers.

4. Prove that sin. $75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$ to rad. 1.

5. Find tan. 5 A in terms of tan. A.

6. Shew that, if two circles cut each other, and, from any point in the straight line produced which joins their intersections, two tangents be drawn, one to each circle, they shall be equal to one another.

7. Solve the equation $x^3 - 6x^2 + 11x - 6 = 0$, the roots being in arithmetical progression.

8. Integrate $\frac{dx}{x^3\sqrt{(1-x^2)}}$;

$\frac{dx}{x^2 - 5x = 6}$.

9. Trace the spiral in which $r = a \cdot \cos. \theta$; and find its area between the values $\theta = 0$ and $\theta = \frac{\pi}{2}$.

10. If P and Q are two weights connected by a string passing over a fixed pulley, and acting in a direction parallel to two given inclined planes having a common altitude, and P descends, find the space described in t'' and the velocity acquired.

11. Explain the construction of a Fire-engine.

12. Compare the resistance upon the surface of a cone, moving in a fluid with a given velocity in the direction of its axis, with the resistance upon its base.

13. In a given latitude, find the altitude of the Sun on the day of the equinox at 9 in the morning.

14. Trace the image of an indefinite straight line *in contact* with a spherical reflector.

15. Find the space through which a body must fall externally that it may acquire the velocity with which it moves in an ellipse about the *centre*.

16. If the mass of a planet is four times that of the Earth, and the distance of its satellite 16 times that of the Moon from the Earth, in how many months will the satellite revolve?

17. Find the position of the centre of gravity of the area of a semiparabola.

18. Find the time in which a straight line of given length will oscillate when suspended at its extremity.

MONDAY AFTERNOON.—MR. WARREN.

Third and Fourth Classes.

1. Find one of the roots of the equation $3x^3 - 26x^2 + 34x - 12 = 0$ by the method of divisors.

2. In a parabola, in the focal distance SP, Sp is taken equal to the ordinate PN. Find the equation to the curve traced out by the point p.

3. If a body oscillate in a cycloid beginning at the highest point, the tension of the string at any point arising from the centrifugal force equals the tension arising from gravity.

4. Find the time in which a vessel formed by the revolution of a cycloid about its axis, placed with its axis vertical and its vertex downwards, will empty itself through a small orifice at the vertex.

5. Find the point of contrary flexure of a spiral, where the angle varies inversely as the n^{th} power of the radius vector.

6. Express a circular arc in a series in terms of its tangent.

7. When the centripetal force varies inversely as the n^{th} power of the distance, n being greater than 3; find the equation to the spiral, which a body, projected with a velocity equal to the velocity acquired by falling from an infinite distance, describes; and determine the number of revolutions which it makes, before it falls into the centre.

8. If v represent the true anomaly of a planet, reckoning from the perihelion, u the eccentric anomaly, and e the eccentricity; when e is small $v - u = e \sin. u$ nearly. Required proof.

9. Compare the mean addititious force with the force by which P is retained in its orbit round T. (Newton, Prop. 66. Cor. 17.)

10. State the phænomena, from which it appears, that the force, by which the Moon is retained in its orbit, tends to the Earth, and that this force varies inversely as the square of the distance.

11. Find the centre of gyration of an hyperboloid revolving about its axis.

MONDAY EVENING.—MR. WARREN.

1. The present value of an annuity of £1 to continue x years is £10, and the present value of an annuity of £1 to continue $2x$ years is £16. What is the rate of interest?

2. The roots of the equation $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$ are in geometrical progression beginning from unity; given $p = 15$, $q = 70$. Required n, r, &c.

3. If a, b, c, d be the sides of a quadrilateral inscribed in a circle, and $S = \frac{a+b+c+d}{2}$; prove that the area of the quadrilateral $= \sqrt{(S-a) \cdot (S-b) \cdot (S-c) \cdot (S-d)}$.

4. Express the chord of 36° to radius unity in a continued fraction.

5. A body whose elasticity : perfect elasticity :: $e : 1$, projected from a horizontal plane, at a given elevation, with a given velocity, impinges against a perfectly hard vertical plane, whose distance from the point of projection is given. Required the horizontal range of the body; the vertical plane being supposed perpendicular to the plane of the body's motion.

6. Explain the nature of the stereographic projection of the sphere, and shew that the projection of all circles, the planes of which do not pass through the eye, are circles.

7. A body describing a parabola, by a uniform force acting in parallel lines, receives an impulse. Given the line which the body would describe by the impulse alone in a given time T; and the chord of the parabolic arc, which it would have described in the same time T if no impulse had been communicated to it: to find the chord of the arc which it actually does describe.

8. Find the area of the least parabola which can circumscribe a given circle.

9. A solid of revolution, whose axis is perpendicular to the horizon, empties itself through a small given orifice. Required its nature when the velocity of the descending surface is uniform.

10. A paraboloid, generated by the revolution of a parabola, whose equation is $y^n = a^{n-1}x$, placed with its vertex downwards and its axis vertical, empties itself through a small orifice at the vertex; and the value of n is such, that if a sphere empty itself in the same time that the paraboloid does, half the sphere will empty itself in the same time that half the paraboloid does: compare the distance of the descending surface from the vertex when half the paraboloid is emptied with the distance at first.

11. Sum the series

$$\sin. a + \sin.(a+b) + \sin.(a+2b) + \text{\&c. to } n \text{ terms.}$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \text{\&c. to infinity.}$$

12. Determine the position of equilibrium of a uniform rod, one end of which rests against a plane perpendicular to the horizon and the other on the interior surface of a given hemisphere.

13. Supposing the force of gravity to vary inversely as the n^{th} power of the distance; by what law does the density of the atmosphere vary?

14. Shew that the length of a parabola, whose equation is $ay^{2n} = x^{2n+1}$, where n is any whole positive number, may be found in terms of the abscissa in a finite algebraical form.

15. Shew how the polar equation to caustics formed by reflection may be found generally; and apply the method when the reflecting curve is a circular arc, and the radiating point in the circumference of the circle.

16. Find the equation to the curve surface in which the tangent plane at any point intersects the axis of z at a distance from the origin equal to m times the corresponding value of z.

17. A corpuscle placed within a circle is attracted to every particle in the circumference with a force that varies inversely as the square of the distance: prove, when the distance of the corpuscle from the centre is small, that the attraction on the corpuscle varies nearly as its distance from the centre, and draws it from the centre.

18. In the 10th Lemma of the 1st Section of Newton, where the abscissa AD represents the time, the ordinate DB the velocity, and the area ABD the space described; if a straight line be drawn touching the curve AB in B the extremity of the ordinate, the tangent of the angle which this line makes with the axis will represent the force. Required proof.

19. An arch, where the equilibrium is preserved by the weights of the voussoirs, is so constructed, that the centres of gravity of the voussoirs are in a catenary curve and the joints perpendicular to the curve. Required the equation from which the length of the voussoirs may be obtained.

20. Investigate the formula of lunar nutation in right ascension; find the longitude of the Moon's nodes when it is a maximum, and thence its maximum value; and by means of these values express the nutation in right ascension in a more simple formula.

21. Given the ratio of the periodic time of the Moon to the time of the Earth's revolution about its axis and the ratio of the mean distance of the Moon to the mean semi-diameter of the Earth, to find the ratio of the polar and equatorial diameters of the Earth nearly.

22. A weight P raises up another weight Q by means of a string passing over a fixed cylindrical pulley; given P, Q, and the weight of the pulley, to compare the tensions of the two parts of the string; the weight of the string and the friction at the axis of the pulley being neglected.

23. A body oscillates in a cycloid in a resisting medium, where the resistance is as the square of the velocity; find the resistance at any point of the body's motion (Newton, Prop. 29, Book II.).

24. Prove from the preceding proposition; if a space S be taken so as to be represented by the rectangle of the hyperbola in Newton's figure on the same scale that the cycloidal arc is represented by the hyperbolic area, and V be the velocity acquired in falling through S by the force of gravity *in medio non resistente*; that V is the limit of the velocity which can be acquired in the resisting medium by the force of gravity.

TUESDAY MORNING.—Mr. WARREN.

First and Second Classes.

1. Draw a perpendicular to two given straight lines not in the same plane.

2. Sum the series

$$1 + 2^2x + 3^2x^2 + 4^2x^3 + \&c. \text{ to } n \text{ terms,}$$

$$\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \&c. \text{ to infinity.}$$

3. Express cos. nA in terms of cos. A, n being a whole number.

4. When a chain fixed at two points is acted upon by a central attractive or repulsive force, the tension at any point is inversely as the perpendicular let fall from the centre of force on the tangent at that point. Required proof.

5. Determine the point in the curve surface on which a semiparaboloid will rest on an horizontal plane.

6. Having given the refracting powers of two mediums; find the ratio of the focal lengths of a convex and concave lens formed of these substances, which when united produce images nearly free from colour.

7. Compare the time in which a sphere slides down an inclined plane with the time in which it rolls down the same plane.

8. A given uniform rod moves in the same plane in a hemisphere. Determine its motion.

9. Shew how the complete integral of the differential equation

$$Ay + B \cdot \frac{dy}{dx} + C \cdot \frac{d^2y}{dx^2} + D \cdot \frac{d^3y}{dx^3} + \&c. = x.$$

is to be obtained, when A, B, C, D, &c. are constant quantities, and X function of x.

10. Given the height of the spring and neap tides, to compare

the densities of the Sun and Moon; their apparent diameters being considered as equal.

11. Explain the cause of aberration of light; shew how it is to be measured, and distinguish accurately between the aberration of the fixed stars and the aberration of the planets.

12. Determine generally the resistance of a medium, so that a body acted upon by a centripetal force, whose law is known, may move in a given curve, and thence find the resistance when the force is uniform and acts in parallel lines, so that the curve may be a circular arc.

TUESDAY AFTERNOON.—MR. WARREN.

Fifth and Sixth Classes.

1. If the terms of an equation, all whose roots are possible, be multiplied by the terms of the arithmetical progression 0, 1, 2, 3, &c. the resulting equation will be a limiting equation to the former, with this exception, that no root of the limit will lie between the positive and negative roots of the proposed equation.

2. The centre of an ellipse coincides with the vertex of a common parabola, and the axis major of the ellipse is perpendicular to the axis of the parabola. Required the proportion of the axis of the ellipse, so that it may cut the parabola at right angles.

3. Several bodies are projected from a given point, with the same velocity, in different directions, being acted upon by the force of gravity. Find the focus of them all at the end of a given time.

4. Explain the method of drawing asymptotes to spirals, and apply it to the hyperbola considered as a spiral having the pole in the focus.

5. Find the elasticity of two bodies A and B and their proportion to each other, so that when A impinges upon B at rest, A may remain at rest after impact and B move on with an n^{th} part of A's velocity.

6. Given two sides of a triangle and the difference of the angles opposite to them. Required the other angle.

7. Find the geometrical focus of a small pencil of rays diverging from a given point in the axis and incident nearly perpendicularly on an elliptic reflector.

8. Determine at what angle the wind must strike against the sails of a mill, so that the effect to put them in motion may be the greatest possible.

9. Given the latitude of the place and the declination of the Sun. Find the time that the Sun is above the borzion.

10. Explain the principal advantages of Brigg's system of logarithms.

11. The equation $x^3-7x^2+16x-12=0$ has two equal roots. Find all the roots.

12. Find the law of the force tending to the pole of the logarithmic spiral.

13. The sum of the three angles of a spherical triangle is greater than two right angles and less than six right angles. Required proof.

14. The *whole* surface of a cone is three times the area of the base. Find its vertical angle.

15. When the force varies inversely as the n^{th} power of the distance, compare the velocity acquired by falling from an infinite distance with the velocity of a body revolving in a circle.

16. Given the hypothenute of a right-angled triangle and the side of an inscribed square. Required the two sides of the triangle.

17. Investigate the differential expression for the radius of curvature, and apply it to find the radius of curvature of the logarithmic curve.

TUESDAY AFTERNOON.—MR. CHEVALLIER.

Third and Fourth Classes.

1. What is the present value of a freehold estate of £. 150 a year allowing the purchaser 6 per cent. compound interest?

2. Shew that the greater two consecutive numbers are, the less is the difference between their logarithms.

3. Compare the curvature of an ellipse at the extremity of the major and minor axis.

4. A given weight is to be supported at a given point upon a straight lever of uniform density by a power acting at its extremity, on the same side of the fulcrum. Required the least power which will support the system, and the corresponding length of the lever.

5. Find the point in a given hyperbola where the velocity of a body acted on by a force tending to its focus is twice as great as the velocity in a parabola at the same distance.

6. If a solid cylinder and a thin hollow cylinder of the same *weight* and radius roll together from rest down a given inclined plane, how far will they be separated after a given time?

7. If the Earth's motion about its axis were to cease, how

much would a clock keeping true time in a given latitude gain in 24 hours?

8. Given the time of Sun-rise and the altitude of the Sun when due east on the same day, to find the latitude of the place and the declination of the Sun.

9. State what advantage is obtained by substituting a glass rectangular prism for a plane reflector, in the construction of Newton's telescope: and trace the course of a pencil of rays on that supposition.

10. A spherical bubble composed of matter the specific gravity of which is S, and filled with gas of the specific gravity s, just floats in air, specific gravity σ. Required the thickness of the bubble.

11. Solve the equation $x^6 + 1 = 0$.

12. Given the declination of the Moon, to find the duration of the superior or inferior tide occasioned by her action alone.

TUESDAY EVENING.—MR. CHEVALLIER.

1. The sum of two numbers is 6, and the sum of their cubes 72. Reqired the numbers.

2 Upon a given straight line as an hypothenuse, describe a right-angled triangle which shall have its three sides in continued proportion.

3. Find a series of fractions converging to $\frac{41}{72}$.

4. Sum the following series:

$$\frac{1}{2} + \frac{3}{4} + 1 + \text{\&c. to 8 terms.}$$

$$\frac{1}{1\cdot3} + \frac{1}{2\cdot4} + \frac{1}{3\cdot5} + \text{\&c. to } n \text{ terms.}$$

$$\frac{5}{1\cdot2\cdot3\cdot4} + \frac{7}{2\cdot3\cdot4\cdot5} + \frac{9}{3\cdot4\cdot5\cdot6} + \text{\&c. to } n \text{ terms.}$$

$$1^3 + 2^3 + 3^3 + \text{\&c.} + x^3.$$

5. A circular hoop is supported in a horizontal position, and 3 weights of 4, 5, and 6 pounds respectively are suspended over its circumference by 3 strings meeting in the centre. What must be their position so that they may sustain one another?

6. A cylinder of given altitude has its lower half filled with mercury and the rest with water. Find the time in which it will empty itself through a small orifice in the base.

7. The N. P. D. of α Aquilæ being 81°. 38″, 25″, and its observed zenith distance when on the meridian 43°. 50′. 45″,

find the latitude of the place: and state the several corrections which must be applied to obtain an accurate result.

8. Find the focal length of a double convex glass lens of inconsiderable thickness when the radii of the surfaces are equal; and sin. I : sin. R :: 3 : 2.

9. Shew that a horizontal dial, constructed for North latitude l, will be a vertical meridional dial for South latitude $(90 - l)$.

10. Two bodies are projected from the same point in a horizontal plane with equal velocities, and have the same horizontal range. Required the directions of projection so that the area included between the parabolas described may be the greatest possible.

11. Find the time in which a pendulum would oscillate in a hypocycloid within the earth, the diameter of the wheel being half the earth's radius.

12. P descends drawing Q over a fixed pulley. Find the space described in a given time, the string being conceived to have weight.

13. If an ordinate be drawn to the axis of a cycloid from any point P, cutting the circle described upon the axis in D, and from P and D there be drawn tangents to the cycloid and circle, intersecting one another in T; Required the curve which is the locus of T, and the area which is contained between it and the cycloid.

14. A straight bar of given length is made to oscillate in its own plane about an axis situated in a line which bisects it at right angles. Required the point of suspension so that the time of oscillation may be the least possible.

15. Integrate the equations

$$(x^2 + yx)\, dy = (x - y)\, dx.$$

$$(2y^2x + 3y^3)\, dx + (2x^2y + 9xy^2 + 8y^3)\, dy = 0.$$

$$x^3\, dx^3 + dy^3 = axdx^2\, dy.$$

16. If seven balls be drawn from a bag, in which are four white balls and eight black, what is the probability that three white balls will be taken?

17. Required the nature of the curve which cuts perpendicularly a series of similar concentric ellipses.

18. A weight being suddenly removed from the deck of a vessel, the area of which at the surface of the water is given, she is observed to make a small vertical oscillation in t''. Required the weight of the vessel.

19. There is a fixed centre of force which varies inversely as the square of the distance; and about this as a focus an ellipse is described, the axes of which are to one another in the

proportion of $\sqrt{2} : 1$. A perfectly elastic body falls from a distance equal to the axis major in the direction of the radius vector passing through the extremity of the axis minor, and impinges on the ellipse. Required the motion after impact.

20. The earth being supposed a sphere revolving about its axis with a given angular velocity, find the curve, in a meridional plane, which is the locus of a body, the centrifugal force of which opposed to gravity is every where equal to the force of gravity acting upon it.

21. State the methods by which the masses of all the planets and of their satellites may be compared with that of the Sun.

22. If a body be revolving in an ellipse about the focus, and the force be suddenly made to vary inversely as the cube of the distance, the *actual* force at the mean distance being unaltered, what will be the curve described?

23. Find the curve described by a body projected in a medium, the resistance of which varies as the velocity, and acted upon by gravity.

24. Prove that an arc of a circle which does not exceed 60°. is a curve of quicker descent than any other curve which can be drawn within the same arc: and the arc of 90°. is a curve of quicker descent than any other curve which can be drawn without the same arc.

THE SENATE HOUSE PROBLEMS,

Proposed by the Two Moderators, to the Candidates for Honors, during the Examination for the Degree of B. A.

MONDAY, JANUARY 16, 1826.

MONDAY MORNING,—MR. KING.

First and Second Classes.

1. FIND the equations to a straight line drawn from a given point perpendicular to a given plane.

2. In the cubic equation $x^3-qx-r=0$ where $\frac{r^2}{4} < \frac{q^3}{27}$, shew that the solution may be effected by means of a table of natural sines, and explain why the accuracy of this method cannot be depended upon in practice.

t-angled spherical triangle where c is the hypo- and b the sides, prove that

$$\tan. \frac{c+a}{2} \,.\, \tan. \frac{c-a}{2} = \tan.^2 \frac{b}{2},$$

the time of a small oscillation of an oblique-angled vibrating in its own plane about an axis passing of its angular points.

from a point in a horizontal plane any number of projected in the same vertical plane with such velocities uch directions that the areas of the parabolas described equal to one another; find the curve which shall them all.

Trace the curve whose equation is $y^2 = \frac{x^3 - bx^2}{x+c}$, and rmine the nature of its singular points.

7. A rod of given length and uniform density is supported a fluid, the density of which varies as the depth, by a string ttached to it at a given point; find the position of equilibrium, upposing one extremity of the rod coincident with the surface.

8. If P be the pole of the heavens, Z the zenith and S a given star, find when the angle ZSP increases fastest.

9. Integrate the following differentials;

$$\frac{dx}{x^4+1}, \quad e^x . \sin. mx \,.\, dx,$$

$$dy - \overline{x+y}|^n \,.\, dx = a^n dx.$$

10. Sum the following series:

$$\frac{\sin. \theta}{1} \,.\, \frac{1}{2} - \frac{\overline{\sin. \theta}|^3}{1.2.3} \,.\, \frac{1}{3} + \frac{\overline{\sin. \theta}|^5}{1.2\ 3.4.5} \,.\, \frac{1}{4} - \&c. \textit{ ad inf.}$$

$$2 + 5 + 24 + 83 + 334 + \&c. \text{ to } n \text{ terms.}$$

11. Find the shortest line which can be drawn touching a given ellipse, and intercepted by the tangents drawn at the extremities of the axes of the ellipse.

12. Find the equation to the curve surface in which the tangent plane at any point cuts from the axis of z a portion equal to the distance of the point of contact from the origin of the co-ordinates.

MONDAY AFTERNOON.—MR. KING.

Fifth and Sixth Classes.

1. Investigate the rule for finding the least common multiple of any two quantities, and apply it to find the least common multiple of 174 and 336.

2. Having given the n^{th} term of an arithmetic series, and also the sum of n terms, find the series.

3. Having given the sine of 12°, find the sine of 48°.

4. Two circles are situated in the same vertical plane; determine analytically and geometrically the straight line of swiftest descent from one to the other; and shew that the two results agree.

5. At what hour on a given night, in a given latitude, will the vertical circle passing through a known star cut the equator in a given angle.

6. Find the value of $\frac{\sin. 2x + \overline{2 \sin. x}|^2 - 2 \sin. x}{1 - \cos. x}$, when $x = 0$.

7. Two straight rods inclined to each other at a given angle are immersed vertically in a fluid in a given position; find the angle formed by their images.

8. Trace the curve whose equation is $a^2 y = \frac{x^5}{x^2 - b^2}$.

9. The resultant and sum of two forces are given, and also the angle which one of them makes with the resultant; it is required to determine the forces and the angle at which they act.

10. Having given two conjugate diameters of an ellipse and the angle contained between them; find the magnitude and position of the axes.

11. Find the resistance to a cycloid moving in a fluid in the direction of its base.

12. Enumerate the arguments by which the diurnal rotation of the Earth round its axis and its annual motion round the Sun are established.

13. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are α, β, γ into one whose roots are $\alpha^2 + \beta^2, \alpha^2 + \gamma^2, \beta^2 + \gamma^2$.

14. Investigate the equation to the orbit in which the centripetal force is always n times as great as the centrifugal, and find the time of one revolution.

15. Determine that point in an ellipse described round a centre of force situated in the focus where the linear velocity is n times as great as the paracentric.

16. Find the moment of inertia of a rectangle revolving in its own plane round an axis passing through its centre of gravity.

17. Sum the following series:

$$1 . 2^2 + 2 . 3^2 + 3 . 4^2 + \&c. \text{ to } n \text{ terms.}$$

$$\frac{1}{1 . 2} - \frac{1}{3 . 4} + \frac{1}{5 . 6} - \frac{1}{7 . 8} + \&c. \text{ to infinity.}$$

18. Integrate the following differentials:

$$\frac{x+4}{x^3-x^2}\,.\,dx \qquad \frac{x^4\,dx}{\sqrt{1-x^2}},$$

$$\sqrt{x}\,.\,dx + \sqrt{y}\,.\,dx = \sqrt{y}\,.\,dy.$$

MONDAY AFTERNOON.—MR. HIND.

Third and Fourth Classes.

1. A sum of money P is left among A, B and C, in such a manner that at the end of a, b and c years, when they respectively come of age, they are to possess equal sums; required the share of each at compound interest.

2. Prove that 1, 3, 5, 7, &c. is the only arithmetic progression beginning from 1, in which the sum of the first half of any even number of terms bears to the sum of the second half the same constant ratio, and find that ratio.

3. Shew that $\text{vers.}\,(\pi - A) = 2\ \text{vers.}\left(\frac{\pi + A}{2}\right)\text{vers.}\left(\frac{\pi - A}{2}\right)$ to a radius $= 1$.

4. Two equal parabolas have a common axis; prove that the area included between one of them and a straight line touching the other is a constant magnitude.

5. From the top of a tower two bodies are projected with the same given velocity at different given angles of elevation, and they strike the horizon at the same place. Find the height of the tower.

6. Determine the magnitude of a sphere of given specific gravity which will rest just immersed in a fluid whose density varies as its depth.

7. Find the principal focus of a reflector generated by the revolution of a cycloid about its axis, and determine the relation between the distances of the conjugate foci from the vertex.

8. Find all the angles in which the curve whose equation is

$$\left(\frac{x}{y} + \frac{y}{x}\right)^2 = 2a^2\,.\left(\frac{1}{y^2} - \frac{1}{x^2}\right)$$

cuts the axis, and determine the value of its greatest ordinate.

9. If a body describe a logarithmic curve by a force acting perpendicularly to its axis; prove that the force at any point varies as the body's distance from the axis, and the velocity as the square root of the chord of curvature parallel to it.

10. Integrate $\frac{\tan.\,\theta\,.\,d\theta}{1-\tan.^2\theta}$.

11. Three given points are taken in the circumference of a given circle; find its vertical position on a horizontal plane, that the sum of their altitudes may be the greatest or least possible.

12. In any latitude find when the time of the rising of the Sun's disk bears the least ratio to the time of its crossing the meridian.

13. If a body describe a circle by a force in the circumference and at the same time the circle revolve about the centre of force in its own plane with an angular velocity varying inversely as the square of the body's distance; prove that the path traced out in fixed space is the spiral of Archimedes, and find the law of force by which it may be described.

14. Construct and graduate a clepsydra in which the weight of the fluid discharged measures the time.

MONDAY EVENING.—MR. HIND.

1. Express the secant and cosecant of the sum and difference of two arcs in terms of the secants and cosecants of the simple arcs.

2. Shew that the number of permutations of (m) things taken (n) together is equal to $m\,.\,(m-1)\,\ldots\,.\,(m-r+1)$ times the number of permutations of $(m-r)$ things taken $(n-r)$ together.

3. In how many different ways is it possible to pay £10. in crowns, seven shillings, and moidores?

4. Two balls, of given magnitude and elasticity, not affected by gravity are projected at the same instant from given points with given velocities in opposite directions in the same straight line; find *when* and *where* their impact takes place, and their positions at the end of any assigned time after impact.

5. The vertex of a parabola is A, and the axis AN; and in the ordinate NP a point Q is taken always equidistant from A and P; find the equation to the curve which is the locus of Q; trace it, and determine the angles in which it cuts the axis and the arcs of the parabola.

6. If (t) be the length of a part of the catenary, whose weight is equal to the tension at a point whose abscissa is x and corresponding arc s; prove that

$$s+x : s-x :: \sqrt{(t+x)} : \sqrt{(t-x)}.$$

7. The abscissa and double ordinate of a common parabola are a and b, and the diameters of its circumscribed and inscribed circles D and d; prove that $(D+d)=(a+b)$.

8. Find the time of emptying the frustum of a cone, the radii of whose bases are R and r, and altitude (h), through an orifice in its smaller end.

9. If a quadrantal arc be divided into two parts in the ratio of $2n : 1$, of which θ is the less; prove that

$$\sin.\theta.\sin.3\theta.\sin.5\theta\ldots\ldots\sin.(2n+1)\theta = \frac{1}{2^n}.$$

10. Required the law of force when the space due externally to the velocity in a circle : space due internally :: $\sqrt{e} : 1$; e being the base of the hyperbolic system of logarithms.

11. The quantities a, b, c, d, &c. are in arithmetical progression; prove that the terms of any order of the differences of the quantities $\frac{a}{b}, \frac{b}{c}, \frac{c}{d}$, &c. increase or decrease according as the progression decreases or increases.

12. If any number of hyperbolas have a common centre, and at distances proportional to their major axes double ordinates be drawn; shew that bodies acted upon by the same absolute force situated in the centre will describe any of the arcs thus cut off in equal times.

13. Prove that Des Cartes' solution of a biquadratic equation succeeds when all the roots are possible and two of them equal, and apply it to solve the equation $x^4 - 6x^3 + 8x^2 + 6x - 9 = 0$.

14. An ellipse whose semi-axes are (a) and (b), and eccentricity (e) revolves about its major axis; shew that the surface of the solid thus generated is

$$= 2\pi b\left\{b + \frac{a}{e}.A\right\};$$ A being a circular arc whose sin. $= e$ to a rad. $= 1$.

15. Three straight lines revolve about three given points not in the same straight line and intersect one another in three points; prove that if the loci of two of these intersections be straight lines, the locus of the third will be a conic section.

16. A pencil of parallel rays incident upon a transparent medium bounded by parallel plane surfaces, is partly reflected at the upper surface, partly refracted by the medium, and afterwards reflected at the lower surface; prove that all the rays of the emergent pencil are parallel, and find the angle of incidence that its breadth may be the greatest possible.

17. If the force be repulsive and vary as the distance from the centre of the globe; prove that the oscillations in an epicycloid are isochronous; and having given the radii of the globe

and wheel, find the velocity at any point, and the actual time of an oscillation.

18. If $\cos(\lambda + \delta) : \cos(\lambda - \delta) :: 1 : 3$ where λ is the latitude of the place, and δ the declination of the Moon; prove that the time of the ebbing or flowing of the superior tide : the time of the ebbing or flowing of the inferior tide :: 2 : 1.

19. If the roots of the equation

$$x^n - px^{n-1} + qx^{n-2} - \&c. \ldots + Qx^2 - Px + L = 0$$

be in harmonical progression, then will the greatest and least be

$$\frac{n\sqrt{n+1} \, . \, L}{\sqrt{n+1} \, . \, P - \sqrt{3 \, . \, (n-1)^2 P^2 - 6n \, . \, (n-1) QL}},$$

and

$$\frac{n\sqrt{n+1} \, . \, L}{\sqrt{n+1} \, . \, P + \sqrt{3 \, . \, (n-1)^2 P^2 - 6n \, . \, (n-1) QL}}.$$

Required a proof.

20. Integrate $\dfrac{(\sec\theta - \tan\theta) \, . \, d\theta}{\sec\theta + \tan\theta}$, and $\dfrac{e^\theta \, . \sin m\theta \, . \, d\theta}{1 - 2e^\theta \cos m\theta + e^{2\theta}}$;

and find the relation of x and y in the equation

$$\frac{d^2y}{dx^2} - a \, . \, \frac{dy^2}{dx^2} + bx \, . \, \frac{dy^3}{dx^3} = 0.$$

21. A globe of given weight and magnitude, after descending by the force of gravity (a) feet in air, passes into another medium whose density is (n) times as great; required the relations between the space described, the velocity acquired, and the time of motion.

22. Find the motions of two equal balls connected by an inflexible rod without weight, one of them being attached to a given weight by means of a string passing over a fixed pulley, and the other moving on a perfectly smooth horizontal plane.

23. If (r) be the radius vector of an ellipse from the centre, and θ the angle which it makes with the major axis; it is required to express (r) in a series of the form $A_0 + A_1 \cos\theta + A_2 \cos 2\theta + \&c. \ldots.$ and to show particularly how A_0, A_1, A_2 may be determined.

24. A straight rod which is always parallel to the horizon descends freely by the force of gravity, and at the same time revolves uniformly about one of its extremities; required the equations to the surface traced out by it, and to the tangent plane at any point.

TUESDAY MORNING.—MR. HIND.

First and Second Classes.

1. Find the sum of the projections of the three sides of a plane triangle upon three planes at right angles to each other.

2. A body is projected perpendicularly upwards and the time between its leaving a given point and returning to it again is given; find the velocity of projection and the whole time of motion.

3. Given (a) the area corresponding to any rectangular co-ordinates in the figure of secants; to find the content of the solid generated by its revolution round the axis.

4. Sum the series

$$\sin^2 a + \sin^2(a + b) + \sin^2(a + 2b) + \&c. \text{ to } (n) \text{ terms.}$$

5. Through any point in the straight line joining the centre, and intersection of the tangents to any two points, of an ellipse two straight lines are drawn respectively parallel to its diameters, passing through the points of contact; prove that the triangles formed by these lines and the tangents are equal.

6. If λ be the true latitude of a place, and θ the latitude on Mercator's chart constructed to a rad. $= 1$; prove that

$$2\tan\lambda = (e^{\theta} - e^{-\theta}).$$

7. A globe of given weight and radius *rolls* down the surface of a hemispherical bowl from rest; find the velocity acquired at any point of its descent.

8. A chain suspended at its extremities from two tacks in the same horizontal line forms itself into a cycloid; prove that the density at any point $\propto \sec^3\left(\frac{\theta}{2}\right)$, and the weight of the corresponding arc $\propto \tan\left(\frac{\theta}{2}\right)$, θ being the arc of the generating circle measured from the vertex.

9. A small orifice is made in the side of an upright cylindrical vessel, and the vessel revolves about its axis with a given uniform velocity; find the path traced out by the fluid on the horizontal plane.

10. The bias and velocity of projection of a bowl are such as to cause it to describe a given logarithmic spiral; find the direction of projection, so that after having described a path of given length, it may impell the jack in a given direction.

11. Prove that the force by which a body may describe any of the conic sections round a centre of force in the vertex varies

inversely as the square of the distance, and directly as the cube of the secant of the angle which it makes with the axis.

12. Find the interval between the heliacal rising and setting of a given star, to a spectator in a given latitude.

13. The velocities of the different parts of a river vary as their distances from the bank; required the path described by a boat moving in a course inclined to the stream at an angle of 45°: also the velocity at any place, and the time of reaching a point at a given distance from the bank.

TUESDAY AFTERNOON.—MR. HIND.

Fifth and Sixth Classes.

1. Required the discount of £100 due three years hence, at 5 per cent. per annum, compound interest.

2. Find the values of x, y and z, which satisfy the equations

$$xy = a \,.\, (x + y),$$
$$xz = b \,.\, (x + z),$$
$$yz = c \,.\, (y + z).$$

3. If a fraction in its lowest terms be converted into a recurring decimal, prove that the number of figures which recur is always less than the denominator of the fraction.

4. Shew that the parameter belonging to any diameter of a parabola varies inversely as the square of the sine of the angle at which the corresponsing ordinates are inclined to it.

5. Divide the angle A into two parts, so that their versed sines may be in the given ratio of $m : n$.

6. A body projected in the direction of the action of a constant force, describes P and Q feet in the p^{th} and q^{th} seconds; find the magnitude of the force and the velocity of projections.

7. Tarce the curve whose equation is $\sqrt{y} = \dfrac{a}{\sqrt{x}} + \sqrt{x}$, and determine the position of its asymptote.

8. A parabola with its axis vertical, has its vertex coincident with the surface of a fluid in which it is immersed, divide it by lines parallel to the surface into four parts, so that the pressures upon them may be equal.

9. Differentiate $\dfrac{a}{a+x} + \dfrac{b}{b+x} - \dfrac{a+b}{a-b} \,.\, \log.\left(\dfrac{a+x}{b+x}\right)$, and find the value of $\dfrac{a^x \sin ax - b^x \sin bx}{c^x \,.\, \text{vers}\, cx - e^x \,.\, \tan ex}$, when $x = 0$.

10. Given one of the angles and the perimeter of a plane triangle, to find the sides, when the area is the greatest possible.

11. Prove that the semi-axes major and minor and the semi-latus rectum of an elliptic orbit are respectively an arithmetic, geometric, and harmonic mean between the aphelion and perihelion distances.

12. Integrate $\sin\theta \,.\, \cos\theta \,.\, \text{vers}\,\theta \,.\, d\theta$.

13. Two bodies begin to descend at the same instant down two given inclined planes from given points in the same vertical line; find their distance from each other at the end of any assigned time.

14. In any recurring equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, whose roots are a, b, c, &c.; prove that

$$\frac{a^2}{b^2} + \frac{b^2}{a^2} + \frac{a^2}{c^2} + \frac{c^2}{a^2} + \&c. = (p^2 - 2q + \sqrt{n}) \,.\, (p^2 - 2q - \sqrt{n}).$$

15. A body projected in a given direction with a given velocity and attracted towards a given centre of force, has its velocity at every point : the velocity in a circle at the same dist. :: $1 : \sqrt{2}$; find the orbit described, the position of its apse, the magnitude of its axis, and the law of force.

16. Prove that the image of a straight line formed by rays refracted through a sphere is a conic section; and find the magnitudes and positions of its axes and the latus rectum.

17. Find the sum of the series:

$$\frac{1}{1\,.\,2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} + \frac{1}{3\,.\,4\,.\,5} + \&c. \textit{ in infinitum.}$$

and of $\dfrac{nx^3}{1\,.\,2\,.\,3} + \dfrac{n\,.\,(n-1)x^4}{1\,.\,2\,.\,3\,.\,4} + \dfrac{n\,.\,(n-1)\,.\,(n-2)x^5}{1\,.\,2\,.\,3\,.\,4\,.\,5} + \&c.$ to (n) terms.

18. Find that section of a sphere which attracts a particle placed at a given point in the axis produced with the greatest possible force. Force of each particle $\propto \frac{1}{D^2}$.

TUESDAY AFTERNOON.—MR. KING.

Third and Fourth Classes.

1. Insert 6 harmonic means between 1 and 20.
2. Find $\sin\theta$ from the equation

$$\sin 3\theta - 2\sin 2\theta + \sin^2\theta + 4\sin^3\theta = 0.$$

3. If A, B and C be the angles, and a, b and c the sides of a spherical triangle, and if $b + c = \pi$, prove that

$$\sin 2B + \sin 2C = 0.$$

4. How far must a give frustum of a sphere be immersed in a fluid, with its axis vertical, that the pressure on its two ends may be equal to n times the pressure on its curve surface?

5. An eye is situated in a given point of the straight line joining the centres of two given spheres, but not between them; find the visible surface of each.

6. P_1 and P_2 be two sums due respectively at the end of times t_1 and t_2, prove that the equated time of payment is expressed by

$$\frac{P_1t_1+P_2t_2}{P_1+P_2} - \frac{P_1P_2}{(P_1+P_2)^2} \cdot \overline{t_1 - t_2}|^2 \cdot r, \text{ very nearly,}$$

where r is the interest of £1 for one year.

7. Having given the radius of a circle, find the area and perimeter of a regular octagon inscribed in it, and compare them with the area and perimeter of the circumscribing octagon.

8. Having given the contemporaneous altitudes of the Sun and a known Star on a given day and also the angular distance between them; find the latitude of the place and the hour of the day.

9. An uniform rod is made to vibrate about a point so that the time of its oscillation is a minimum; find the force exerted on the point of suspension in any given position.

10. Trace the curve whose equation is

$$x^4y^4 - a^4y^4 + b^8 = 0, \text{ and find its area.}$$

11. Integrate the following differentials:

$$\frac{dx \cdot \sqrt{a^2 + x^2}}{x}, \quad \frac{\overline{\sin x}|^5 \cdot dx}{\overline{\cos x}|^3}$$

$$(a^2y + x^3) \cdot dx + (b^3 + a^2x) \cdot dy = 0.$$

12. If a body describe an equilateral hyperbola, round a centre of force, situated in the centre, and if θ be the angle described by the body from an apse in time t, prove that

$$\sin 2\theta = \frac{e^{4\sqrt{k}\cdot t} - 1}{e^{4\sqrt{k}\cdot t} + 1},$$

the force at distance 1 being represented by k.

TUESDAY EVENING.—MR. KING.

1. Find the least whole numbers which satisfy the equation

$$11x - 18y = 63.$$

2. Prove that $n \,.\, \overline{n-1} \,.\, \overline{n-2} \,.\, \&c. \ldots \overline{n-r}$ is divisible by $1 \,.\, 2 \,.\, 3 \,.\, \&c. \ldots \overline{r+1}$, n being a whole number.

3. Determine the locus of a point so situated within a plane triangle, that the sum of the squares of the straight lines drawn from it to the angular points is constant; if the curve has a centre, determine its position.

4. Determine when the sum of the zenith distances of two known stars in a given latitude is a maximum.

5. One end of a beam is connected to a horizontal plane by a hinge about which the beam is suffered to revolve in a vertical plane; the other end is attached to a weight by means of a string passing over a pulley in the same vertical plane; find the position of equilibrium.

6. A parabola and hyperbola have the same vertex and the same axis; draw a tangent to the former which shall cut the latter in a given angle.

7. A cone containing a given quantity of fluid has its axis inclined to the horizon at a given angle; find the time of emptying through a small orifice in its vertex.

8. Having given six straight lines, of which each is less than the sum of any two determine how many tetrahedrons can be formed, of which these straight lines are the edges

9. Two rods of given length are erected perpendicular to a given plane; find the locus of an eye in that plane to which the sum of their apparent magnitudes will always be the same.

10. If A, B and C be the angles of a spherical triangle, a, b and c the opposite sides, and δ the distance of a point on the surface of the sphere equally distant from the angular points; prove that

$$\tan^2 \delta = \frac{\tan^2 \frac{a}{2} \,.\, \tan^2 \frac{c}{2} - 2 \tan \frac{a}{2} \,.\, \tan \frac{c}{2} \cos B}{\sin^2 B}.$$

11. A semicircle, the plane of which is vertical and base horizontal has an uniform chain of given length placed in a given position upon its circumference; find the velocity of the chain at the end of a given time.

12. If $ay^2 + bxy + cx^2 + dy + ex + f = 0$ be the equation to a curve of the second order; prove that the angles which its principal diameters make with the axis of x may be determined from the equation $\tan 2\theta = -\dfrac{b}{a-c}$.

13. It is required to graduate a horizontal dial, the style of which is in the meridian, and inclined to the horizon at a

given angle, so that on a given day it shall shew the apparent time in a given latitude.

14. Find the centre of pressure of the sector of a circle, the axis of the sector being supposed to be vertical.

15. Find that point in the surface of a given paraboloid through which if two planes be drawn, one perpendicular and the other parallel to the axis, the sum of the areas of the sections shall be a maximum.

16. Apply the differential equations of motion to determine the density of the medium so that a body may describe a given curve about a given centre of force: and the law of the density when the curve is a circle and a force is situated in circumference varying as $\frac{1}{D^n}$.

17. Two given weights are connected by a string passing through a hole in a horizontal plane; one of them is projected in any direction in the horizontal plane, the other descends vertically by the action of gravity; find the motion of the bodies and the curve described on the plane.

18. If a body be projected in a medium the resistance of which varies as the velocity and be acted on by gravity, and another be projected in vacuo at the same angle and with the same velocity, and acted upon by the same constant force, and if t_1 and t_2 be the times of describing two arcs in the medium and in vacuo, so related to each other that the tangents at their extremities shall cut the axis at the same angle; then $(e^{kt_1} - 1) = kt_2$, k being the resistance to velocity 1.

19. If a body describe an ellipse uniformly round two centres of force situated in the foci; prove that the forces at any point of the ellipse are equal, and inversely proportional to the square of the corresponding conjugate diameter.

20. Find the locus of the vertices of all right-angled spherical triangles having the same hypothenuse, and from the equation obtained prove that the locus is a circle when the radius of the sphere is infinite.

21. Three points move with equal velocities in three rectangular axes; one of them commences its motion from the origin, the other two from two given points equally distant from the origin; find the equation to the surface to which the plane passing through the contemporaneous positions of the points shall always be a tangent.

22. Integrate the following differentials:

$$\frac{x^2 dx}{(x+a)(x^2+b^2)}, \quad \frac{dx}{x^5+2x^3+3x},$$

$$x^3ydx - y^3dy = x^3dy,\ x^3 \cdot \frac{d^2y}{dx^2} - x \cdot \frac{dy}{dx} + y = 0,$$

$$z - px - qy = 3a \cdot p^{\frac{1}{3}} \cdot q^{\frac{1}{3}},\ \text{where } p = \left(\frac{dz}{dx}\right),\ \text{and } q = \left(\frac{dz}{dy}\right).$$

23. Sum the following series:

$$\frac{1}{1 \cdot 3 \cdot 5} + \frac{2}{3 \cdot 5 \cdot 7} + \frac{3}{5 \cdot 7 \cdot 9} + \text{\&c. to } n \text{ terms and to infinity.}$$

$$\frac{1^2}{1^2 - k^2} \cdot \cos\theta + \frac{2^2}{2^2 - k^2} \cdot \cos 2\theta + \frac{2^2}{3^2 - k^2} \cdot \cos 3\theta + \text{\&c. to infinity.}$$

24. Integrate the following equation of differences:

$$u_{x+1} - au_x = x^2.$$

THE SENATE HOUSE PROBLEMS,

Proposed by the Two Moderators, to the Candidates for Honors, during the Examination for the Degree of B. A.

MONDAY, JANUARY 22, 1827.

MONDAY MORNING,—MR. MADDY.

First and Second Classes.

1. THERE were 5 Sundays in February, 1824, in what year will this happen again?

2. If C be the centre of an ellipse, and in the normal to any point P, PQ be taken equal to the semi-conjugate at P, Q will trace out a circle round C.

3. A cord, the ends of which are joined, is suspended freely over two pegs in the same horizontal line, so as to form two catenaries, of which the arcs are $2s$ and $2s'$, and the tensions at the lowest points a and a'; prove that

$$s - s' : a' - a :: a' + a : s + s'.$$

4. Draw through a given point in the side of a spherical triangle, an arc of a great circle, cutting off a given part of the triangle.

5. If u be a homogeneous function of x, y, z, &c. of n dimensions, and p, q, r, &c. the values of $\frac{du}{dx}$, $\frac{du}{dy}$, &c.

$$(n-1)\, du = x\, dp + y\, dq + \text{\&c.}$$

6. The curve surface of a conical vessel of water, placed with its axis vertical and base uppermost, is composed of an infinite number of triangular sectors, admitting of revolution round hinges at the vertex of the cone, and confined by a string passing round the base; find the tension of this string.

7. Find when the inclination of the ecliptic to the horizon increases fastest.

8. If an equation has n equal roots, the equation formed by multiplying the terms by the terms of an arithmetical progression has $(n - 1)$ of them.

9. A ray of light which passes through two media, bounded by parallel plane surfaces, will emerge parallel to its first direction, if the deviation in passing out of one medium into another under a *given* angle of incidence be supposed proportional to the difference of the densities of the media.

10. A body projected from a given point in a plane is attracted by forces $\frac{\alpha}{x^3}$ in the direction of x, and $\frac{\beta}{y^3}$ in the direction of y; prove that if the velocity and direction of projection be rightly assumed, it will describe a circle round the origin as a centre, and find how the areas described in equal times vary.

11. There are two heaps of cards, one of which contains three black and four red ones, and the other five black and two red; what is the probability that a person who takes up one card will draw a red one.

12. What probable and adequate cause has been assigned for the secondary planets always turning the same face towards their primaries?

13. If all possible ellipsoids be described of which the axes a, b, c are subject to the condition $c : a :: a : b :: b :$ a given line (n); shew how the equation to the surface they all touch may be found, and find the equations of the curve in which it is touched by any one of the ellipsoids, and of the curve which is the locus of the intersections of all such curves. The centre and the directions of the axes being the same in all the ellipsoids.

MONDAY AFTERNOON.—MR. MADDY.

Fifth and Sixth Classes.

1. Sum the series $15 + \frac{44}{3} + \frac{43}{3} + \&c$ .. to 16 terms,

$$x^{\frac{3}{2}} - ax + \frac{a^2}{\sqrt{x}} - \&c. \text{ .. to } n \text{ terms,}$$

and find the value of the recurring decimal 1.23434....

2. If a body be projected downwards with a given velocity, what is its velocity after describing a given space?

3. Find the differentials of log (sin x) and $\frac{\sqrt[n]{(1+x^2)^3}}{\sqrt[n]{(1-x^2)^2}}$, and the integrals of

$$\frac{dx}{\sqrt{ax-bx^2}},\quad \frac{x^2dx}{x^2+1},\quad \text{and } e^{\log x}dx.$$

4. Solve the equations $\left.\begin{array}{l}(a^2)^x-(b^2)^y=c\\ a^x-b^y=d\end{array}\right\}.$

$$x-3=\frac{3+4\sqrt{x}}{x}.$$

5. A body weighs four ounces in vacuo, and if another body which weighs three ounces in water be attached to it, the whole in water weighs two ounces and a quarter, find the specific gravity of the former body.

6. Compare the difference of the forces in the fixed and moveable orbits, with the force in a circle at the same distance described with the same angular velocity.

7. There are two air pumps, one with a receiver A and barrel B, the other with a receiver B and barrel A; compare the quantities of air exhausted by them in t turns.

8. If a star whose right ascension is 19° 25′ pass over the meridian 2h 18′ of sidereal time before the Sun, what is the Sun's right ascension when on the meridian?

9. Given the base, the vertical angle, and the difference of the sides of a plane triangle; find the remaining angles.

10. If $y=x^3-2x^2+x+4$, find the maximum and minimum values of y, distinguish them from each other, and shew that they are not the greatest and least values that y admits of.

11. Trace the curve whose equation is $y=\sqrt[3]{\frac{a^4}{x}}-x$, and find the area and the solid formed by its revolution from $x=0$ to $x=a$.

12. Expand sin x in powers of x, state whether in your result x is expressed in seconds or in parts of the radius, and convert it into the other.

13. Given the radius vector at any point of a parabola, and the angle it makes with the curve; find the latus rectum and the place of the vertex.

14. Find the pressure which a given power exerts by means of a common vice, the dimensions of which are given.

15. Extract the square roots of $7+4\sqrt{3}$ and $2\sqrt{-2}-1$.

16. Prove that an object seen through the astronomical telescope appears inverted, but may be made erect by two additional glasses, and find the magnifying power of the telescope thus formed.

17. Explain the Moon's phases, and why part of the disk is always visible.

18. To what depth will a given paraboloid placed with its axis vertical sink in a fluid of three times the specific gravity of itself?

MONDAY AFTERNOON.—MR. CODDINGTON.

Third and Fourth Classes.

1. According to what law must the centripetal force vary, that the areas *dato tempore* in all circles uniformly described about the centre may be equal?

2. What are the lines traced by the vertex and the focus of a parabola rolling on another equal to it, the vertices coinciding in one position?

3. At what distance from a luminous sphere must a point be situated so as to receive the greatest quantity of light from it?

4. How is the multiplication of two high numbers facilitated by a table of squares, or by one of cosines?

5. Prove that a spheroidal mirror may be made to reflect diverging rays accurately to one point: and by the help of this proposition find the common formula for a spherical mirror.

6. The times of the Sun's rising and setting being calculated for a certain place, what correction is necessary to make them serve for another place not far distant from it?

7. Prove, that when a mass entirely free is struck at any point, the motion of a translation is the same as if the direction of the impact passed through the centre of gravity, and that of rotation as if the centre were fixed by an axis. By this proposition find the distance of the centre of percussion from a fixed axis.

8. Compare the volume of a sphere with that of the least cone that can be described about it.

9. The style of a horizontal dial being bent down, its edge coincides with the 9 o'clock hour line. For what latitude was it constructed?

10. Supposing the Sun to remain above the horizon a given number of days, find the latitude.

11. Differentiate the continued fraction $\cfrac{x^2}{1 - \cfrac{x^2}{1 - \cfrac{x^2}{1 - \dots}}}$

12. Find, geometrically, the inclination of the path of a projectile to the horizon at a given time after the beginning of its motion, and prove by that means that the trajectory is a parabola.

MONDAY EVENING.—MR. CODDINGTON.

1. Explain the method of geometrical analysis, and by it solve the problem: In a given square to inscribe another square having its side equal to a given straight line. To what limitation is this line subject?

2. In how many ways can an equivalent for thirteen dollars, at three shillings, be given in English crowns and seven shilling pieces?

3. Shew that in general a parabola may be found which shall have a much more intimate contact with a given curve than any circle whatever.

4. If L be the length in miles of an arc of a great circle of the Earth, D the depression of it below a tangent drawn at the other, $D = \frac{2}{3}L^2$ nearly.

5. The voussoirs of a bridge being very small, and the equilibrium maintained by a vertical pressure of the masonry above them, of what portion of a circle must the intrados consist, that the ascent of the bridge from a level road may be continuous, supposing the thickness of the arch at its summit to be the to radius of the intrados as 1 to $5\frac{1}{3}$?

6. Prove that Brinkley's formula for the mean refraction is reducible to the same form as Bradley's.

7. Shew that Saturn's ring cannot be a homogenous and regular *solid* of revolution.

8. Given the position of the Moon's nodes, and the inclination of her orbit to the ecliptic, to find when her latitude and declination are equal.

9. An uniform elastic string being of such a length that when it hangs vertically if an equal quantity were appended to the lowest point it would stretch it to twice that length, what weight must be appended at the middle point that the increase of length may be three quarters of the original?

10. In a chart on Mercator's projection the length of the meridian from the parallel of 30° to that of 60° is to the radius of the sphere as the natural logarithm of $\frac{\sqrt{3}+1}{3-\sqrt{3}}$ to 1.

11. Show that with any four lines, each of which is greater than the sum of the others, it is possible to construct a trapezium which may be inscribed in a circle.

12. How may the square of 50978 be found approximately by a table of squares of whole numbers from 1 to 1000?

13. If *AB* be the diameter of a circle, *C* the centre, *FAG* a tangent at *A*, *ACF* one-third of a right angle, and *FG* triple of the radius, *BG* being joined will be very nearly equal to the half circumference.

14. Compare the probabilities of taking an odd or an even number of balls from a heap containing a given number.

15. How must the equidistant seats of a lecture room be constructed, so that persons of equal height sitting on them, and directing their eyes to the same given point may be equally able to see over each other's heads?

16. Find the disturbing forces of Venus on the Earth when their heliocentric longitudes differ by 45°.

17. If the refraction of light be the effect of any causes which act throughout a certain distance from the surface of a medium, and the intensity of which depends solely on the distance from that surface, the ratio of the sines of incidence and refraction must be constant.

18. Supposing the latitude of a star to be 60°, its longitude 95°, and that of the Sun 65°, what is the aberration in longitude? In what sense is 20″.25 the maximum of aberration?

19. In a stereographic projection of the sphere it is required to draw a great circle through two given points.

20. Explain the manner of using Gunter's logarithmic scales.

21. Solve the equation $\phi x^3 = \phi(2x) + 2$.

22. Find two whole numbers of which the product shall be divisible by the sum.

23. Given $z = x + e^z$. Required z in terms of x.

24. Explain the division of a string which produces the several notes of the diatonic scale. What alteration would be made in the general pitch by assuming 256 instead of 240 for the number of vibrations constituting the tenor *C*?

TUESDAY MORNING.—MR. CODDINGTON.

First and Second Classes.

1. Explain the method of finding the refracting power of a soft substance by placing it between glass lenses.

2. Of all conical surfaces of equal altitude, determine that which exerts the greatest attraction on a particle at its vertex.

3. A circle being described on the axis major of an ellipse, and a tangent drawn to each curve at the points where an ordi-

nate to the axis meets them, find where the angle between these tangents is greatest: and show what is the ultimate point of contact in this case, when the eccentricity of the ellipse is diminished *sine limite.*

4. Prove the rule for multiplying decimals together without any reference to vulgar fractions.

5. If two bodies S and P attracting each other with forces varying inversely as the square of the distance, revolve about each other, S being much greater than P, the actual time of one revolution will be less than if S were immoveable in the ratio of 1 to $1 + \frac{P}{3S}$.

6. On the supposition of a homogeneous atmosphere, the refraction may be expressed by the formula

$$r = \frac{m-1}{\sin 1''} \tan\left(Z - \frac{\delta}{(m-1)(1+\delta)}\, r\right).$$

δ being the ratio of the height of the homogeneous atmosphere to the radius of the Earth.

7. Find the equation to a conical surface in general, and deduce from it that of a common right cone.

8. In a plane triangle ABC, when the side b is much less than a, the angle B may be found by the formula

$$B = \frac{b}{a} \cdot \frac{\sin C}{\sin 1''} + \frac{b^2}{2a^2} \cdot \frac{\sin 2C}{\sin 2''} + \frac{b^3}{3a^3} \cdot \frac{\sin 3C}{\sin 3''} - \ldots.$$

9. What curve is that in which the perpendicular from the origin on the tangent is always equal to the abscissa?

10. Find the value of the fraction $\frac{3 \cdot 5 \cdot 9 \cdot 17 \ldots}{2 \cdot 4 \cdot 8 \cdot 16 \ldots}$ when its numerator and denominator are continued *sine limite.*

11. If S be half the sum of the sides of a spherical triangle, D its area,

$$\tan\frac{D}{4} = \sqrt{\left\{\tan\frac{S}{2} \cdot \tan\frac{S-a}{2} \cdot \tan\frac{S-b}{2} \cdot \tan\frac{S-c}{2}\right\}}.$$

12. Supposing the orbit of a comet to lie in one plane, if the force attracting it towards the Sun vary as that power of the distance whose index is $2 - \delta$ (δ being a small fraction) the heliocentric angle between two successive perihelia will be $360° \cdot \frac{\delta}{1+\delta}$.

TUESDAY AFTERNOON.—MR. CODDINGTON.

Fifth and Sixth Classes.

1. Prove geometrically that if A, B be any two arcs, rad × cos $(A - B) = \cos A \,.\, \cos B + \sin A \,.\, \sin B$.

2. The length of a floor being 10 feet 6 inches, and the breadth 9 feet 3 inches, find the area by duodecimal multiplication, and define the several terms of the product.

3. The second and third terms of a geometrical progression are together equal to 24, and the two next to 216: what is the first?

4. Investigate the rule of Alligation, in which the prices of the ingredients and of the mixture are given, to find the proportions of the former.

5. Given a, b, c the sides of a plane triangle, find the radius of the inscribed circle.

6. Draw a straight line touching a circle at a given point, without any other instruments besides a parallel ruler and a pencil.

7. By what modification may formulæ for spherical triangles be adapted to plane ones?

8. If n be any prime number greater than 3, $n^2 - 1$ is divisible by 12.

9. Given the altitudes of two known stars at the same instant of time: required the latitude of the place. How may this problem be solved geometrically on a sphere?

10. If two elastic balls in the ratio of 1 to 3 meet directly, the larger one will remain at rest.

11. Continue in both directions the harmonic progression of which 4 and 6 are adjacent terms.

Prove geometrically that $\frac{1}{2}(ydx - xdy)$ is the element of a sectorial area about the origin of the co-ordinates.

13. Compare the force at a given point of an ellipse, described about the focus, with that in a circle at the same distance described with the velocity in the ellipse at that point.

14. Two given weights being attached to given points in the circumference of a wheel, find the position in which the greatest weight will be supported on the axle.

15. Prove the following formula for small arcs,

$$l \sin x = lx + \tfrac{1}{3} l \cos x.$$

16. A pair of conjugate hyperbolas being given, find their centre.

17. If $\dfrac{dx}{1+x^2} + \dfrac{dy}{1+y^2} = 0$, find an algebraical value of y in terms of x.

TUESDAY AFTERNOON,—MR. MADDY,

Third and Fourth Classes.

1. Convert 17° 25′ 8″ into time at the rate of 15° to one hour.

2. If $C_0 = \cos a \,.\, \cos b \,.\, \cos c \,\ldots$, $C_n =$ the sums of the products of all the cosines but n multiplied by the sines of those n

$$\cos(a+b+c+\&c.) = C_0 - C_2 + C_4 - \&c.$$
$$\sin(a+b+c+\&c.) = C_1 - C_3 + C_5 - \&c.$$

3. Find at what angle a plane must be inclined to the side of a cone in order that the section may be a rectangular hyperbola; and determine the least vertical angle of the cone for which the problem is possible.

4. Shew that a body projected on an inclined plane, but not in the direction in which it would naturally fall, describes a parabola, and find its latus rectum, having given the inclination of the plane, and the velocity and direction of projection.

5. In the ordinate PN of a parabola πN is taken proportional to the curvature of the parabola at P; find the whole area of the curve which is the locus of π.

6. If $\tan\theta$ be assumed $= \frac{b}{a}$,

$$\sqrt{a \pm b\sqrt{-1}} = (a^2+b^2)^{\frac{1}{2n}} \left\{\cos\frac{\theta}{n} \pm \sin\frac{\theta}{n}\sqrt{-1}\right\}.$$

7. Find where the space due externally to the velocity in an ellipse, force in focus, is thrice the space due internally.

8. Trace the curve whose equation is $a^{\frac{3}{2}} y = (x-a)^2 \sqrt{x}$.

9. What must be the form of a surface of revolution in which, when filled with water, which runs out by a small orifice at the lowest point, the surface descends from its greatest altitude with an uniformly accelerated motion?

10. A small pencil of rays parallel to the axis of a hemisphere of denser medium, is incident nearly perpendicularly on the convex surface; find the geometrical focus of emergent rays.

11. If ordinates $y_1, y_2 \ldots . y_n$ be drawn, at equal intervals, beginning from the origin, to the catenary whose equation is $\frac{y}{a} = \frac{1}{2}.\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$; prove that

20. Integrate the differentials and differential equations

$$\frac{x \log x \,.\, dx}{\sqrt{1-x^2}}, \quad \frac{dx}{\sqrt[3]{1+x}-\sqrt{1+x}}, \quad \frac{d^n y}{dx^n}=y,$$

$$dx + 2xdy = x^2y^2dy.$$

21. Sum the series $\cos x - \cos 2x + \cos 3x -$ &c. *ad infinitum*.

$$\frac{1}{2\cdot3\cdot5}+\frac{2}{3\cdot5\cdot9}+\frac{4}{5\cdot9\cdot17}+\frac{8}{9\cdot17\cdot33}+\text{\&c. to } n \text{ terms},$$

and find the integral of $a^x \,.\, (1+x)^2$.

22. If x and y be co-ordinates of any point of the shortest line drawn between two given points on a surface formed by the revolution of a plane curve round the axis of z, and ds the differential of the line, prove that $xdy - ydx = cds$, c being some constant; and find the equations to the line when the surface is a paraboloid.

23. If the particles of a hollow elastic cylinder be so arranged that on its being subjected to a given internal pressure they may all be in the same given degree of dilation, shew that in order for the strength of the cylinder to increase in arithmetical progression its thickness must increase in geometrical.

24. Prove that there are generally either two homogeneous fluid spheroids of equilibrium or none, for the same time of rotation, and supposing the eccentricity of the one spheroid very small, find the ratio of the axes in the other*.

Questions Proposed by PROFESSOR AIRY, *at the* EXAMINATION FOR SMITH'S PRIZES.

JANUARY 22, 1827.

Five Hours allowed.

[*It is expected that the numerical calculations will be completed.*]

1. PROVE that

$$\Sigma u_x = \int_x u_x - \frac{1}{2}u_x + \frac{1}{12}\cdot\frac{du_x}{dx} - \frac{1}{720}\cdot\frac{d^3u_x}{dx^3} + \frac{1}{30240}\cdot\frac{d^5u_x}{dx^5} - \&c$$

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* In an oblate spheroid whose minor axis is b, eccentricity e, and density k, the attractions on a particle at the pole and at the equator are respectively

$$4\pi kb \,.\, \left\{\frac{1}{e^2} - \frac{\sqrt{1-e^2}}{e^3}\sin^{-1} e\right\}$$

$$2\pi kb \left\{\frac{1}{e^3}\sin^{-1} e - \frac{\sqrt{1-e^2}}{e^2}\right\}.$$

the coefficients being the same as those in the expansion of $\frac{1}{e^x - 1}$: and sum the series

$$\frac{1}{21} + \frac{1}{25} + \frac{1}{29} + \text{\&c. to 820 terms.}$$

2. By the observations of Captain Sabine, the length of the second's pendulum at Sierra Leone, latitude 8° 29′ 28″, is 39.01997 inches: that at Drontheim, latitude 63° 25′ 54″, is 39.17456 inches. Calculate from these data the ellipticity of the Earth.

3. Given the dispersive powers of two kinds of glass; find the relation between the focal lengths of two lenses, whose combination will produce an achromatic object-glass, 1st when the lenses are in contact, 2nd when they are separated by a given interval.

4. Express differences in terms of differential coefficients, and calculate to 7 decimals, the 1st, 2nd, 3rd, and 4th differences of the common logarithm of 1000.

5. Explain why the gnomon of a dial must be parallel to the Earth's axis.

6. A vertical dial for latitude 52° 12′ 43″ N. faces a point of the horizon 15° west of the south point: find the angle made by the hour-line of 4^h P. M. with the vertical.

7. State what is meant by the *great inequality* of Jupiter and Saturn: and mention the peculiarities in the motions of those planets to which it is owing.

8. Prove that the portion of the sky seen within the primary rainbow, *cæteris paribus*, is brighter than that seen without it.

9. Explain the transformation of the independent variable: and transform the equation

$$0 = \frac{d^4y}{dx^4} - \frac{4}{x} \cdot \frac{d^3y}{dx^3} + \frac{12}{x^2} \cdot \frac{d^2y}{dx^2} - \frac{24}{x^3} \cdot \frac{dy}{dx} + \frac{24}{x^4} y,$$

where x is the independent variable, into one where z is the independent variable, z being = common log. x.

10. The object-glass of an astronomical telescope has 7 feet focal length: its eye-piece consists of two lenses of 1 and 3 inches focal length respectively, separated by an interval of 2 inches: the lens of 3 inches focal length, which is nearest to the object-glass, is distant from it 6 feet 10½ inches; find the magifying power.

11. Explain what is meant by the variation of parameters, and apply it to solve the equation $\frac{d^4y}{dx^4} - y = a^x$.

12. The N.P.D. of Regulus is 77° 11′ 40″; his right ascension $9^h\ 59'\ 11''.4$: his corrected altitude 33° 17′, E. of the meridian; as seen at Cambridge Observatory, whose latitude = 52° 12′ 43″. Find the right ascension of the point of the equator on the meridian at the instant of the observation.

13. In the example above, find the error produced in the result by an error of 11″ in the altitude.

14. Explain the precession of the equinoxes, and the nutation of the Earth's axis, without analogical reference to the disturbances produced by the Sun in the Moon's motion: and point out a method of calculating theoretically the coefficients.

15. Prove the process for taking an integral between limits; and find the value of $\int_x \frac{1}{x^3+1}$ between the limits $x = 0$, $x = 1$.

16. In an air-pump, similar in construction to the common water-pump, an interval is left between the piston and the lower valve at the lowest position of the piston; find the density of the air in the receiver after n strokes, and after an infinite number.

17. In finding the sine of half an arc, shew that, when θ is small, a large error may be expected in applying the formula $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$, and a small one in using the formula $\sin\frac{\theta}{2} = \frac{1}{2}\sqrt{1+\sin\theta} - \frac{1}{2}\sqrt{1-\sin\theta}$.

18. Give the method of determining the variation of the elements of the planetary orbits arising from the mutual attraction of the planets

19. Find the alteration of gravity at the top of a conical mountain 3 miles high, and 24 miles in diameter: the Earth being supposed a uniform sphere whose diameter = 8000 miles.

20. In throwing a quoit, it is found necessary to give it a motion of rotation in its plane, in order that it may preserve its parallelism during its motion. Explain this on mechanical principles.

21. Every point of a fluid is acted on by equal and constant forces directed to two centres: find the figure which it will assume.

22. The evolute of a curve is similar to the curve itself: find an equation to the curve.

23. Give the necessary theorems for constructing Mercator's chart for an oblate spheroid of small ellipticity.

24. Investigate theoretically the quantity of astronomical refraction, the density of the atmosphere being variable.

14. The height of a homogeneous atmosphere is the same for whatever distance above the Earth's surface we find it.

15. Compare the resistance on the arc of a plane curve moving in a fluid in the direction of its axis, with the resistance on the base; and apply the formula to the case of a semicircle.

16. Given three altitudes of a known star observed very near the meridian, and the differences of the times of observation; determine the latitude of the place.

17. Describe the manner in which Bradley discovered Aberration, and in which he distinguished Nutation from it.

18. If z be the true zenith distance, P the horizontal parallax, p the parallax in seconds,

$$p = \sin P \cdot \frac{\sin z}{\sin 1''} + \sin^2 P \cdot \frac{\sin 2z}{\sin 2''} + \sin^3 P \cdot \frac{\sin 3z}{\sin 3''} + \&c.$$

EVENING PROBLEMS.

1. If to the square of any number not divisible by 3 the number 2 be added, the result will be divisible by 3.

2. Reduce $a\sqrt[4]{-1} + b\sqrt[8]{-1}$ to the form of $\alpha + \beta\sqrt{-1}$.

3. Two straight lines are inclined to each other at a given angle, find the area of all the circles which can be described touching each other and the two given lines, the position of the centre of the last circle being given.

4. Find the m^{th} differential coefficient of $\sqrt{\cos x}$.

5. Find that point in the surface of a spherical triangle from which if straight lines be drawn to the angular points the pyramid thus formed shall be a maximum.

6. Find the equation to the curve from any point of which if two tangents be drawn to a given ellipse, the angle contained between them shall be constant.

7. Having given the latitude of the place, find the day of the year on which the shadow of a given ellipse placed perpendicular to the meridian with its major axis vertical will be an ellipse of half the excentricity.

8. The interval between the passages of a known circumpolar star through the plane of a vertical instrument with a given azimuth is observed; find the latitude of the place and the area of the spherical surface contained between the vertical circle and the apparent path of the star.

9. A uniform rod is at liberty to move freely in a vertical plane about a horizontal axis; find the nature of the circumference of a wheel which revolving uniformly about a given horizontal axis shall cause the rod to revolve uniformly also: the point of con-

tact of the wheel and the rod being always at the same distance from the point of suspension.

10. Perpendiculars are drawn from a given point upon an infinite number of planes all passing through another given point; find the locus of the intersections of the perpendiculars with the planes.

11. Apply the differential expression for the volume of any solid referred to three rectangular co-ordinates to find the volume of a portion of a paraboloid whose equation is

$$x^2 + y^2 = 2az$$

cut off by a plane whose equation is

$$By - Cz = 0.$$

12. A heavy piston descends by its own weight in a close cylinder filled with atmospheric air; find the velocity at any point of its descent and shew how to approximate to the whole length of the oscillation.

13. The particles of a fluid mass are attracted to two equal constant centres of force and a uniform motion of rotation is given to the mass about the line joining those centres; find the equation to the surface which the fluid will assume.

14. A hexagonal pyramid whose sides are isosceles triangles is placed with its base on the plane of x and y; find the sum of the projections of its sides on the three co-ordinate planes.

15. A ring slides down a perfectly smooth rod revolving uniformly in a vertical plane; find the motion of the ring.

16. An oblique parallelopiped oscillates about one of its edges which is in a horizontal position; determine its motion and the pressure it exerts against the axis of suspension in any position.

17. If an ellipsoid, whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, be cut by a plane passing through the origin perpendicular to the plane of xy, prove that the normals to the surface drawn from every point of the intersection of the plane with the ellipsoid will cut the plane of xy in a straight line, and find the equation to that line.

18. Explain the use of observations made by reflexion on the Polar Star in adjusting a transit instrument.

19. Prove that $1 \,.\, 2 \,.\, 3, \&c.\; x = \sqrt{2\pi x} \,.\, \left(\frac{x}{e}\right)^x$ very nearly when x is large, and shew the utility of this formula in the solution of the following problem.

20. In a pack of 52 cards, containing an equal number of red and black cards, determine the probability that in drawing any even number of cards, there shall be an equal number of red and

black cards: supposing that the probability of drawing any even number is the same. A numerical result is required.

21. Solve the following equation of differences,

$$u_{x+2} + 2au_{x+1} + a^2 u_x = Ax^3 + Bx^2 + Cx.$$

22. A particle is placed any where within a triangle, the sides of which are composed of particles attracting with forces varying as $\frac{1}{D^2}$; find the direction in which it will begin to move.

23. If a body oscillate in a cycloid, in a medium the resistance of which varies as the velocity, and s be the first arc of descent, prove that the whole space described by the body before the motion ceases

$$= s \cdot \frac{e^{\frac{k\pi}{\sqrt{\left(\frac{g}{4a} - k^3\right)}}} + 1}{e^{\frac{k\pi}{\sqrt{\left(\frac{g}{4a} - k^2\right)}}} - 1}$$

where $2k$ = resistance to velocity 1, g = gravity, $2a$ = diameter of the generating circle, and π = the semi-circumference of a circle the radius of which is 1.

24. Shew how very small secular inequalities in the mean motion of two planets may be introduced when their mean motions are nearly commensurable.

MORNING PROBLEMS.

1. If n be greater than 3, shew that $\sqrt{n} > \sqrt{n+1}$.

2. If any two circles, the centres of which are given, intersect each other, the greatest line which can be drawn through either point of intersection and terminated by the circles is independent of the diameters of the circles.

3. If a and b be the semi-axes of an ellipse, and θ and ϕ the angles which any two conjugates make with the major axis, prove that $\tan\theta \cdot \tan\phi = \frac{b^2}{a^2}$.

4. Sum the series

$$\left.\begin{array}{l} \frac{1}{1\cdot 3} \cdot \frac{1}{2^2} - \frac{1}{2\cdot 4} \cdot \frac{1}{2^4} + \frac{1}{3\cdot 5} \cdot \frac{1}{2^6} + \&c. \\ \cos\theta - \frac{1}{2^2} \cdot \cos 2\theta + \frac{1}{3^2} \cdot \cos 3\theta - \&c. \end{array}\right\} \textit{ad infinitum.}$$

5. Given the radius of the circumscribed circle and the three angles of a triangle; find expressions for the three sides.

6. Develope $\sin \overline{\alpha + \beta x + \gamma x^2}$ in a series of the form $A + Bx + Cx^2 + Dx^3 + \&c.$

7. In an ellipse in which the semi-axes are CA, CB, and the abscissa and ordinate CM and MP; in MP take

$$MQ = \frac{CB^2}{CM + MP},$$

trace the curve which is the locus of Q; find its maximum and minimum ordinates, and the angles made by its two extremities with the axes.

8. Integrate the differentials

$$\frac{x^4\,dx}{\sqrt{2ax - x^2}}, \quad \frac{\sin^2 x\,dx}{\cos^4 x}, \quad e^{az} \cdot \cos^m z\,dz,$$

and the differential equations

$$\sqrt{x}dx + \sqrt{y} \cdot dy = \sqrt[4]{xy} \cdot dy, \quad \frac{d^2y}{dx^2} + \frac{y}{x^2} = x^3 + ax^2,$$

9. Give a construction depending upon the cycloid, for determining an arc equal to its cosine.

10. State the most recent and approved experiments, whereby it is ascertained that the decrement of velocity arising from friction is the same for all velocities.

11. A bent lever, of which the arms are a and b and the angle θ, makes small oscillations in its own plane; the length of the isochronous simple pendulum is

$$\frac{2}{3} \cdot \frac{a^3 + b^3}{\sqrt{a^4 + b^4 + 2a^2b^2 \cdot \cos\theta}}.$$

12. Find the actual velocity of the point P, (Newt. Sect. vii. Prop. 39.) the force tending to C being supposed to vary directly as the distance.

13. The force in an orbit $\propto \frac{m}{\sqrt{a^2 + r^2}}$, where r is the radius vector; find the angle between the apsides when the orbit is nearly circular.

14. Why do objects appear further off and smaller, when viewed through the wrong end of a telescope?

15. A given hemispherical vessel filled with fluid is whirled round its vertical axis, so that the surface of the fluid which remains, touches the lowest point of the hemisphere; find the angular velocity, and the quantity of fluid remaining.

16. Compare the portion of the surface of the Earth illuminated by the Sun in perigee with that illuminated in apogee; taking into account the magnitude of the Sun.

17. A given quantity of matter is to be formed into a cone; find its form, that its attraction on a particle at its vertex may be a maximum, the attraction of each particle varying as $\frac{1}{D^2}$.

18. If the true centre of the Moon's orbit move uniformly in a circle about the mean centre, the result is a change of the Moon's place of the form $m \,.\, \sin 2 \,.\, \overline{(\text{☽} - \odot) - A}$, where A is the Moon's anomaly.

MONDAY AFTERNOON, 1 o'clock to 3.

First and Second Classes.

1. Explain the method of drawing a tangent plane at any proposed point of a given curve-surface, and find the equation to that plane in the case of an ellipsoid.

2. Explain the relation which exists between the curves which the complete integral of any differential equation of the first order represents, and that which is defined by a particular solution of that equation; and shew how the particular solution may be deduced from the complete integral. Exemplify in the case where $y = x \,.\, \tan\alpha - \frac{x^2}{4h \,.\cos^2\alpha}$ is the complete integral, (α) being the arbitrary constant.

3. Prove the following theorem in finite differences;

$$\Sigma u_x v_x = u_x \Sigma v_x - \Delta u_x \Sigma^2 v_{x+1} + \Delta^2 u_x \Sigma^3 v_{x+2} - \&c.$$

and apply it to find the sum of x terms of the series

$$1 \,.\, 2 \,.\, 3a - 2 \,.\, 3 \,.\, 4a^2 + 3 \,.\, 4 \,.\, 5a^3 - \&c.$$

4. State and explain D'Alembert's principle, and apply it to determine the pressure on the axis about which a body revolves when acted on by a single force in a plane perpendicular to the axis.

5. Give Newton's construction for determining the path of a projectile acted upon by gravity in a medium whose resistance $\propto$ velocity, and apply the differential equations of motion to determine the actual equation.

6. Shew from Newton's construction for determining the horary increment of the area described by the Moon in a circular orbit round the earth at rest, that the velocity generated by the tangential ablatitious force between quadrature and syzygy is to that which would be generated in the same time by the mean addititious :: $3 : \pi$.

7. Investigate the general equation of equilibrium of any

fluid: and shew from the equation that the resultant of the forces at any point in the surface of a fluid incompressible and perfectly free, is a normal to the surface.

8. If the whole force at the pole of an oblate spheroid be to that at the equator as the equatorial radius to the polar, and to any point within the spheroid canals of any form be drawn, the pressure on that point will be the same whatever be the form or direction of the canal.

9. Apply Lagrange's Theorem to the determination of the three first terms of the series expressing the true anomaly in terms of the mean, the series ascending by powers of (ϵ) and the anomaly being measured from the perihelion.

MONDAY AFTERNOON, 1 o'clock to 3.

Third and Fourth Classes.

1. Shew that the limiting equation has at least as many possible roots as the original equation, wanting one; and determine the nature of the roots of the equation

$$x^7 - a^5 x^2 + c^7 = 0.$$

2. State Napier's rules for the solution of right-angled spherical triangles, and prove the two cases in which the complement of the hypothenuse is the middle part.

3. Shew generally how to find the evolute of any curve whose equation is given, and find that of the common parabola.

4. Solve the differential equation

$$(a + bx + cy)\, dx = (a' + b'x + c'y)\, dy.$$

5. When any number of bodies move uniformly in straight lines in different planes, their centre of gravity also moves uniformly and in a straight line.

6. Having given the moment of inertia round any axis passing through the centre of gravity of a body, to determine that round any axis parallel to the former.

7. To determine the horary motion of the Moon's nodes in a circular orbit. (Newton, Book III. Prop. 30.)

8. If the object placed before a spherical reflector be a straight line, the image will be a conic section. Prove this, and shew how the different parts of the image are formed, when the object is placed between the principal focus and the surface of the reflector.

9. A globe is projected vertically upwards with a given velocity c, in a medium where the resistance is $= k \times (\text{vel.})^2$, and is acted on by gravity; determine the relation between the time, space, and velocity.

10. Given three distances of a planet from the Sun, and the corresponding arguments of latitude, to find the place of the perihelion, and the true anomaly at the first observation.

TUESDAY MORNING, 9 o'clock to 11.

First Class.

1. Given a solution of a differential equation of the first order, find whether it is included in the complete integral.

2. Given the solution of the equation $\frac{d^2u}{dv^2} + u = 0$, solve

$$\frac{d^2u}{dv^2} + u + a \cdot \cos v = 0$$

by the method of the variation of parameters.

3. Determine $\Delta^n u_x$ in a series involving

$$u_{x+n},\ u_{x+n-1},\ u_{x+n-2},\ \&c.$$

4. When any number of forces act on a body, shew that the plane on which the sum of the projections of the *moments* is a maximum, is perpendicular to the planes with respect to which this sum is 0.

5. Find the attraction of a spheroid of finite eccentricity on a particle in its equator.

6. Explain fully the principles on which Newton calculates the correction in the motion of the nodes due to the unequable description of areas, and shew that the mean decrement when the nodes are in quadratures is equal to ¼ decrement in syzygy.

7. A body may revolve in the equiangular spiral in a medium of which the density is inversely as the distance from the centre by a force varying inversely as the square of the distance from the centre, (Newton, Book II. Prop. 15.) Prove this geometrically and analytically.

8. If a uniform force act upon a body tending to give it a motion of rotation about an axis always perpendicular to the axis about which it is at each instant revolving, and always in the same plane, the angular velocity is unaltered. Shew hence that the angular velocity of the Earth is not affected by the action of the Sun and Moon.

9. Construct for the place of high water in a given position of the Sun and Moon, and find an expression for the actual height of the compound tide.

10. Mention the facts from which it appears that the phenomena of the extraordinary ray in a double refracting crystal can be accounted for on the supposition of a repulsive force emanating from the axis.

TUESDAY MORNING, 9 o'clock to 11.

Second and Third Classes.

1. Approximate to the greatest root of the equation

$$x^3 - 7x = 1.$$

2. Having given the equation to an ellipse referred to its principal axes, transform it into one in which the axes are inclined at an angle θ, and in which the axis of y' makes with that of y, a given angle ϕ. Find also the relation between ϕ and θ when the transformed equation is of the same form with the original equation, and shew that in this case each of the new axes is parallel to the tangent drawn at the extremity of the other.

3. Shew that in a curve surface the sections of the greatest and least curvature are at right angles to each other.

4. Shew that every recurring series may be resolved into a certain number of geometric series, and exemplify the method by resolving the following series, and finding the sum of any number of its terms

$$1 + 4 + 18 + 80 + 356 + \ldots\ldots$$

5. If gravity act upon a system of bodies m', m'', ... and h', h'', ... be the vertical spaces described, and v', v'', ... be the actual velocities of the bodies, prove that

$$m'v'^2 + m''v''^2 + \ldots. = 2g(m'h' + m''h'' + \ldots.).$$

6. Required the geometrical focus of a thin pencil of rays after being refracted at a curved surface.

7. Find the place of a body in a parabolic orbit at any assigned time. (Newton, Book I. Prop. 30.)

8. Find the horary variation of the inclination of the lunar orbit to the plane of the ecliptic. (Newton, Book III. Prop. 34.)

9. Shew how to determine the altitude of mountains by the barometer, and explain the corrections to be applied in consequence of a change in temperature.

10. The centre of gravity of the Earth and Moon describes an orbit round the Sun much more nearly elliptical than that described by the Earth or Moon.

TUESDAY MORNING, 9 o'clock to 11.

Fourth Class.

1. Prove the rule for transforming a number from one scale of notation to another. Transform 1828 to local value 3.

THE SENATE HOUSE EXAMINATION.

MODERATORS,

Mr. WHEWELL, Trinity. Mr. KING, Queen's.

EXAMINERS,

Mr. MARTIN, Trinity. Mr. MELVILL, St. Peter's.

FRIDAY MORNING.—*January* 11, 1828.

9 o'clock to 11.

First, Second, Third, and Fourth Classes.

1. IN a given circle to inscribe an equilateral and equiangula pentagon.

2. If four quantities of the same kind be proportional, the greatest and least together are greater than the other two.

3. What is the amount of 37cwt. 2qrs. 14lbs., at £7 10s. 9d. per cwt. ?

4. Find $\sqrt{3\sqrt{3}+2\sqrt{6}}$, in the form of a binomial surd.

5. Find an expression for the sum of a decreasing geometric series, and explain clearly the possibility of an infinite number of terms having a finite sum.

6. Find the value of an annuity of £100 to commence 10 years hence and to continue for ever, allowing compound interest.

7. Determine the number of permutations of n things taken altogether, supposing the same quantities to recur.

8. Prove the binomial theorem when the index is fractional. Apply it to expand $(a^2 - \frac{2}{3}x^2)^{\frac{3}{2}}$ to five terms.

9. Given the sines and cosines of two arcs, find the sine of their sum and difference.

10. In the equation $x^4 + 8x^3 + x^2 - x - 10 = 0$ take away the second term, and then find the reducing cubic.

11. Find the sum of the sixth powers of the roots of the equation $x^3 - x - 1 = 0$.

12. Shew that Cardan's solution applies only to those cases in which the equation has two impossible roots, unless two of the roots be equal.

13. Transform the continued fraction $\dfrac{1}{a + \dfrac{1}{b + \dfrac{1}{c + \&c.}}}$ int a series of converging fractions, and prove that each of the lat-

ter approaches the value of the original fraction more nearly than the preceding.

14. If two chords of a parabola move parallel to themselves intersecting each other, the rectangles of their segments are in a constant ratio.

15. In the ellipse all the circumscribing parallelograms are equal.

16. Define the radius of curvature and prove that in an ellipse it $= \frac{CD^3}{PF}$.

17. Find the algebraic equation to the cissoid of Diocles, trace the curve, and deduce the polar equation, the cusp being the pole.

18. In a spherical triangle the sines of the angles are as the sines of the opposite sides.

19. Prove without the use of the integral calculus that the solid content of a cone is one-third that of a cylinder of the same base and altitude.

20. What is the logarithm of any number? Why is the common system selected? What is the base? Explain and prove the rule for proportional parts.

FRIDAY AFTERNOON, Half-past 12 to 3.

First, Second, Third, and Fourth Classes.

1. If a point be kept at rest by three forces acting upon it at the same time, any three lines which are in the direction of those forces and form a triangle will represent them.

2. A body is placed on a horizontal plane; find when it will be supported.

3. By what experiments is the third law of motion established?

4. Two bodies of given magnitude and elasticity, impinge directly upon each other with given velocities; find the velocity of each after impact.

5. A body is projected from a given point in a given direction with a given velocity, and acted upon by gravity; find where it will strike a given plane.

6. How is it shewn that fluids press equally in all directions? Apply this principle to the explanation of the Hydrostatical Paradox and Bramah's Press.

7. Find the density of the air in a common condenser after (t) descents of the piston.

8. When different planes move in directions perpendicular to their surfaces in different fluids and with different velocities,

the resistances will be as the squares of their velocities × densities of the fluid × areas of the planes.

9. If parallel rays be incident nearly perpendicularly upon a spherical refracting surface, the distance of the geometrical focus of refracted rays from the surface, is to its distance from the centre, as the sine of incidence to the sine of refraction.

10. Prove that the image of a straight line formed by a plane refracting surface is a straight line, and having given the inclination of the line to the refracting surface, find the inclination of the image.

11. Construct the Solar microscope.

12. Construct Gregory's telescope, find its magnifying power, and its greatest field of view.

13. Define continuous curvature, and shew that the arc, chord, and tangent of any curve of continuous curvature are ultimately equal.

14. Enunciate and prove Newton's eleventh Lemma,

15. If equal areas be described by a body in equal times about a given point in a given plane, the body is urged by a force tending to that point. (Newt. Book I. Prop. 2.)

16. If a body be projected from a given point in a given direction with a given velocity about a centre of force varying as the distance, shew that it will describe an ellipse having its centre in the centre of force, and find the magnitude and position of the axes. (Newt. Book I. Prop. 10. Cor. 1.)

17. Find the law of the force tending to the focus of an hyperbola. (Newt. Book I. Prop. 12.)

18. State the principal arguments for the diurnal rotation of the Earth round an axis, and its annual motion round the Sun.

19. Explain the theory of the aberration of light, and define clearly the plane in which it takes place.

20. Having giving the right ascension and declination of a star; find its latitude and longitude, and adapt the formulæ to logarithmic computation.

EVENING PROBLEMS.

1. The sums of the coefficients of the even and odd terms of any power of $a+b$ are equal.

2. If the side of a pentagon inscribed in a circle be 1, the radius is $\frac{\sqrt{5+\sqrt{5}}}{\sqrt{10}}$.

Prove this, and hence find the sine of 36° to 5 places of decimals.

3. A box is full of small spherical shot: what portion of the space is empty?

4. A ladder of uniform thickness rests with its lower end on a horizontal plane, and its upper end on a slope inclined 60° to the horizon. The ladder makes an angle of 30° with the horizon; find the force which must act horizontally at the foot to prevent sliding.

5. Find the velocity and direction of projection of a ball, that it may be 100 feet above ground at 1 mile distance and may strike the ground at 3 miles.

6. If a right cone of which the semi-angle is γ be cut by a plane making an angle δ with its axis, the ellipse thus obtained will have minor-axis : major-axis :: $\sqrt{\sin(\delta+\gamma)\sin(\delta-\gamma)} : \cos\gamma$.

7. If a line be drawn through the focus of an ellipse making an angle θ with the major-axis, and tangents be drawn at the extremities of this line: these tangents will make an angle ϕ, such that $\tan\phi = \dfrac{2e\sin\theta}{1-e^2}$.

8. What must be the relation of the distances from the Sun, of a superior and an inferior planet, that their synodical revolutions may be equal?

9. A telescope consists of 3 convex lenses whose focal lengths are $30m$, $3m$, m, the two latter being at a distance $2m$. Find its magnifying power, and the distance of the two first lenses. Trace the course of the rays.

10. Find the n quantities $x_1\, x_2\, x_3 \ldots x_n$, from the n equations,

$$\begin{aligned}
x_1 + x_2 \ldots\ldots\ldots\ldots + x_n &= A\\
a_1 x_1 + a_2 x_2 \ldots + a_n x_n &= 0\\
a_1^2 x_1 + a_2^2 x_2 \ldots + a_n^2 x_n &= 0\\
\ldots\ldots\ldots\ldots\ldots\ldots\\
a_1^{n-1}x_1 + a_2^{n-1}x_2 \ldots + a_n^{n-1}x_n &= 0,
\end{aligned}$$

and obtain symmetrical expressions for $x_1\, x_2$, &c.

11. Shew that

$$\frac{1}{(a+x)^m} = \overline{p+1} \text{ terms of the expansion of } (a+x)^{-m},$$

$$+\frac{x^{p+1}}{(a+x)^m} \times m \text{ terms of the expansion of } (\overline{a+x}-a)^{-\overline{p+1}}.$$

12. Draw the curve whose equation is $y = \sin x + 2\sin 2x$. Find all its points of maximum, flexure, and intersection; and shew after what values of x its form will recur.

13. Integrate

$$x^3 dx \sqrt{\frac{1+x^2}{1-x^2}}; \quad x^{nx} dx \begin{bmatrix} x = 0 \\ x = 1 \end{bmatrix};$$

$$\frac{d^2x}{dt^2} = 2x - 5y, \quad \frac{d^2y}{dt^2} = x + 2y;$$

and sum the series

$$1 + \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 3}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 4}x^3 + \text{\&c. } \textit{in infinitum.}$$

14. In a given sphere rests a given plane triangle of uniform thickness; find the angle which it makes with the horizon.

15. Find the form of a uniform chain suspended from any two points on the surface of an upright cone, and resting on the curve surface. Find the tension when it becomes a horizontal circle.

16. On a given triangle a pyramid is to be constituted of a given content. Determine it so that its surface may be the least possible.

17. *A* throws 6 dice, *B* throws 12, *C* throws 18. Compare the chances of *A* throwing one six, *B* two sixes, and *C* three sixes.

18. Three stars *A*, *B*, *C* are very nearly in a great circle, the angle made by *A* and *C* at *B* being 180 — β, where β is small. Shew that if t be the time which elapses between *AB* and *BC* being vertical,

$$t = \frac{\sin z}{\cos. a \cos l} \cdot \frac{\beta}{15},$$

where z is the zenith distance and a the azimuth of *B*, and l the latitude of the place.

19. A style projects from the vertex of an upright cone; trace the hour lines on the surface of the cone; and find the time in each day during which the dial will serve.

20. In a steam engine working *expansively*, the influx of steam is stopped when it has filled $\frac{1}{m}$ of the cylinder, and the piston is afterwards driven by the expansion of the steam. Compare the effect of *a given quantity* of steam so employed with its effect when it is not stopped; the effect being measured by the force × space moved through.

21. A straight rod moves on a smooth horizontal plane, subject to the condition of always passing through a given point: determine its motion. Prove that the varying centre of gyration of the rod with respect to the fixed point will describe areas uniformly about that point.

22. If a body fall to the Earth in the time t'', the deviation

to the east of the point from which it fell will be $\frac{1}{8} g \alpha t^3 \cos l$; where l is the latitude, and α the angle described by the Earth in $1''$.

23. If the mass of the Earth increase slowly and uniformly, find the resulting *equation* of the Moon's place at any given time, the orbit being nearly circular.

24. Explain briefly the optical experiments and theories to which the following terms refer: *Fits of easy transmission and reflexion; Polarization; Plane of Polarization; Depolarization; Depolarizing Axis; Fringes; Interferences*.

25. Mention the steps of the proof by which Newton shewed that every particle of matter gravitates to every other particle with a force which is inversely as the square of the distance.

SATURDAY MORNING, 9 o'clock to 11.

First, Second, Third, and Fourth Classes.

1. Similar triangles are to one another in the duplicate ratio of their homologous sides.

2. If a straight line be at right angles to a plane, every plane passing through that straight line is at right angles to the same plane.

3. If a, b, c, be the sides of a plane triangle, and A the angle opposite to (a), and $S = \frac{a+b+c}{2}$, prove the four following formulæ:

$$\sin A = \frac{2}{bc} \sqrt{S \cdot (S-a)(S-b)(S-c)},$$

$$\sin \frac{A}{2} = \sqrt{\frac{(S-b) \cdot (S-c)}{bc}},$$

$$\cos \frac{A}{2} = \sqrt{\frac{S \cdot (S-a)}{bc}},$$

$$\tan \frac{A}{2} = \sqrt{\frac{(S-b)(S-c)}{S \cdot (S-a)}}.$$

State under what circumstances each of these formulæ may be used with the greatest advantage.

4. Sum the following series:

$$\left.\begin{array}{r} \cos a + \cos 3a + \cos 5a + \ldots \\ \text{and } \tan a + 2 \tan 2a + 2^2 \tan 2^2 a + \ldots \end{array}\right\} \text{to } (n) \text{ terms.}$$

5. If A, B, C, be the angles of any spherical triangle, and (a) the side opposite to A, prove that

$$\cos a = \frac{\cos A + \cos B \cdot \cos C}{\sin B \cdot \sin C}.$$

6. The sum of the squares of any two conjugate diameters in an ellipse is constant.

7. Investigate the rule for determining the greatest common measure of any two quantities, and apply it to find the greatest common measure of

$$x^3 - 11x^2 + 39x - 45, \quad \text{and } 3x^2 - 22x + 39.$$

8. The coefficient of the second term of an equation with its proper sign, is the sum of the roots with their signs changed; the coefficient of the third term is the sum of the products of every two roots with their signs changed; the coefficient of the fourth term is the sum of the products of every three roots with their signs changed, &c. &c.

9. Solve the equation $x^3 - 6x = 4$, by Trigonometry, and obtain a numerical result.

10. In how many years will a sum of money treble itself at $4\frac{1}{2}$ per cent. compound interest?

$$\log 3 = \cdot 4771213,$$
$$\log 1\cdot 045 = \cdot 0191163.$$

11. Find the least whole positive numbers which will satisfy the equation $7x - 9y = 29$.

12. Assuming the expansion of a^x, deduce a converging series for the logarithm of any number; and apply it to compute the Naperian logarithm of 5 to 7 places of decimals, that of 2 being $\cdot 6931471$; and shew how these two logarithms determine the modulus of Brigg's system.

13. Define the differential coefficient of any function, and from that definition find the differential coefficient of $\frac{a^2 + x^2}{a - x}$, and of $\tan x$.

14. Investigate Maclaurin's theorem, independently of Taylor's theorem, and apply it to find the series for a circular arc in terms of its sine.

15. If (y) be a function of (x), determine the conditions requisite for (y) to be a maximum or minimum; and exemplify the theory when $y = (x - a)^n$, both when (n) is even, and when it is odd.

16. Shew how to determine when a curve is concave, and when convex to the axis. Trace the curve whose equation is

$$a^3 y = x^4 - bx^3 - b^2 x^2,$$

and determine the number and nature of its singular points.

17. Find the differential of the arc of any curve, and apply it to determine the length of the common parabola.

18. Explain the method of resolving any rational fraction into its simple or quadratic factors, and shew the use of such resolution in the integration of

$$\frac{dx}{(x^3 - x^2)(x^2 + x + 1)}.$$

SATURDAY AFTERNOON, Half-past 12 to 3.

First, Second, Third, and Fourth Classes.

1. The equilibrium of the screw will take place when the power is to the weight as the distance of two contiguous threads to the whole circle described by the point where the force is applied.

2. Find the time of a body's descent down any arc of a cycloid, and shew that the times of the whole oscillations are as the square roots of the lengths of the strings.

3. If any number of forces act in the same plane upon a rigid body, determine their resultant, and the equation of the straight line in which the resultant acts.

4. Prove the formula for the place of the centre of gravity of any body, viz. $h = \frac{\int x\,dm}{m}$, and apply it to find the centre of gravity of a common parabola.

5. If a rigid body oscillate about a horizontal axis, find the length of a simple pendulum which shall oscillate in the same time.

6. A body falls towards a centre of force which varies as some power of the distance, determine the cases in which we can integrate so as to find the time of descent.

7. Knowing the force, which varies as $\frac{1}{D^2}$, and the velocity of projection from a given point, to find the path described. (Newton, Prop. 17.)

8. State and prove Newton's construction for the path of a body projected from an apse with a velocity less than that acquired by falling from an infinite distance, and acted upon by a force varying inversely as the cube of the distance. (Newton, Prop. 41. Cor. 3.)

9. If the force vary as $\frac{1}{A^2} + \frac{1}{A^3}$, find the angle between the apsides by Newton's method. (Prop. 45.)

10. Describe the variations which take place in the inclination of P's orbit during one revolution of the line of Nodes. (Newton, Prop. 66. Cor. 10.)

11. When a ray of light passes through a prism in a plane perpendicular to its axis, the deviation is a minimum when the incident and emergent rays make equal angles with the sides.

12. Find the field of view in Galileo's Telescope.

13. Find the longitudinal aberration of parallel rays refracted by a spherical surface.

2. Transform the equation $x^3 - 6x^2 + 11x - 6 = 0$, whose roots are α, β, γ, into the equation whose roots are

$$\frac{1}{\alpha^2 + \beta^2}, \quad \frac{1}{\alpha^2 + \gamma^2}, \quad \frac{1}{\beta^2 + \gamma^2}.$$

3. In the expression $y = 2x^3 - 15x^2 + 36x$, find for what values of x, y is a maximum or minimum, and in each case which.

4. Integrate $\dfrac{dx}{\sqrt{1 + x + x^2}}$, and $\dfrac{x^{2n}\,dx}{1 + x^2}$.

5. A pendulum is taken to the top of a hill; how many seconds a day does it lose?

6. Enunciate and prove Newton, Lemma 9.

7. If diverging rays fall upon a concave spherical surface of a rarer medium, to find the geometrical focus of refracted rays.

8. Construct the common pump, and find the height which the water rises at each stroke.

9. Construct a vertical south dial for a given latitude.

10. Shew how a planet, superior or inferior, may have its motion direct or retrograde, or may be stationary.

TUESDAY AFTERNOON, 1 o'clock to 3.

First Class.

1. Shew the method of integrating $Pp + Qq = R$, when neither $Pdy - Qdx = 0$, nor $Pdz - Rdx = 0$, are integrable separately and independently, and explain the process fully.

2. Define conical surfaces, and investigate their general equation.

3. Find the variation of $\int V dx$, and explain the use of that part of the result which is without the integral sign, and exemplify it by finding the shortest distance between two given straight lines not in the same plane.

4. In a lottery of (m) tickets, (n) of which are prizes, if (p) tickets be drawn at each time, what is the probability that all the prizes will be drawn after (q) drawings?

5. Give an analysis of the reasoning by which Newton explains the theory of the tides, and deduce a numerical comparison between the force of the Sun on the tides, and the force of gravity.

6. If a body float on a fluid, determine its stability at a small angle of inclination from a given position of equilibrium; and the time of one of its small oscillations.

7. Find the general equation for the motion of a vibrating cord.

8. Explain the method of determining the Sun's parallax by observation made on the transit of Venus over the Sun's disc.

TUESDAY AFTERNOON, 1 o'clock to 3.

Second and Third Classes.

1. From the equation $z = \int\left(\frac{y^2 - x^2}{x}\right)$ to eliminate by differentiation the quantity $\int\left(\frac{y^2 - x^2}{x}\right)$.

2. If the equations to two planes be $\left\{\begin{matrix} ax + by + cz = d \\ \alpha x + \beta y + \gamma z = \delta \end{matrix}\right\}$; find the angle made by the planes.

3. $y^2 = \frac{x^3 - a^3}{x + a}$; trace the curve and draw its asymptotes.

4. If u be a homogeneous function of x and y of m dimensions, shew that $mu = \frac{du}{dx} \cdot x + \frac{du}{dy} \cdot y$; and hence find a factor which renders $M \cdot dx + N \cdot dy = 0$ integrable, M and N being homogeneous functions.

5. At points of greatest and least curvature, the osculating circle will have with the curve a contact of a higher than the second order.

6. Find the velocity acquired by a cylinder unrolling and descending vertically through a given space.

7. Compare the times of bodies oscillating in a hypocycloid, revolving about the centre, and falling to the same centre, the force varying as the distance. (Newton, Prop. 52. Cor. 3,).

Find the attraction of a spherical shell in which the attraction of each particle $\propto \frac{1}{\overline{\text{dist.}}|^2}$ according to Newton's method, and analytically.

9. Find the aberration of a given star in R. A. in terms of the R. A., declination, obliquity, and Sun's longitude.

10. If m be the maximum lunar nutation in N. P. D., (l) the longitude of the Moon's ascending node, shew that when the longitude $= l'$, the nutation $= m \cdot \cos(l' - l)$.

TUESDAY AFTERNOON, 1 o'clock to 3.

Fourth Class.

1. Find the cosine of the angle contained between two straight lines whose equations are

$$y = ax + b, \text{ and } y = a'x + b'.$$

2. Shew how to determine the value of a vanishing fraction in all cases, and find the value of

$$\frac{1 - \frac{2x}{\pi}}{\text{cotan}\, x}, \text{ when } x = \frac{\pi}{2},$$

$$\text{and of } \frac{\sqrt{a} - \sqrt{x} + \sqrt{a - x}}{a - \sqrt{2ax - x^2}}, \text{ when } x = a.$$

3. Integrate the following differentials:

$$\frac{dx}{(x + a)(x^2 + a^2)}, \quad \frac{x^5 dx}{\sqrt{1 - x^3}}.$$

4. Sum the following series:

$$\left.\begin{array}{l} \frac{1}{1 \cdot 2} + \frac{3}{2 \cdot 3} + \frac{5}{3 \cdot 4} + \frac{7}{4 \cdot 5} + \\ 1^3 + 2^3 + 3^3 + \ldots\ldots \end{array}\right\} \text{to } n \text{ terms.}$$

5. Divide a cylinder filled with fluid into two such parts, that the times of emptying the fluid contained in each, through a small orifice in the base, may be the same.

6. A body descends in a straight line towards the centre of force varying as the distance, find the velocity acquired in descending through a given space, and the time of descent. (Newton, Book I. Prop. 38.)

7. Find the difference of the forces in the fixed and moveable orbit. (Newton, Book I. Prop. 44.)

8. Prove analytically that the areas described by a body about any centre of force are in the same plane, and proportional to the time.

9. Find the precession in north polar distance.

10. If a small pencil of parallel homogenial rays be refracted into a sphere, and the ratio of the sine of incidence to the sine of refraction be known, to find at what angle the rays must be incident, that they may emerge parallel after any given number of reflections within the sphere.

THE SENATE HOUSE EXAMINATION.

MODERATORS,
Mr. JEFFREYS, St. John's. • Mr. BOWSTEAD, Corpus.
EXAMINERS,
Mr. WHEWELL, Trinity. Mr. KING, Queen's.

FRIDAY MORNING.—*January* 11, 1829.
9 o'clock to 11.

First, Second, Third, and Fourth Classes.

1. IN a circle the angle in a semicircle is a right angle: but the angle in a segment greater than a semicircle is less than a right angle: and the angle in a segment less than a semicircle is greater than a right angle.

2. Extract the square root of 235.6 to two places of decimals: and the cube root of .000079507.

3. Having given two sides and the included angle of a plane triangle, find the angles at the base: find also the third side by an independent method, and adapt the trigonometrical expressions to logarithmic computation.

4. Prove the formula

$$\sin \overline{a+b} = 2 \sin a - \sin(a-b) - 4 \sin a \,.\, \sin^2 \frac{b}{2}.$$

and explain fully its use in the construction of the Trigonometrical Canon.

5. In an ellipse prove that

$$CP^2 + CD^2 = AC^2 + BC^2.$$

6. Prove that the chord of curvature of an hyperbola through the focus $= \dfrac{2CD^2}{AC}$.

7. Expand a^x in a series ascending by the powers of x.

8. Prove that the Napierian logarithm of $N + z$

$$= \text{Nap. log.}\, N + 2 \left\{ \frac{z}{2N+z} + \frac{1}{3} \frac{z^3}{\overline{2N+z}|^3} + \&\text{c.} \right\}$$

and from this formula shew how the logarithm of a number of six places of figures may be found from a table computed only to five places of figures.

9. Prove the rule for finding the greatest common measure of two algebraical quantities and apply it to find the greatest common measure of

$$x^3 - 8x^2 - 12x + 144, \text{ and } 3x^2 - 16x - 12.$$

10. Prove the binomial theorem for any value of the index.

11. A ratio of greater inequality is diminished and of less inequality increased, by adding any quantity to both its terms.

12. The reciprocals of quantities in harmonical progression are in arithmetical progression.

13. Every equation whose roots are possible, has as many changes of sign from + to −, and from − to +, as it has positive roots; and as many continuations of the same sign, from + to + and from − to −, as it has negative roots.

14. If the equation $x^3 - px^2 + qx - r = 0$ has two equal roots, one of them is $\frac{9r - pq}{6q - 2p^2}$, but the converse is not necessarily the case.

15. Explain the method of finding those roots of an equation which are whole numbers, by the *method of divisors*, and apply it to solve the equation

$$x^3 - 9x^2 + 22x - 24 = 0.$$

16. State Napier's rules for the solution of right-angled spherical triangles, and prove them when the complement of the hypothenuse is the middle part.

17. Find the equation to the curve in which the distance of any point from a given fixed point, is equal to the perpendicular drawn from the same point in the curve upon a given line.

18. Find the present value of an annuity to be paid for n years, allowing compound interest.

19. Having given the equation to a straight line, find the equation to another straight line drawn perpendicular to it from a given point; find also the length of the perpendicular.

20. Every square number is of the form $5n$ or $5n \pm 1$.

FRIDAY AFTERNOON.....Half-past 12 to 3.

First, Second, Third, and Fourth Classes.

1. If two weights acting perpendicularly upon a straight lever on opposite sides of the fulcrum, or two forces in opposite directions on the same sides of it, are inversely as their distances from the fulcrum, they will balance each other.

2. If on an isosceles wedge of which the angle is 2α, a power P acting perpendicular to the base, balance a resistance

W acting on each of the sides in a direction making an angle ϵ with a perpendicular to the side

$$P : W :: \sin\alpha : \cos\epsilon.$$

3. When a system is in equilibrium, if a small motion be given to its parts, the centre of gravity will neither ascend nor descend.

4. If two imperfectly elastic bodies impinge obliquely on each other with given velocities and directions, find the velocities and directions of their motions after impact.

5. If a body acted on by gravity be projected from a given point A with a given velocity so as to strike a given point Q, find the direction of projection: and if AI bisects the angle which AQ makes with the vertical, shew that there are two such directions equally inclined to AI.

6. Explain how fluids press equally in all directions; and from this shew that in all tubes communicating with each other, a fluid will stand at the same altitude.

7. Shew that when a body floats in a fluid, the weight of the body is equal to that of the fluid displaced, and that their centres of gravity are in the same vertical line; and hence explain the construction and use of the hydrometer.

8. If a given quantity of air be left in the tube of a barometer, find the depression below the standard altitude.

9. If Q and q be the conjugate foci of rays incident nearly perpendicular on a spherical reflector; E the centre, and F the principal focus,

$$FE^2 = FQ \cdot Fq.$$

10. A *convex* spherical refracting surface of a *denser*, and a *concave* of a *rarer* medium, diminish the divergency and increase the convergency of all pencils of rays incident nearly perpendicularly, unless the focus of incident rays be between the surface and centre of the refractor.

11. Construct Gregory's telescope. Draw accurately the course of the extreme ray. Express the magnifying power in terms of the focal lengths and the distance between the principal foci of the reflectors.

12. Prove and explain Newton, Lemma II.

13. Enunciate and prove Newton, Lemma X.; and if the force be constant shew that it is true when the time is finite.

14. Find the law of force acting in parallel lines by which a body will be made to describe a portion of the circle. (Newt. Prop. 8.)

15. Find the law of force tending to the focus, by which a body may be made to describe an ellipse. (Newt. Prop. 11.)

16. Explain the nature of centrifugal force, and shew that in

a body revolving about a centre it varies inversely as the cube of the distance.

17. If a body be projected from a given point with a given velocity in a given direction, and acted on by a force which varies inversely as the square of the distance; find the conic section described.

18. Explain the cause of change of seasons, and of the different lengths of day and night. Shew that the full Moon in winter is longer above the horizon than in summer.

19. Explain by what observations the path of the Sun among the fixed stars is determined.

20. Shew that the aberration of a star takes place towards a point of the ecliptic 90° before the Earth's place, and that it varies as the sine of the angle of the Earth's way.

21. Explain the cause of twilight, and shew how its duration may be found by the declination of the Sun, and the latitude of the place. Find also on what day it is shortest at a given place.

SATURDAY MORNING......9 o'clock to 11.

First, Second, Third, and Fourth Classes.

1. If two triangles have the sides about equal angles reciprocally proportional, they are equal.

2. If a straight line be at right angles to two straight lines at their point of intersection, it is at right angles to the plane passing through them.

3. Investigate a true rule for the equated time of payment of two sums due at different times. Who is the gainer by the common rule?

4. Expand the n^{th} power of $\sin A$ in terms of sines and cosines of multiples of A, and write down the last term with its proper sign when n is of the form $4m + 2$.

5. State the construction of the polar triangle, and shew that its sides and angles are respectively the supplements of the angles and sides of the original triangle.

6. If a line, intersecting an hyperbola in the point P and its assymptotes in R, r, move parallel to itself, the rectangle $RP \cdot Pr$ is constant.

7. The section of a right cone made by a plane will be an ellipse, hyperbola, or parabola. Prove this and determine the position for the plane in each case.

8. Shew that any recurring equation of $2m$ or $2m + 1$ dimensions may be solved by an equation of m dimensions.

9. In the solution of a biquadratic by Des Cartes's method,

whatever root of the reducing cubic is employed the same values of the roots of the biquadratic will be obtained.

10. If $\frac{N}{N'}$ and $\frac{P}{P'}$ be successive approximations to the value of a continued fraction,

$$NP' - PN' = \pm 1.$$

11. Investigate the formulæ for the transformation of rectangular co-ordinates to oblique co-ordinates in the same plane; and hence, having given the equation to a parabola referred to its principal axis, find its equation referred to any other diameter; the abscissa being measured from a point in the curve, and the ordinate being parallel to the tangent at that point.

12. Find the limiting ratio of the corresponding *increments* of $\sqrt{a^2 + x^2}$ and x; and those of $\frac{e^x + 1}{e - 1}$ and x.

13. Expand $\sin x$ and $\cos x$ by Taylor's Theorem; and find limits of the value of the terms after the n^{th} term in each case.

14. Find the rectangular equation to the conchoid of Nicomedes, and draw a tangent to the curve.

15. Trace the curve of which the equation is

$$y = x \cdot \frac{x - a}{x - 2a};$$

and find its maximum and minimum ordinates.

16. Define the differential; and hence find the differential of a solid of revolution. Also apply this to prove that the sphere is $\frac{2}{3}$ of the circumscribing cylinder.

17. Find the sum of the series

$$1 \cdot 3 \cdot 4 + 2 \cdot 4 \cdot 5 + 3 \cdot 5 \cdot 6 + \&\text{c. to } n \text{ terms.}$$

$$\text{and of } \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \&\text{c. } \textit{in infinitum.}$$

18. Integrate $\frac{dx}{x^3 + 1}$; and shew that between the limits $x = 0$, and $x = 1$ it is

$$\frac{12}{3} + \frac{\pi}{3\sqrt{3}}.$$

SATURDAY AFTERNOON, Half-past 12 to 3.

1. Find the ratio of the power to the weight in that system where each pulley hangs by a separate string; first, when the strings are parallel; secondly, when they are not.

2. Find the resultant of any number of parallel forces acting

on a rigid body, and shew that they cannot in all cases be reduced to a single force which shall have the same effect.

3. Determine the equation to the catenary, the force of gravity being supposed constant.

4. Find the limit of the velocity communicated by a body *A* to *C*, through an indefinite number of mean proportionals between *A*, *C*, the bodies being supposed perfectly elastic.

5. Explain D'Alembert's principle, and apply it to determine the motion of two bodies connected together by a wheel and axle; the inertia of the machine being taken into account.

6. At similar points in similar curves described round centres of force similarly situated, the forces are as the squares of the velocities directly, and the distances inversely.

7. Find the place of a body in an elliptic trajectory after a given time. (Newt. Prop. 31.)

8. State and prove the proportion which Newton has given in Prop. 44, Cor. 1, for determining the actual value of the difference of the forces in the fixed and moveable orbit, and apply it to determine the whole force on *p*, when the fixed orbit is an ellipse, and the force in the focus.

9. If *P* and *S* attract each other, the curve which *P* describes relatively to *S*, may be described by *P* round *S* fixed. (Newt. Prop. 58.)

10. Prove that the pressure upon any portion of a vessel filled with a fluid of uniform density is equal to the weight of a column of fluid whose base is the area of the surface pressed, and altitude the perpendicular depth of its centre of gravity below the surface of the fluid; and find the whole pressure on the surface of a spherical segment filled with fluid.

11. If a floating body revolve round an horizontal axis and so pass through all its positions of equilibrium; they will be alternately stable and unstable.

12. If the distances above the surface of the Earth increase in arithmetrical progression, the corresponding densities of the air will decrease in geometrical progression; the force of gravity being invariable.

13. Explain the formation of the primary and secondary Rainbows, and why the red arc is exterior in the former case and interior in the latter.

14. Converging rays are incident upon a concave spherical reflector; find the longitudinal and lateral aberrations.

15. Explain Flamstead's method of determining the right ascension of the Sun, by observations made near the equinoxes; and show that it will serve to determine the place of the equinox.

16. On a given day and hour, find the azimuth of the ascending point of the ecliptic, and the longitude and altitude of the nonagesimal degree.

17. Find the times of the beginning and end of a lunar eclipse, and the number of digits eclipsed.

18. Prove that the stereographic projection of any circle on the sphere, is a circle; and find the centre and radius of the circle.

EVENING PROBLEMS.

1. Find the value of the circulating decimal 3·42753753 &c. also perform the same operation with the fraction 45·2534534 &c. where the radix is 6.

2. The sides of a triangle are in arithmetical progression, and its area is to that of an equilateral triangle of the same perimeter as 3 : 5. Find the ratio of the sides, and the value of the largest angle.

3. A cone and sphere of given weights, support each other between two given inclined planes, the cone resting on its base. Determine what must be the vertical angle of the cone, that the equilibrium may subsist.

4. Trace the curve, of which the equation is $y^5 = ax^4 + mx^5$; draw its asymptote, and determine its singular points.

5. An hemisphere with its base downwards is filled with fluid, and inclined at a given angle to the horizon: the base being moveable about a tangent at its upper extremity, find the force which applied at the centre will keep it at rest.

6. If tangents be drawn from a given point to each of a given system of curves, shew generally how to determine the curve which is the locus of all the points of contact; and apply the method when tangents are drawn from a given point, to a system of concentric similar ellipses.

7. Find an expression in terms of the sides of a spherical triangle for the arc drawn from one angle (C) bisecting the opposite side (c), and adapt the expression to logarithmic computation,

8. If the object placed before a lens, or a spherical reflecting or refracting surface, be a conic section, of which the focus and axis coincide respectively with the centre and axis of the reflector or refractor; the image will be a conic section. Prove it in each case, and find the eccentricity and axes of the image.

9. Find the equation between the angle and radius vector in a spiral, in which the radius vector is always equal to n times the

chord of curvature drawn through the pole. Find also the value of the radius of curvature in such a spiral.

10. The style of a horizontal dial, which is accurately graduated for a given place, is bent through a small given angle (δ); if (α) be the hour angle, at which there is no error in the time denoted by the dial, find the error in the time denoted at any hour angle (h), on a given day; and shew that at six o'clock the error is independent of the latitude of the place, and vanishes at sun-set.

11. To inscribe the greatest ellipse in a given semicircle, one axis of the ellipse being parallel to the diameter of the semicircle.

12. If there be n bags containing each (a) white and (b) black balls, and a ball be drawn out of each successively, shew how to determine beforehand what is the most probable number of white balls that will be drawn; and apply the process to the case of 12 bags containing each 2 black and 7 white balls.

13. Explain the use of the Astronomical Clock, and shew how we may determine whether it goes uniformly at all hours, Also explain clearly why the *mean daily rate* determined by the assistance of tables, differs from that determined by direct observations on the transits of stars, and state for what stars this disagreement is perceptible.

14. A body urged towards a plane by a force varying as the perpendicular distance from it, is projected at right angles to the plane from a given point in it with a given velocity. Find what force must act at the same time on the body parallel to the plane, that it may describe an arc of a given parabola having its axis in the plane; and determine the circumstances of the motion.

15. If the distance CP in an ellipse be a mean proportional between the semi-axes, prove that the semi-conjugate CD divides the quadrant of the ellipse into two arcs, whose difference is equal to the difference of the semi-axes.

16. AM, MN, NP being the co-ordinates to the point P in a curve surface, find the equation to the surface when the foot of the normal lies always in the centre of gravity of the triangle AMN.

17. Integrate the following differentials and differential equations:

$$\frac{dx}{\sqrt[3]{1+x}+\sqrt[5]{1+x}} \quad \frac{dx}{(x^2+1)\sqrt{x^2-2}},$$

$x^6 . \sin x dx$ from

$$x = 0 \text{ to } x = \frac{\pi}{2},$$

$$\frac{dx}{dy} = xy + x^2y^3,$$

$$2x\frac{dz}{dx} + y\frac{dz}{dy} = 2xy,$$

and $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} + y = 0$, in a rational form,

and sum the series

$$\frac{16}{2\cdot 3\cdot 4} - \frac{2\cdot 21}{3\cdot 4\cdot 5\cdot 3} + \frac{2^2\cdot 26}{4\cdot 5\cdot 6\cdot 3^2} - \&c.$$

to n terms and to infinity.

18. Find the whole longitudinal aberration when a pencil of parallel rays passes through a plano convex lens of inconsiderable thickness, the rays being incident on the convex side in a direction parallel to the axis of the lens.

19. If a solid hemisphere rests on its base wholly immersed in a fluid of less specific gravity than itself, whilst the fluid flows horizontally against it with a given velocity, find the whole pressure of the hemisphere against the bottom of the vessel, the solid being kept at rest by the friction.

20. Find the attraction on a particle placed within an heterogeneous spheroidal shell of small eccentricity, the density being the same throughout concentric spheroidal surfaces of different eccentricities, and the internal and external surfaces being of given eccentricity, and the density uniform throughout them.

21. A uniform rod is oscillating about one extremity; find the tendency of the vis inertia in any given position to bend the rod at any point, and determine at what point that tendency is the greatest.

22. A body is oscillating in a cycloid, in a medium where the resistance varies as the $(\text{vel.})^2$, and the density varies inversely as the arc measured from the lowest point: prove by a method similar to Newton's in Book II. Prop. 26. that the time of descent to the lowest point will be the same from all altitudes, and apply the integral calculus to find the whole time, supposing the resistance at the highest point corresponding to any velocity (v), to be less than $\frac{v^2}{l}$, where (l) is the length of the pendulum.

23. If a straight line DCP be made to revolve about C, and cut the curve PP_1P_2 in as many points as it has dimensions; and if $\frac{1}{CD^2}$ be made always equal to $\frac{1}{CP^2} + \frac{1}{CP^2_1} +$

&c., the locus of the point D will be a conic section whose centre is C.

24. Prove that the motion of a body P round T when disturbed by a body S is determined by the equation

$$\frac{d^2x}{dt^2} + \frac{P+T}{r^3}\cdot x + \frac{dR}{dx} = 0.$$

together with two similar equations in y and z, where $r = PT$, and

$$R = \frac{S\cdot r}{ST^2}\cdot \cos \angle STP - \frac{S}{SP};$$

and by means of them exemplify fully the method of determining the variation of the elements of P's orbit arising from the disturbance, when that disturbance is small.

EVENING PROBLEMS.

1. A banker borrows money at $3\frac{1}{2}$ per cent. per annum, and pays the interest at the end of the year: he lends it out at the rate of 5 per cent. per annum, but receives the interest quarterly, and by this means gains £200 a year. How much does he borrow?

2. Given $\tan 3A = n \tan A$. Find A in terms of n: find also the value of n that A may be 15°.

3. AP is the arc of a conic section, of which the vertex is A. PG the normal, and PK a perpendicular to the chord AP, meets the axis in G and K. Shew that GK is equal to half the latus rectum.

4. All objects from the zenith to the horizon are visible to an eye under water, and appear to be bounded by a conical surface of which the eye is the vertex. Explain this, and, having given the refracting power of water, find the vertical angle of the conical surface. State also what is seen beyond the limits of the conical surface.

5. The equation to a conic section is

$$5y^2 + 2xy + 5x^2 - 12x - 12y = 0.$$

Find its centre, and the magnitude and position of its principal axes.

6. A weight W is suspended from a point P of an uniform catenary APA'. O and O' are the lowest points of two uniform catenaries, of which AP and $A'P$ are parts. Shew that W is equal to the *difference* or *sum* of the weights of the portions OP, $O'P$ of the catenaries according as AP and $A'P$ are *one*, or *both* less than a semi-catenary.

7. The coefficient of x^n in the expansion of

$$(1 + x + 2x^2 + 3x^3 + \dots\dots \textit{ad inf.})^2$$

is equal to $\frac{n^3 + 11n}{6}$.

8. A body revolving in a given ellipse, force in focus, leaves the higher apse; and when it arrives at the lower apse A, the absolute force is suddenly altered, so that the body describes a similar ellipse, of which A is the higher apse; and a like alteration takes place when the body arrives at the lower apse of the new ellipse: and so on at each successive apse. Find the time to the centre of force.

9. The heights of the ridge and eaves of a house are H and h, and the roof is inclined at 30° to the horizon. Find where a sphere *rolling* down the roof from the ridge will strike the ground, and also the time of descent from the eaves.

10. If $A_1, A_2 \dots . A_n$ and $B_1, B_2 \dots . B_n$ be two series of positive numbers arranged in order of magnitude, of which A_1 and B_1 are respectively the greatest, shew that $\frac{A_1}{B_1} + \frac{A_2}{B_2} + \dots + \frac{A_n}{B_n}$ is less, and $\frac{A_1}{B_n} + \frac{A_2}{B_{n-1}} + \dots . + \frac{A_n}{B_1}$ greater than if the denominators $B_1, B_4 \dots . B_n$ be arranged in any other order under $A_1, A_2 \dots . A_n$.

11. A hemispherical vessel filled with fluid revolves round its vertical diameter with such an angular velocity that half the fluid is thrown out: required the pressure on the surface.

12. The first of February of the present year (1829) falls on a Sunday. Find generally when this will happen again; and write down all the years in which it will occur during the present century.

13. The content of any segment of a right or oblique prismatic solid, is equal to the area of one end of the segment multiplied into the perpendicular let fall upon it from the centre of gravity of the area of the other end.

14. A river, of which the breadth is a, flows with a velocity u, and a swimmer whose velocity is nu, always aims at a mark on the farther bank directly opposite to the place where he entered the river. Find the curve in which he swims, and shew that the time of his arriving at the mark is equal to

$$\frac{a}{u} \cdot \frac{n}{n^2 - 1}.$$

15. Two vessels, of which the capacities are a and b, are filled, the one with wine and the other with water; equal quan-

ties c are taken from each and poured into the other; and this operation is repeated n times. Find the quantities of wine and water remaining in each vessel.

16. Prove that the integral of $\frac{x^{m-1}dx}{x^n+1}$ between the limits $x=0$ and $x=\infty$, is equal to $\frac{\pi}{n\sin\frac{m}{n}\pi}$, n being greater than m.

17. A uniform rod, of which the elasticity is e, falls upon a smooth horizontal plane: given the altitude from which it falls, and its inclination to the horizon, find its motion after rebounding.

18. If rays are incident upon a common looking glass in an oblique direction from a candle, one faint image is observed before the principal image, and a row of them behind it. Explain this; and find the caustic formed by the rays emergent from the glass after reflection at the quicksilver, the thickness of the glass and its *refracting* power being given.

19. A and B sit down to cards, the former having p shillings and the latter q, and they agree each to stake a shilling on every game, and to play till one has lost all his money. Find the value of the expectation of each before they begin to play, supposing the skill of A : that of B :: m : n.

20. Integrate

$$\frac{dx}{(a^2+x^2)^4},\quad \frac{dx}{a+b\sin x+c\cos x},\quad \frac{dy}{dx}=\frac{10+6y-4x}{6x-9y+3},$$

$$\left.\begin{aligned}\frac{d^2y}{dt^2}+2\frac{d^2x}{dt^2}&=2y-3x-1\\ \frac{d^2x}{dt^2}-2\frac{d^2y}{dt^2}&=x+16y-3\end{aligned}\right\},$$

and sum the series

$$\frac{1}{2^2-3^2}+\frac{1}{4^2-3^2}+\frac{1}{6^2-3^2}+\ldots\text{to } n \text{ terms, and to infinity,}$$

$$1\cdot 3\sin\theta+3\cdot 5\cdot\sin 3\theta+5\cdot 7\cdot\sin 5\theta+\ldots\text{to } n \text{ terms.}$$

$$\frac{1}{1^3\cdot 3^3}+\frac{1}{1^3\cdot 5^3}+\&c.+\frac{1}{3^3\cdot 5^3}+\&c.\ \textit{ad inf.}$$

21. If x_1, y_1, z_1, be the distances from the origin of the coordinates at which a tangent plane to a curve surface cuts the axes x, y, z, and if $x_1{}^n+y_1{}^n+z_1{}^n=a^n$, prove that the equa-

tion to the surface touched is

$$x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} + z^{\frac{n}{n+1}} = a^{\frac{a}{n+1}}.$$

22. $u_x \,.\, u_{x+\pi} = a(u\ u_{x+\pi})$ Give a complete solution of this equation, and determine the particular value of the constants when it is the equation to a conic section about the focus.

23. If u_1, u_2, be values of u which satisfy the differential equation

$$\frac{d^2u}{dt^2} + M\frac{du}{dt} + Nu = 0,$$

M and N functions of t, shew that the integral of
$\frac{d^2u}{dt^2} + M\frac{du}{dt} + Nu + \Pi = 0$ is $c_1u_1 + c_2u_2 - u_1\int\frac{u_2}{u_1}\int\frac{\Pi dt^2}{u_1 d\,.\,\frac{u_2}{u_1}}$.

24. The parallax of the Moon

$$= P\left\{1 + e\cos(c\theta - \alpha) + m^2\cos(\overline{2 - 2m}\,.\,\theta + 2\beta) + \frac{15}{8}me\cos(\overline{2 - 2m - c}\,\theta + 2\beta + \alpha)\right\}$$

where P = mean parallax, $m = \frac{\text{Sun's mean motion}}{\text{Moon's mean motion}}$, θ = long. of Moon, α = long. of perigee, $-\beta$ = Sun's mean long. when $\theta = 0$. Explain fully the effects of the several terms in the above expression on the Moon's orbit.

MORNING PROBLEMS.

1. Transform the equation $x^3 - px^2 + qx - r = 0$, whose roots are a, b, c, into one whose roots are

$$\frac{c}{a + b - c}\quad \frac{b}{a + c - b}\quad \frac{a}{b + c - a}.$$

2. If I have 9 half guineas and 6 half crowns in my purse, how may I pay a debt of £4 11*s.* 6*d.*?

3. If from two fixed points in the circumference of a circle, straight lines be drawn intercepting a given arc and meeting without the circle, the locus of their intersections is a circle.

4. Given the equations of two straight lines, find the equation to a third, which shall pass through their point of intersection, and make equal angles with them; and shew from the result that there are two straight lines at right angles to each other, which satisfy the question.

5. If three parallel forces acting at the angular points A, B,

C, of a plane triangle are respectively proportional to the opposite sides a, b, c; prove that the distance of the centre of parallel forces from A

$$= \frac{2bc}{a+b+c} \cos \frac{A}{2}.$$

6. In the ellipe if the distances CP, CQ be drawn at right angles to each other, prove that

$$\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{AC^2} + \frac{1}{BC^2}.$$

7. If R and r be the radii of the circumscribed, and inscribed circles of a regular polygon of m sides, and R', r' the corresponding radii for a regular polygon of $2m$ sides and of the same perimeter as the former, then $Rr' = R'^2$, and $R + r = 2r'$.

8. The chord of curvature at any point (x, y) of a curve drawn through a point whose co-ordinates are α. β,

$$= \frac{2(1+p^2).(y-\beta-p.\overline{x-\alpha})}{q\sqrt{(x-\alpha)^2+(y-\beta)^2}}, \text{ where } p = \frac{dy}{dx}\, q = \frac{d^2y}{dx^2}.$$

9. A body acted on by a centripetal force varying partly as $\frac{1}{D^3}$, and partly as $\frac{1}{D^5}$, is projected with the velocity which would be acquired in falling from infinity at an angle with the distance, whose tangent $= \sqrt{2}$, the forces being equal at the point of projection. Required the orbit described, at the time of descent to the centre.

10. An imperfectly elastic body slides down a smooth plane of given length, and is reflected from the horizontal plane. Find the inclination of the plane that the range may be a maximum, and find the range.

11. If b_n represent the co-efficient of x^n in the expansion of any function of x by Maclaurin's Theorem, and a_n in a similar expansion of the hyperbolic logarithm of that function, prove that

$$a_n = \frac{1}{n.b_0} . \left\{-\overline{n-1}b_1 a_{n-1} - \overline{n-2}b_2 a_{n-2} - \&c. - b_{n-1}a_1 + nb_n\right\}$$

and apply this theorem to determine the relation between the coefficients in the expansion of hyp. log. cos x.

12. The shadow cast by an oblate spheroid resting on its vertex in the Sun, on an horizontal plane, is an ellipse; aud the spheroid stands in its focus.

13. Integrate

$$\frac{dx}{(x+2)^2.(1+x^2)}, \quad \frac{dx}{x^4\sqrt[3]{a^3-x^3}}, \text{ and if } \frac{dy}{dx} = 1 + xy,$$

shew that

$$y = 1 + \frac{1}{1 \cdot 3} + \frac{1}{1 \cdot 3 \cdot 5} + \text{\&c. between } x = 0, \text{ and } x = 1.$$

Also integrate $\frac{z + y}{y} dx - \frac{x + y}{z} dz - \frac{y^2 - xz}{y^2} dy = 0$, by applying the criterion of integrability.

14. What is the least depth of fluid, in which a given cone can rest permanently with its axis vertical, the vertex of the cone resting on the base of the vessel, and the specific gravities of the cone and fluid being given.

15. If at a place between the tropics, (c) be the zenith distance of a known star, when the ecliptic comes upon the zenith of the place of observation, and (θ) the longitude of the Earth, when the corresponding aberration in zenith distance vanishes, prove that θ is determined by the equation.

$$\cot(\theta - L) = \frac{\sin^3 \lambda \, . \cos c}{\sqrt{\sin(c + \lambda) \, . \sin(c - \lambda)}},$$

where L and λ represent the longitude and latitude of the star.

16. Find the ratio of the height to the diameter of the base of a cylinder, that the moment of inertia may be the same about any axis whatever passing through its centre of gravity.

17. Find the length of the tide-day, the Sun and Moon being in the equator, and shew how the densities of the Sun and Moon may be compared, by observing the lengths of the greatest and least tide-days.

18. State the general nature of developable surfaces. Investigate the partial differential equation of the second order, which belongs to them, and integrate for the partial differentials of the first order. Shew also how such surfaces may be obtained from given curves of double curvature.

MONDAY AFTERNOON, 1 o'clock to 3.

First and Second Classes.

1. Integrate the differentials $e^x \, . \sin^m x dx$, and the differential equation

$$x^2 dy = \overline{x^2 - ay^2} \, . dx.$$

2. Draw a tangent plane to any curve surface, and determine the angle it makes with the plane of xy.

3. Find Σx^n in a series proceeding according to the descending powers of x.

4. In a common pump find the height through which the water ascends at any stroke.

5. A spherical surface being constituted of particles, the

forces of which vary as $\frac{1}{D^2}$; shew that the attraction of the whole surface on a particle without it, varies inversely as the square of the distance of the particle from the centre. (Newt. Prop. 71.)

6. If a body oscillate in a medium in which the resistance varies as the square of the velocity; the difference between the times of oscillation in the medium and in vacuo are proportional to the arcs nearly. (Newt. Book II. Prop. 27.)

7. The elevation of the summit of the spheroid produced by the attraction of the Sun or Moon on a fluid sphere is double of the depression of the equator below the sphere.

8. Investigate the motion of the pole of the Earth produced by the Moon in one sidereal revolution.

9. If V be any function of x, y, p, q, &c. prove that when $\int V dx$ is a maximum or minimum,

$$N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \&c. = 0.$$

MONDAY AFTERNOON, 1 o'clock to 3.

First and Second Classes.

Trace the curve of which the equation is

$$y^2(x-a) = x^3 - b^3;$$

when $a > b$ and when $a < b$. Find its asymptotes and singular points.

MONDAY AFTERNOON, 1 o'clock to 3.

Third and Fourth Classes.

1. Having given the equation to a plane; find the equations to a straight line perpendicular to it, and passing through a given point.

2. If the observed angular distance of two points be a, their observed elveations above the horizon being H and h, and θ the angular distance reduced to the horizon, shew that

$$\sin^2\frac{\theta}{2} = \frac{\sin\frac{1}{2}(a+H-h) \, . \, \sin\frac{1}{2}(a+h-H)}{\cos H \, . \, \cos h}.$$

3. Integrate $\frac{(x+2) \, . \, dx}{(x^2+1) \, . \, (x-3)^2}$, and the following differential equation $dy + ydx = ax^3dx$.

4. Explain the method of summing any recurring series, and sum the series 2, 5, 13, 35, &c. to n terms.

5. A body being acted upon by any number of forces in the same plane; find the equations of equilibrium.

6. When a body is acted on by forces X and Y in the directions of the co-ordinates x and y,

$$\text{prove } \frac{d^2x}{dt^2} = X \text{ and } \frac{d^2y}{dt^2} = Y.$$

7 Find the specific gravity of a body lighter than the fluid in which it is weighed.

8. In Newton, Prop. 66. explain the effect of the disturbing force of S in producing a motion of the nodes of P's orbit, and a variation of the inclination.

9. Find when the altitude of a known star increases fastest.

10. If a small pencil of parallel homogeneal rays be refracted into a sphere; find at what angle the rays must be incident that they may emerge parallel after any given number of reflections within the sphere.

TUESDAY MORNING, 9 o'clock to 11.

First Class.

1. Explain Newton's rule for discovering impossible roots in any equation.

2. Assuming the ordinary expansion of $\delta \int V dx$, determine the requisite addition to be made to it when V involves the limiting values of x, y, p, &c. and apply the method to find the positions of the curve of quickest descent from one curve to another, when the motion commences from the first curve.

3. Shew that the solution of the equation of differences

$$u_{x+n} + A_x u_{x+n-1} + \text{\&c.} + P_x u_x = Q_x$$

may be made to depend upon the solution of the equation

$$u_{x+n} + A_x u_{x+n-1} + P_x u_x = 0.$$

4. Investigate the general form of the equation to cylindrical surfaces, and apply it when the directrix is a curve of which the equations are

$$x^2 + y^2 = r^2 \text{ and } z = cx.$$

5. Apply Lagrange's theorem to determine the least root of the equation

$$x^3 - 5x + 7 = 0.$$

6. A body moveable about a fixed axis is acted upon by a single force in a plane perpendicular to the axis: find the pressure on the axis arising from that force, and thence determine fully the co-ordinates of the centre of percussion.

7. Find the horary motion of the nodes in an eliptic orbit. (Newt. Book iii. Prop. 31.)

8. Determine the attraction of an oblate spheroid on a particle situated in its equator.

9. Explain *fully* the method of determining the Sun's parallax by the transit of Venus over the Sun's disk.

TUESDAY MORNING, 9 o'clock to 11.

Second and Third Classes.

1. If there be a chances for an event happening and b for its failing in one trial, find the probability of its happening t times *at least* in n trials.

2. In the equation $x^2 - px + q = 0$, find the sums of the n^{th} powers of the roots in terms of the coefficients.

3. In the general equation of the second degree

$$ay^2 + bxy + cx^2 + ex + fy + g = 0,$$

shew in what cases the curve will be an ellipse, hyperbola, and parabola; and find the co-ordinates of the centre in the former case.

4. Investigate the conditions requisite in order that a function of two variables x, y may be a maximum or minimum. Apply them to find when

$$u = x^4 + y^4 - 4axy^2$$

is a maximum or minimum.

5. Shew that in a small spherical triangle, if $\frac{1}{3}$ of the *spherical excess* be subtracted from each of the angles, the resulting angles will be those of a plane triangle having the same sides as those of a spherical triangle.

6. When a body moves upon a surface of revolution, find the re-action of the surface.

7. If a body be acted upon by any forces, the motion of the centre of gravity will be the same as if all the forces were applied at that point: and the motion of rotation will be the same as if the centre of gravity were fixed and the same forces applied.

8. Explain accurately what is meant by the *equation of time;* and shew that it vanishes four times in a year.

9. Find the horary increment of the area described by the Moon; and compare its values at quadrature and syzygy. (Newt. B. iii. Prop. 26.)

10. Define the *metacentre* of any floating body; and investigate a formula for its position.

TUESDAY MORNING, 9 o'clock to 11.

Fourth Class.

1. If a, b, c, &c. be the roots of an equation, find the value of

$$a^2b + a^2c + b^2a + \&c.$$

2. Investigate the polar equation to the hyperbola, the focus being the pole (given that $SP - HP = 2AC$), and draw the asymptote by means of this equation.

3. Trace the curve whose equation is $y = \frac{a^2 x}{a^2 + x^2}$, and find the number and nature of its singular points.

4. Integrate $\frac{dx}{\sqrt[3]{1+x^3}}$, $\frac{xdx}{\sqrt{a+bx+cx^2}}$, $\frac{dx}{\cos^3 x}$.

5. Define the radius of gyration of any body or system of bodies moveable about a fixed axis, and investigate an expression for determining its magnitude: apply also this expression to a sphere revolving about a diameter.

6. Determine by the principles of Newton's seventh Section the spaces due to the velocity externally and internally in an ellipse, the force being in the centre.

7. Find the effect of the disturbing force of the Sun on the apsides of the lunar orbit during one revolution of the Moon. (Newt. Prop. 66. Cor. 7.)

8. Explain the construction of the fire engine, and shew the use of the *air vessel*.

9. Find the caustic when the reflecting curve is a logarithmic spiral and the luminous point in the pole.

10. Construct a vertical south dial, and determine how much of it must be graduated.

TUESDAY AFTERNOON, 1 o'clock to 3.

First Class.

1. Prove that

$$\Delta_n u_x = \frac{d^n u_x}{dx^n} + \frac{\Delta_n o^{n+1}}{1.2\ldots n+1}\,\frac{d^{n+1} u_x}{dx^{n+1}} + \frac{\Delta_n o^{n+2}}{1.2\ldots n+2}\,\frac{d^{n+2} u_x}{dx^{n+2}} + \&c.$$

2. In any surface of the second order which has a centre, the sum of the squares of any system of conjugate diameters is equal to the sum of the squares of the principal diameters.

3. Shew under what condition the equation

$$Pdx + Qdy + Rdz = 0$$

is integrable. Find also the factor which will render it integrable when homogeneous.

4. Explain the method of integrating the partial differential equation $Pp + Q = R$; when $Pdy - Qdx = 0$, and $Pdz - Rdx = 0$ are integrable separately. Apply the method to the equation $py + qx = z$.

5. Find the angles which the axis of instantaneous rotation makes with the co-ordinates x, y, z.

6. If an incompressible fluid contained in any vessel flow through an orifice k with a velocity u, z' being the vertical co-ordinate of the upper surface, and y' its area, z, y the same quantities for any other horizontal section, p the pressure on a

unit of this section,

$$p = \Pi + g(z - z') - k\frac{du}{dt}\int\frac{dz}{y} - \frac{k^2u^2}{2}\left(\frac{1}{y^2} - \frac{1}{y'^2}\right),$$

where Π is the atmospheric pressure on the surface. Also from this expression find the velocity of the issuing fluid when the orifice is small.

7. If a point move through one or more spaces bounded by parallel planes, and be acted upon by a force which is perpendicular to the planes, and which is the same at the same distance from them; the angle of incidence is to the angle of emergence in a given ratio. (Newt. Prop. 94.)

8. In a homogeneous spheroid attracting a point on the surface, the effect of the force parallel to the equator is as the distance from the axis.

9. Having given three geocentric places of a comet; find the corresponding heliocentric and geocentric distances.

TUESDAY AFTERNOON, 1 o'clock to 3.

Second and Third Classes.

1. Find the sum of the m^{th} powers of the roots of an equation in terms of the coefficients and the sums of the inferior powers.

2. If a curve have as many asymptotes as it has dimensions, and a right line be drawn which cuts them all, the parts of the line measured from the asymptotes to the curve will together be equal to the parts measured in the same direction from the curve to the asymptotes.

3. Shew the method of extracting the cube root of the binomial surd $a + \sqrt{b}$, and apply it to the solution of the equation $x^3 - 3x - 18 = 0$ by Cardan's rule.

4. Expand $\frac{1}{1 + e\cos x}$ into a series of the form $A + B\cos x + C\cos 2x$, and explain the law of the coefficients.

5. Eliminate by differentiation the constants from the equation $y^2 = ax + bx^2$, and shew how many *derivatives* of the m^{th} order there are to an equation containing n arbitrary constants.

6. Find the horary variation of the inclination of the lunar orbit to the plane of the ecliptic, and thence by Newton's construction determine its mean monthly value.

7. Explain the nature and use of the Ballistic pendulum, and perform the requisite calculations in the experiment.

8. Find the Moon's parallax by observations made out of the plane of the meridian, and shew how the effect of refraction may be avoided.

9. If a body move in a surface of revolution acted upon by a centre of force situated in the axis: the areas described, projected on a plane perpendicular to the axis, are proportional to the times. (Newt. Prop. 55.)

10. Find the attraction of a homogeneous spheroid of small excentricity on a particle situated on its pole.

TUESDAY AFTERNOON, 1 o'clock to 3.

Fourth Class.

1. If a circle be described on AM the axis major of an ellipse, and if an ordinate to the axis meet the ellipse in P and the circle in Q; S being a point in the major axis, the areas ASP, ASQ are in a constant ratio.

2. Solve by Cardan's rule $x^3 + 3x^2 + 9x - 13 = 0$.

3. If $5x + 21y = 2000$, find x and y, and the number of positive integer solutions which the equation admits of.

4. Integrate $dx \,.\, (a^2 + x^2)^{\frac{5}{2}}$, and $d\theta \,.\, \sin m\theta \,.\, \cos n\theta \,.\, \cos p\theta$.

5. Shew that when a body moves in an inverted cycloid, the force by which it is urged along the curve varies as the arc to the lowest point; and hence shew that the oscillations are isochronous.

6. Construct the air pump, and shew that the quantities of air expelled by successive strokes are in geometrical progression.

7. A person can see distinctly at the distance of four inches; find the focal length and nature of a lens which will enable him to see distinctly at the distance of sixteen inches.

8. If the velocities of two bodies at any equal distances from the centre of force be the same, and if one body move in a straight line to or from the centre and the other in a curve; their velocities will be the same at all other equal distances. (Newt. Prop. 40.)

9. If S and P revolve round their common centre of gravity by their mutual attraction, shew that each will move in the same manner as if a body were placed in the centre of gravity exerting a force varying according to the same law. What is the magnitude of this body for each of the two S and P, the force varying inversely as the square of the distance?

10. Find the length of the longest day in latitude $52° 13'$ the obliquity of the ecliptic being $23° 28'$.

Having given $10.1105786 = \log \tan 52° 13'$,
$9.6376106 = \log \tan 23° 28'$,
$9.7481230 = \log \cos 55° 57'$,
$9\ 7483099 = \log \cos 55° 56'$.